Sheet 3

3.1 (Nice for Lie's Theorem later) Let L be a Lie algebra.

1. Identify the kernel of the adjoint homomorphism
$$\text{ad}\colon L \to \mathfrak{gl}(L).$$
2. Let $\text{ad}\, L$ denote the image of the above homomorphism. Prove that:
$$L \text{ is a solvable Lie algebra} \iff \text{ad}\, L \text{ is a solvable subalgebra of } \mathfrak{gl}(V).$$
3. Show the above also holds if we replace 'solvable' with 'nilpotent'.

3.2 (Standard example of nilpotent) Consider $L = \mathfrak{n}_n$, strictly upper triangular matrices. Show that L^k has basis $\{e_{ij} \mid j > i + k\}$, and hence show that L is nilpotent.

Note that in pictures, for $n = 4$ this can be visualised as:

$$\underset{L}{\begin{pmatrix} 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{pmatrix}}, \quad \underset{L^1}{\begin{pmatrix} 0 & 0 & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}, \quad \underset{L^2}{\begin{pmatrix} 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}$$

3.3 (Standard example of solvable) Consider $\mathfrak{b}_n$, upper triangular matrices.

1. Show that $\mathfrak{b}_n' = \mathfrak{n}_n$.
2. By combining this with Q3.2, or otherwise, prove that $\mathfrak{b}_n$ is solvable.
3. For $n \geq 2$, justify why $\mathfrak{b}_n$ is not nilpotent.

3.4 Let L be a simple Lie algebra, and let K be a solvable Lie algebra.

1. Prove that if $\varphi\colon L \to K$ is a homomorphism, then $\varphi = 0$.
2. On the other hand, find an example of a simple L, solvable K, and a non-zero homomorphism $K \to L$.

3.5 Give an example of a Lie algebra homomorphism $\varphi\colon L \to M$ such that $\varphi(Z(L))$ is not contained in $Z(M)$.

3.6 (Part 2 is important, we will also prove the converse is true later)

1. Let L be a Lie algebra, and let I be an ideal of L. Prove that if $\text{rad}(I) = 0$ and $\text{rad}(L/I) = 0$, then $\text{rad}(L) = 0$.
2. Show that if $L = L_1 \oplus ... \oplus L_n$ where each L_i is simple, then $\text{rad}(L) = 0$.