For Rewriting Systems The Topological Finiteness Conditions FDT, FHT Are Not Equivalent*

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Abstract

A finite rewriting system is presented that does not satisfy the homotopical finiteness condition FDT, although it satisfies the homological finiteness condition FHT. This system is obtained from a group G and a finitely generated subgroup H of G through a monoid extension that is almost an HNN-extension. The FHT property of the extension is closely related to the FP_2 property for the subgroup H, while the FDT property of the extension is related to the finite presentability of H. The example system separating the FDT property from the FHT property is then obtained by applying this construction to an example group considered by Bestvina and Brady (1997).

1 Introduction

In the theory of finite string rewriting systems, two related geometric properties have been introduced. The first, a homotopical property, *finite derivation type* (FDT), was defined and studied by Squier [SOK]. The second, *finite homological type* (FHT), was introduced in [WP]. Both these properties are invariants in the sense that if two finite rewriting systems are Tietze equivalent (that is, if they represent the same monoid), then if one has the property so does the other. This allows us to talk about FDT or FHT *monoids*. In general, FDT implies FHT, and the properties are equivalent for groups [WP]. It has been an open question whether the properties are equivalent in general. Here we will show that this is not the case.

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Let G be a finitely presented group, and let H be a finitely generated subgroup of G. From G and H we obtain a monoid M by forming the free product $G * \{t\}^*$ of G and the free monoid on t, and then factoring by the congruence generated by the pairs (th, ht) $(h \in H)$. This monoid can be defined by a finite rewriting system. Here we will establish the following result, relating the FDT and FHT properties of M to certain properties of G and H.

Theorem 1 (i) *M is FDT if and only if G is FDT, and H is finitely presented.*

(ii) *M is FHT if and only if G is FHT (= FDT), and H is of type FP_2.*

Bestvina and Brady [BB] have exhibited a finitely generated subgroup H of a right-angled Coxeter group (or graph group) G which is of type FP_2, but not finitely presented. As a graph group, G is FDT. Thus, our construction applies to this group and its subgroup H. Further, as we will see in Section 5, the monoid M obtained from Bestvina and Brady's example has word problem decidable in quadratic time. This yields the following consequence.

Corollary 2 *There is a finite rewriting system R that has all of the following properties:*

1. *The word problem for R is decidable in quadratic time.*
2. *R is FHT.*
3. *R is not FDT.*

McGlashan [McG01],[McG02] has introduced and studied higher dimensional properties FDT_2, FHT_2 for rewriting systems. At the end of our paper we will briefly discuss these properties, and the relevance of our example to McGlashan's work.

2 Preliminaries

2.1 2-Complexes

It will be convenient throughout this paper to take a combinatorial model of 2-complexes.

A *graph* will consist of a set of vertices, and a set of edges. Each edge e will have an initial vertex ιe, and a terminal vertex τe. Also, each edge e will have an inverse edge e^{-1} $(\neq e)$, with $(e^{-1})^{-1} = e$, $\iota e^{-1} = \tau e$, $\tau e^{-1} = \iota e$. A selection of one edge from each pair $\{e, e^{-1}\}$ will be called an *orientation* $\mathbf{e}^+$ of the edge set $\mathbf{e}$.

A *non-empty path* α will consist of a sequence of edges $e_1\, e_2 \ldots e_n$ with $\tau e_i = \iota e_{i+1}$ $(1 \leq i < n)$. We then define $\iota\alpha$, $\tau\alpha$, α^{-1} to be ιe_1, τe_n, $e_n^{-1} \ldots e_2^{-1} e_1^{-1}$, respectively. For each vertex v there is also the *empty path* 1_v at v with no edges and $\iota(1_v) = \tau(1_v) = v$, $1_v^{-1} = 1_v$. A path α will be said to be *closed* (at v) if $\iota\alpha = \tau\alpha = v$. If α, β are paths with $\tau\alpha = \iota\beta$, then we can form the *product path* $\alpha\,\beta$ consisting of the edges of α followed by the edges of β.

A *2-complex* $\mathcal{K}$ will consist of a graph, together with a set $\mathbf{s}$ of closed paths ("defining paths") in the graph. The underlying graph of $\mathcal{K}$ will be called the *1-skeleton* of $\mathcal{K}$. Two paths will be said to be *homotopic* (in $\mathcal{K}$) if one can be transformed to the other by a finite number of the following operations ("homotopy moves"):

(i) insert/delete a subpath ee^{-1} (e an edge);

(ii) replace a subpath α by β if $\alpha\,\beta^{-1}$ is a cyclic permutation of some path in $\mathbf{s} \cup \mathbf{s}^{-1}$.

A path will be said to be *null-homotopic* if it is homotopic to an empty path.

The *fundamental group* $\pi_1(\mathcal{K}, v)$ of $\mathcal{K}$ at a vertex v will consist of all homotopy classes (α) of closed paths α at v, with multiplication $(\alpha)(\beta) = (\alpha\,\beta)$.

Let A be a group, and let λ be a function which associates to each edge e of $\mathcal{K}$ an element $\lambda_e \in A$, such that $\lambda_{e^{-1}} = \lambda_e^{-1}$. For a non-empty path $\alpha = e_1 e_2 \ldots e_n$ we then define λ_α to be $\lambda_{e_1}\lambda_{e_2}\ldots\lambda_{e_n}$ (product in A). If α is an empty path we define λ_α to be the identity of A. We will say that λ is a *mapping of* $\mathcal{K}$ *to* A if $\lambda_\sigma = 1$ for each $\sigma \in \mathbf{s}$. For such a mapping we will then have $\lambda_\alpha = \lambda_\beta$ whenever α and β are homotopic. In particular, for any vertex v of $\mathcal{K}$ we will have a well-defined group homomorphism

$$\pi_1(\mathcal{K}, v) \longrightarrow A, \quad (\alpha) \mapsto \lambda_\alpha.$$

We have the chain complex

$$C(\mathcal{K}) : 0 \longrightarrow C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \longrightarrow 0$$

where C_2 is the free abelian group on $\mathbf{s}$, C_1 is the free abelian group on an orientation $\mathbf{e}^+$ of the edge set of $\mathcal{K}$, and C_0 is the free abelian group on the vertex set of $\mathcal{K}$. The maps ∂_2, ∂_1 are defined by

$$\begin{array}{lcll} \partial_1 e & = & \tau e - \iota e & (e \in \mathbf{e}^+) \\ \partial_2 \sigma & = & \sum_{i=1}^{n} \varepsilon_i e_i & (\sigma \in \mathbf{s}, \sigma = e_1^{\varepsilon_1} \ldots e_n^{\varepsilon_n}, e_i \in \mathbf{e}^+, \varepsilon_i = \pm 1 \text{ for } 1 \le i \le n). \end{array}$$

Then

$$H_1(\mathcal{K}) = \frac{\text{Ker } \partial_1}{\text{Im } \partial_2}$$

is the *first homology group* of the chain complex $C(\mathcal{K})$.

A combinatorial 2-complex as described above is a blueprint for a 2-dimensional CW-complex, where the 1-skeleton is the geometric realization of the underlying graph and the defining paths are used as attaching maps for 2-cells.

For further information on combinatorial 2-complexes see the Appendix to [NP], or §1 of [Pr88].

2.2 The Squier complex

Let $\mathcal{R} = [\mathbf{x}; \mathbf{r}]$ be a string rewriting system. Here $\mathbf{x}$ is an *alphabet* and $\mathbf{r}$ is a set of *rewriting rules.* A typical rule $R \in \mathbf{r}$ has the form $R : R_{+1} = R_{-1}$, where R_{+1} and R_{-1} are distinct words on $\mathbf{x}$. The *monoid* $S = S(\mathcal{R})$ corresponding to $\mathcal{R}$ is the quotient of the free monoid $\mathbf{x}^*$ by the congruence generated by the rewriting rules $\mathbf{r}$. The congruence class of $W \in \mathbf{x}^*$ will be denoted by $[W]$.

There is a 2-complex $\mathcal{D} = \mathcal{D}(\mathcal{R})$ associated with $\mathcal{R}$ as follows.

The 1-skeleton has vertex set $\mathbf{x}^*$, and edge set consisting of quadruples

$$\mathbb{E} = (U, R, \varepsilon, V), \quad U, V \in \mathbf{x}^*,\ R \in \mathbf{r},\ \varepsilon = \pm 1.$$

The initial, terminal, and inverse functions on edges are given by

$$\iota(\mathbb{E}) = UR_\varepsilon V,\ \tau(\mathbb{E}) = UR_{-\varepsilon} V,\ \mathbb{E}^{-1} = (U, R, -\varepsilon, V),$$

respectively. The edge $\mathbb{E}$ is called *positive* if $\varepsilon = +1$. There are left and right actions of $\mathbf{x}^*$ on this graph: if $Y, Z \in \mathbf{x}^*$, then for any vertex W, $Y \cdot W \cdot Z = YWZ$ (the product in $\mathbf{x}^*$), and for any edge $\mathbb{E}$ as above $Y \cdot \mathbb{E} \cdot Z = (YU, R, \varepsilon, VZ)$. These actions naturally extend to paths. Note that an edge corresponds to a single application of a rewriting rule, and so a path $\mathbb{P}$ corresponds to a derivation of $\tau(\mathbb{P})$ from $\iota(\mathbb{P})$. Thus, for $W \in \mathbf{x}^*$, the set of vertices of the connected component containing W is just the congruence class $[W]$. Hence, the connected components are in one-to-one correspondence with the elements of S.

Now it may be that a word W has two *disjoint* occurrences of rewriting rules, that is, $W = UR_\varepsilon VU'R'_{\varepsilon'}V'$, where $U, V, U', V' \in \mathbf{x}^*$, $R, R' \in \mathbf{r}$, and $\varepsilon, \varepsilon' \in \{-1, +1\}$. Let $\mathbb{E} = (U, R, \varepsilon, V)$, $\mathbb{E}' = (U', R', \varepsilon', V')$. Then the path $\mathbb{P} = (\mathbb{E} \cdot \iota\mathbb{E}')(\tau\mathbb{E} \cdot \mathbb{E}')$ corresponds to first rewriting R_ε to $R_{-\varepsilon}$ and then rewriting $R'_{\varepsilon'}$ to $R'_{-\varepsilon'}$, while the path $\mathbb{P}' = (\iota\mathbb{E}\cdot\mathbb{E}')(\mathbb{E}\cdot\tau\mathbb{E}')$ corresponds to first rewriting $R'_{\varepsilon'}$ to $R'_{-\varepsilon'}$ and then rewriting R_ε to $R_{-\varepsilon}$. We will say that the two edges in the path $\mathbb{P}$ are *disjoint*, and similarly for $\mathbb{P}'$. We want to regard the paths $\mathbb{P}$, $\mathbb{P}'$ as "essentially the same." This can be achieved by adjoining the closed path $\mathbb{P}\,\mathbb{P}'^{-1}$ as a defining path.

Thus, for any two edges $\mathbb{E}_1, \mathbb{E}_2$ in the graph we adjoin a defining path

$$[\mathbb{E}_1, \mathbb{E}_2] = (\mathbb{E}_1 \cdot \iota\mathbb{E}_2)(\tau\mathbb{E}_1 \cdot \mathbb{E}_2)(\mathbb{E}_1^{-1} \cdot \tau\mathbb{E}_2)(\iota\mathbb{E}_1 \cdot \mathbb{E}_2^{-1}).$$

The resulting 2-complex $\mathcal{D} = \mathcal{D}(\mathcal{R})$ is the *Squier complex* of $\mathcal{R}$. The left and right actions of $\mathbf{x}^*$ extend naturally to actions on $\mathcal{D}$: for a defining path $[\mathbb{E}_1, \mathbb{E}_2]$,

$$U \cdot [\mathbb{E}_1, \mathbb{E}_2] \cdot V = [U \cdot \mathbb{E}_1, \mathbb{E}_2 \cdot V] \quad (U, V \in \mathbf{x}^*).$$

The fundamental groups of connected components of Squier complexes are called *diagram groups*. They have been extensively studied in [Fa, GS97, GS99, GS02, Ki].

Let $\mathbf{p}$ be a set of closed paths in $\mathcal{D}$. We can form a new 2-complex $\mathcal{D}^{\mathbf{p}}$ by adjoining additional defining paths $U\cdot\mathbb{P}\cdot V$ $(U, V \in \mathbf{x}^*, \mathbb{P} \in \mathbf{p})$. We will say that $\mathbf{p}$ is a *homotopy* (respectively, *homology*) *trivializer*, if all closed paths in $\mathcal{D}^{\mathbf{p}}$ are null-homotopic (respectively, null-homologous). Clearly a homotopy trivializer is also a homology trivializer.

If two paths $\mathbb{P}_1, \mathbb{P}_2$ are homotopic in $\mathcal{D}$, we will write $\mathbb{P}_1 \simeq \mathbb{P}_2$. Moreover, if $\mathbf{p}$ is a set of closed paths, then we will write $\mathbb{P}_1 \simeq_{\mathbf{p}} \mathbb{P}_2$ if $\mathbb{P}_1, \mathbb{P}_2$ are homotopic in $\mathcal{D}^{\mathbf{p}}$. A path which is null-homotopic in $\mathcal{D}^{\mathbf{p}}$ will be said to be null-homotopic *modulo* $\mathbf{p}$.

For any two paths $\mathbb{P}, \mathbb{Q}$ in $\mathcal{D}$ we have the closed path

$$[\mathbb{P}, \mathbb{Q}] = (\mathbb{P} \cdot \iota\mathbb{Q})(\tau\mathbb{P} \cdot \mathbb{Q})(\mathbb{P}^{-1} \cdot \tau\mathbb{Q})(\iota\mathbb{P} \cdot \mathbb{Q}^{-1}).$$

Now if $\mathbb{P} = \mathbb{P}_1\mathbb{P}_2$, then $[\mathbb{P}, \mathbb{Q}]$ is null-homotopic modulo $[\mathbb{P}_1, \mathbb{Q}], [\mathbb{P}_2, \mathbb{Q}]$ (see Figure 1), and similarly if $\mathbb{Q} = \mathbb{Q}_1\mathbb{Q}_2$, then $[\mathbb{P}, \mathbb{Q}]$ is null-homotopic modulo $[\mathbb{P}, \mathbb{Q}_1], [\mathbb{P}, \mathbb{Q}_2]$. It thus follows by induction on the lengths of $\mathbb{P}, \mathbb{Q}$ that $[\mathbb{P}, \mathbb{Q}]$ is null-homotopic in $\mathcal{D}$. These closed paths, and the fact that they are null-homotopic, will be of importance in our computations.

Figure 1: $[\mathbb{P}_1\mathbb{P}_2, \mathbb{Q}]$

Definition 3 *The rewriting system $\mathcal{R}$ is said to be of* finite derivation type (FDT)*, respectively,* finite homological type (FHT)*, if $\mathcal{R}$ is finite and there is a* finite *homotopy, respectively homology, trivializer.*

The FDT property was introduced in a different (but equivalent) form in [SOK]. The FHT property was introduced in [WP]. In both cases, it turns out that if two finite rewriting systems $\mathcal{R}_1, \mathcal{R}_2$ are Tietze equivalent (which, by the Tietze Theorem amounts to the assertion that $S(\mathcal{R}_1) \cong S(\mathcal{R}_2)$), then if one has the property so does the other [SOK],[WP]. This allows us to talk about FDT and FHT *monoids*. It is shown further in [WP] that both properties are invariant under *retraction*, that is, if the monoid S' is a retract of S, then if S is FDT or FHT, so is S'. Recall that S' is a retract of S, if S' is a submonoid of S, and there exists a homomorphism $S \to S'$ which restricted to S' is the identity. Clearly FDT implies FHT. It is shown in [CO, Pr99] that for *groups* the two properties are equivalent.

Next we make some comments on the homology group $H_1(\mathcal{D})$. We have the chain complex

$$C(\mathcal{D}) : 0 \longrightarrow C_2(\mathcal{D}) \xrightarrow{\partial_2} C_1(\mathcal{D}) \xrightarrow{\partial_1} C_0(\mathcal{D}) \longrightarrow 0$$

of $\mathcal{D}$. Here $C_0(\mathcal{D})$ is the free abelian group with basis $\mathbf{x}^*$, $C_1(\mathcal{D})$ is the free abelian group with basis the set of positive edges, and $C_2(\mathcal{D})$ is the free abelian group with basis the set of defining paths. The chain groups are $(\mathbb{Z}\mathbf{x}^*, \mathbb{Z}\mathbf{x}^*)$-bi-modules via the left and right actions of $\mathbf{x}^*$ on the bases of the chain groups, and the boundary maps are bi-module homomorphisms. It is shown in [Pr95] that the first homology group $H_1(\mathcal{D})$ inherits a $(\mathbb{Z}S, \mathbb{Z}S)$-bi-module structure under the actions induced from those of $\mathbb{Z}\mathbf{x}^*$:

$$[U] \cdot (c + \operatorname{Im} \partial_2) \cdot [V] = UcV + \operatorname{Im} \partial_2 \quad (c \in \operatorname{Ker} \partial_1,\ U, V \in \mathbf{x}^*).$$

This bi-module is called the *homology bi-module* of $\mathcal{R}$, and it is denoted by $\pi^{(b)}(\mathcal{R})$. For any closed path $\mathbb{P} = \mathbb{E}_1^{\varepsilon_1}\mathbb{E}_2^{\varepsilon_2} \cdots \mathbb{E}_n^{\varepsilon_n}$ ($\mathbb{E}_i$ a positive edge, $\varepsilon_i = \pm 1$, $i = 1, \ldots, n$) we denote the homology class of the corresponding 1-cycle

$$\sum_{i=1}^{n} \varepsilon_i \mathbb{E}_i$$

by $\zeta_{\mathbb{P}}$.

There is a $(\mathbb{Z}S, \mathbb{Z}S)$-bi-module homomorphism

$$\rho : \pi^{(b)}(\mathcal{R}) \longrightarrow \bigoplus_{R \in \mathbf{r}} \mathbb{Z}S \cdot e_R \cdot \mathbb{Z}S \qquad (1)$$

of $\pi^{(b)}(\mathcal{R})$ into the free $(\mathbb{Z}S, \mathbb{Z}S)$-bi-module with basis e_R ($R \in \mathbf{r}$), where, for a closed path $\mathbb{P}$ as above with $\mathbb{E}_i = (U_i, R_i, +1, V_i)$ ($1 \le i \le n$),

$$\rho(\zeta_{\mathbb{P}}) = \sum_{i=1}^{n} \varepsilon_i [U_i] \cdot e_{R_i} \cdot [V_i].$$

An important result is that this map is actually an *embedding* [GS97, KO].

It is clear from the above discussion that an equivalent reformulation of the FHT property is that $\pi^{(b)}(\mathcal{R})$ is finitely generated as a $(\mathbb{Z}S, \mathbb{Z}S)$-bi-module. Note also that if $\mathbf{p}$ is a set of closed paths in $\mathcal{D}$, then $H_1(\mathcal{D}^{\mathbf{p}})$ is isomorphic to $H_1(\mathcal{D})/L$, where L is the submodule of $H_1(\mathcal{D})$ generated by $\zeta_{\mathbb{P}}$ ($\mathbb{P} \in \mathbf{p}$). The isomorphism sends the homology class of a closed path $\mathbb{Q}$ in $\mathcal{D}^{\mathbf{p}}$ to $\zeta_{\mathbb{Q}} + L$.

2.3 Some remarks concerning $\mathbb{Z}M$

The monoid M is obtained by factoring the free product of G and the infinite cyclic *monoid* on t, by the congruence generated by the pairs (th, ht) $(h \in H)$. We also have the HNN extension $\overline{G}$ obtained by factoring the free product of G and the infinite cyclic *group* on t, by the smallest normal subgroup containing the elements $tht^{-1}h^{-1}$ $(h \in H)$. There is then a monoid homomorphism $M \to \overline{G}$ induced by the map

$$g \mapsto g \ (g \in G), t \mapsto t.$$

If Σ is a left transversal for H in G, then it is easily seen that each element of M can be represented in the form

$$\sigma_0 t \sigma_1 t \dots t \sigma_n h \tag{2}$$

$(n \geq 0, \sigma_0, \dots, \sigma_n \in \Sigma, h \in H)$. By taking the image in $\overline{G}$ and using the normal form theorem for HNN extensions [LS], we see that each element of M is *uniquely* represented in the above form (and as a bi-product, we have that the map $M \to \overline{G}$ is injective). Similarly, if Σ' is a right transversal for H in G, then each element of M has a unique representative in the form

$$h\sigma_0' t \sigma_1' t \dots t \sigma_n' \tag{3}$$

$(n \geq 0, \sigma_0', \dots, \sigma_n' \in \Sigma', h \in H)$.

Let Λ be the set of all elements represented in the form (2) with $h = 1$. Then, as an abelian group,

$$\mathbb{Z}M = \bigoplus_{\lambda \in \Lambda} \mathbb{Z}\lambda H.$$

Moreover, $\mathbb{Z}\lambda H$ is isomorphic to $\mathbb{Z}H$ as a right $\mathbb{Z}H$-module over itself, so $\mathbb{Z}M$ is free (with basis Λ) as a right $\mathbb{Z}H$-module. Similarly, using (3), we find that $\mathbb{Z}M$ is free as a left $\mathbb{Z}H$-module. Consequently we have that

$$\textit{the functors } \mathbb{Z}M \otimes_{\mathbb{Z}H} - \textit{ and } - \otimes_{\mathbb{Z}H} \mathbb{Z}M \textit{ are exact.} \tag{4}$$

Also, the following general lemma applies to the rings $\mathbb{Z}H$ and $\mathbb{Z}M$.

Lemma 4 *Let D be a subring of a ring C, and suppose C is free as a left and right D-module. Let K be a (D, D)-bi-module. Then K is finitely generated as a (D, D)-bi-module if and only if $C \otimes_D K \otimes_D C$ is finitely generated as a (C, C)-bi-module.*

Proof. The 'only-if' part is obvious. For the 'if' part, let Ω and Ω' be right, respectively, left bases for C as a free D-module (with 1 the basis element corresponding to D itself). We have a well-defined map

$$\varphi : C \times K \times C \longrightarrow \bigoplus_{\Omega \times \Omega'} {}_{\omega}K_{\omega'} = L,$$

where ${}_{\omega}K_{\omega'}$ is a copy of K, as follows. For $(c, k, c') \in C \times K \times C$, write (uniquely)

$$c = \sum_{\omega \in \Omega} \omega d_{\omega} \quad \text{and} \quad c' = \sum_{\omega' \in \Omega'} d_{\omega'} \omega',$$

where $d_{\omega}, d_{\omega'} \in D$. Then the (ω, ω')-component of $\varphi(c, k, c')$ is $d_{\omega} k d_{\omega'}$. This mapping is tri-linear over D, and so we get an induced map

$$\hat{\varphi} : C \otimes_D K \otimes_D C \to L.$$

Let ψ denote the map $L \to K$ defined through

$$ {}_1K_1 \overset{id}{\longmapsto} K, \; {}_\omega K_{\omega'} \longmapsto 0 \; ((\omega, \omega') \neq (1,1)). $$

If $C \otimes_D K \otimes_D C$ is finitely generated, then it will have a generating set of the form $1 \otimes j \otimes 1$ ($j \in \mathbf{j}$) for some finite subset $\mathbf{j}$ of K. Thus for $k \in K$, $1 \otimes k \otimes 1$ will be a (C, C)-bilinear combination of the elements $1 \otimes j \otimes 1$. Applying $\psi\hat{\varphi}$ will then give $k = \psi\hat{\varphi}(1 \otimes k \otimes 1)$ as a (D, D)-bilinear combination of the elements $j = \psi\hat{\varphi}(1 \otimes j \otimes 1)$ ($j \in \mathbf{j}$). Hence $\mathbf{j}$ generates K. □

3 Homotopy of a rewriting system for M

3.1 A homotopy trivializer

Choose a finite group presentation

$$ \mathcal{P} = \langle \mathbf{a}; \mathbf{y} \rangle $$

for the group G, where $\mathbf{a}$ contains a subset $\mathbf{b}$ representing a set of group generators for H. This gives rise to a rewriting system

$$ \hat{\mathcal{P}} = [\mathbf{a}, \mathbf{a}^{-1}; \, Y = 1 \, (Y \in \mathbf{y}), \, a^\varepsilon a^{-\varepsilon} = 1 \, (a \in \mathbf{a}, \varepsilon = \pm 1)] $$

for G. A rewriting system for M is then given by

$$ \mathcal{M} = [\mathbf{a}, \mathbf{a}^{-1}, t; \, Y = 1 \, (Y \in \mathbf{y}), \, a^\varepsilon a^{-\varepsilon} = 1 \, (a \in \mathbf{a}, \varepsilon = \pm 1), \, tb^\varepsilon = b^\varepsilon t \, (b \in \mathbf{b}, \varepsilon = \pm 1)]. $$

It will be convenient to denote the rule $tb^\varepsilon = b^\varepsilon t$ by $T_{b,\varepsilon}$, and to denote the set of all such rules by $\mathbf{t}$. An edge of $\mathcal{D}(\mathcal{M})$ of the form $(U, T_{b,\varepsilon}, \pm 1, V)$ $(U, V \in (\mathbf{a} \cup \mathbf{a}^{-1} \cup \{t\})^*)$ will be called a $T_{b,\varepsilon}$-*edge*, and an edge will be called a $\mathbf{t}$-*edge* if it is a $T_{b,\varepsilon}$-edge for some $T_{b,\varepsilon} \in \mathbf{t}$.

We let $\mathbf{d}$ denote a homotopy trivializer for $\mathcal{D}(\hat{\mathcal{P}})$.

For any word $Z = b_1^{\varepsilon_1} b_2^{\varepsilon_2} \dots b_n^{\varepsilon_n}$ on $\mathbf{b} \cup \mathbf{b}^{-1}$ we have a path

$$ \mathbb{B}_Z : tZ = Z_0 t Z_0' \xrightarrow{\mathbb{T}_1} Z_1 t Z_1' \xrightarrow{\mathbb{T}_2} \cdots \xrightarrow{\mathbb{T}_{i-1}} Z_{i-1} t Z_{i-1}' \xrightarrow{\mathbb{T}_i} Z_i t Z_i' \xrightarrow{\mathbb{T}_{i+1}} \cdots \xrightarrow{\mathbb{T}_n} Z_n t Z_n' = Zt, $$

in $\mathcal{D}(\mathcal{M})$, where $Z_i = b_1^{\varepsilon_1} \dots b_i^{\varepsilon_i}$, $Z_i' = b_{i+1}^{\varepsilon_{i+1}} \dots b_n^{\varepsilon_n}$ $(0 \leq i \leq n)$, and $\mathbb{T}_i = (Z_{i-1}, T_{b_i, \varepsilon_i}, +1, Z_{i-1}')$ for all $i = 1, \dots, n$. Note that for any two words $Z, U \in (\mathbf{b} \cup \mathbf{b}^{-1})^*$ we have $\mathbb{B}_{ZU} = (\mathbb{B}_Z \cdot U)(Z \cdot \mathbb{B}_U)$.

For $b \in \mathbf{b}$ and $\varepsilon = \pm 1$ we have the closed path

$$ \mathbb{K}_{b,\varepsilon} = (1, T_{b,\varepsilon}, +1, 1)(b^\varepsilon t, (b^{-\varepsilon} b^\varepsilon = 1), -1, 1)(b^\varepsilon, T_{b,-\varepsilon}, +1, b^\varepsilon)(1, (b^\varepsilon b^{-\varepsilon} = 1), +1, tb^\varepsilon), $$

which can be represented as follows:

$$ \mathbb{K}_{b,\varepsilon} : tb^\varepsilon \xrightarrow{\mathbb{T}} b^\varepsilon t \xrightarrow{\mathbb{E}} b^\varepsilon t b^{-\varepsilon} b^\varepsilon \xrightarrow{\mathbb{T}'} b^\varepsilon b^{-\varepsilon} t b^\varepsilon \xrightarrow{\mathbb{E}'} tb^\varepsilon. $$

Modulo this path we can exchange a negative $T_{b,\varepsilon}$-edge for a positive $T_{b,\varepsilon}$-edge (at the expense of adding two additional non-$\mathbf{t}$-edges):

$$ \mathbb{T}^{-1} \simeq_{\mathbb{K}_{b,\varepsilon}} \mathbb{E}\mathbb{T}'\mathbb{E}'. \tag{5} $$

If W_1, W_2 are words on $\mathbf{b} \cup \mathbf{b}^{-1}$ with $[W_1] = [W_2]$, then there is a path $\mathbb{P}_{W_1,W_2}$ in $\mathcal{D}(\hat{\mathcal{P}})$ from W_1 to W_2. We then obtain a closed path $\mathbb{Q}_{W_1,W_2}$ in $\mathcal{D}(\mathcal{M})$ as in Figure 2.

Notice that if $\overline{\mathbb{P}}_{W_1,W_2}$ is another path in $\mathcal{D}(\hat{\mathcal{P}})$ from W_1 to W_2, then $\overline{\mathbb{P}}_{W_1,W_2}^{-1}\mathbb{P}_{W_1,W_2}$ is a closed path in $\mathcal{D}(\hat{\mathcal{P}})$, and is therefore null-homotopic modulo $\mathbf{d}$. Hence, exchanging the two occurrences of $\mathbb{P}_{W_1,W_2}$ in $\mathbb{Q}_{W_1,W_2}$ by $\overline{\mathbb{P}}_{W_1,W_2}$ gives a path $\overline{\mathbb{Q}}_{W_1,W_2} \simeq_{\mathbf{d}} \mathbb{Q}_{W_1,W_2}$. Thus:

up to homotopy modulo $\mathbf{d}$, $\mathbb{Q}_{W_1,W_2}$ *is independent of the choice of the path* $\mathbb{P}_{W_1,W_2}$. (6)

We will use this simple, but important observation frequently, often without mention.

Figure 2: $\mathbb{Q}_{W_1,W_2}$

We will denote by $\mathbf{w}$ the set of *all* words on $\mathbf{b} \cup \mathbf{b}^{-1}$ which define the identity in H. For $W \in \mathbf{w}$, taking $W_1 = 1$ and $W_2 = W$ above, we get a closed path as in Figure 3. Here, for convenience, we denote $\mathbb{P}_{1,W}$ and $\mathbb{Q}_{1,W}$ simply by $\mathbb{P}_W$ and $\mathbb{Q}_W$, respectively.

Figure 3: $\mathbb{Q}_W$

We let

$$\begin{aligned} \mathbf{k} &= \{\mathbb{K}_{b,\varepsilon} : b \in \mathbf{b}, \varepsilon = \pm 1\}, \\ \mathbf{q}_0 &= \{\mathbb{Q}_{b^\varepsilon b^{-\varepsilon}} : b \in \mathbf{b}, \varepsilon = \pm 1\}, \\ \mathbf{q} &= \{\mathbb{Q}_W : W \in \mathbf{w}\}. \end{aligned}$$

Lemma 5 *The set* $\mathbf{d} \cup \mathbf{k} \cup \mathbf{q}$ *is a homotopy trivializer for* $\mathcal{D}(\mathcal{M})$.

Proof. Let $\mathbb{X}$ be a closed path in $\mathcal{D}(\mathcal{M})$. We will proceed by induction on the number $\#_t(\mathbb{X})$ of occurrences of the letter t in $\iota(\mathbb{X})$.

If $\#_t(\mathbb{X}) = 0$, then $\mathbb{X}$ is a path in $\mathcal{D}(\hat{\mathcal{P}})$, which is therefore null-homotopic modulo $\mathbf{d}$.

Suppose that $\#_t(\mathbb{X}) > 0$, and write $\iota(\mathbb{X}) = ZtV$, where $Z \in (\mathbf{a} \cup \mathbf{a}^{-1})^*$. Modulo $\mathbf{k}$, we can assume that all $\mathbf{t}$-edges in $\mathbb{X}$ are positive (see (5)).

We will say that an edge of $\mathcal{D}(\mathcal{M})$ is of:

(a) *Type 1*, if it has the form $Ut \cdot \mathbb{E}$, where $U \in (\mathbf{a} \cup \mathbf{a}^{-1})^*$;

(b) *Type 2*, if it has the form $(U, T, +1, U')$, where $T \in \mathbf{t}$ and $U \in (\mathbf{a} \cup \mathbf{a}^{-1})^*$;

(c) *Type 3*, if it has the form $\mathbb{E} \cdot tW$, where $\iota(\mathbb{E}) \in (\mathbf{a} \cup \mathbf{a}^{-1})^*$.

A path will be said to be of Type i ($1 \leq i \leq 3$) if all its edges are of Type i. Note that a path of Type 2 is of the form $U \cdot \mathbb{B}_W \cdot U'$ for some $W \in (\mathbf{b} \cup \mathbf{b}^{-1})^*$ and $U \in (\mathbf{a} \cup \mathbf{a}^{-1})^*$.

Now observe that if $\mathbb{E}, \mathbb{F}$ are edges with $\tau(\mathbb{E}) = \iota(\mathbb{F})$, where $\mathbb{E}$ is of Type i and $\mathbb{F}$ is of Type j for some $i > j$, then $\mathbb{E}, \mathbb{F}$ are disjoint. Thus, $\mathbb{E}\,\mathbb{F}$ is homotopic in $\mathcal{D}(\mathcal{M})$ to a path $\mathbb{F}'\,\mathbb{E}'$ with $\mathbb{E}'$ of Type i and $\mathbb{F}'$ of Type j. It follows that $\mathbb{X}$ is homotopic to a product $\mathbb{X}_1\mathbb{X}_2\mathbb{X}_3$, where $\mathbb{X}_i$ is a path of Type i ($1 \leq i \leq 3$).

We have $\iota(\mathbb{X}_1) = ZtV$, so $\tau(\mathbb{X}_1) = ZtV'$, where $[V'] = [V]$ in M. Then $\mathbb{X}_2 = Z \cdot \mathbb{B}_W \cdot U$, where $W \in (\mathbf{b} \cup \mathbf{b}^{-1})^*$. We must then have that $V' = WU$. Thus, $\tau(\mathbb{X}_2) = ZWtU =$

$\iota(\mathbb{X}_3)$. Since $\tau(\mathbb{X}_3) = ZtV$ (and since $\mathbb{X}_3$ is of Type 3), $[ZW] = [Z]$ and $U = V$. In particular, $W \in \mathbf{w}$, and so we have the path $\mathbb{Q}_W \in \mathbf{q}$. Modulo $\mathbb{Q}_W$, $\mathbb{X}_2 = Z \cdot \mathbb{B}_W \cdot V$ can be replaced by $(Zt \cdot \mathbb{P}_W^{-1} \cdot V)(Z \cdot \mathbb{P}_W \cdot tV)$, giving $\mathbb{X} \simeq_{\mathbb{Q}_W} \mathbb{X}_1'\mathbb{X}_3'$, where

$$\mathbb{X}_1' = \mathbb{X}_1(Zt \cdot \mathbb{P}_W^{-1} \cdot V) \text{ and } \mathbb{X}_3' = (Z \cdot \mathbb{P}_W \cdot tV)\mathbb{X}_3.$$

Now $\mathbb{X}_1'$ is a closed path of Type 1, and it has the form $Zt \cdot \mathbb{Y}$, where $\#_t(\mathbb{Y}) = \#_t(\mathbb{X}) - 1$, so the inductive hypothesis applies. Also $\mathbb{X}_3'$ is a closed path of Type 3, and so has the form $\mathbb{D} \cdot tV$ for some path $\mathbb{D}$ in $\mathcal{D}(\hat{\mathcal{P}})$. Hence, $\mathbb{X}_3'$ is null-homotopic modulo $\mathbf{d}$. □

3.2 The FDT property for M

We will need some preliminary lemmas.

Lemma 6 *For $W, W' \in \mathbf{w}$ we have $\mathbb{Q}_W\mathbb{Q}_{W'} \simeq_{\mathbf{d}} \mathbb{Q}_{WW'}$.*

Proof. Up to homotopy modulo $\mathbf{d}$ (see (6)) we can take $\mathbb{P}_{WW'}$ to be $\mathbb{P}_W\,(W \cdot \mathbb{P}_{W'})$. Now refer to Figure 4. The boundary of this diagram is labelled with $\mathbb{Q}_{WW'}$. All the rectangles in this diagram are labelled by closed paths of the form $[\mathbb{X}, \mathbb{Y}]$, which are null-homotopic in $\mathcal{D}$ (see Subsection 2.2). Thus, we can perform homotopy moves to eliminate all these rectangles, leaving a diagram (consisting of two triangles) whose boundary label $\mathbb{Q}_W\,\mathbb{Q}_{W'}$ will be homotopic to the boundary label of the original diagram. □

The following two lemmas are more general than needed for our present purposes, but we will require the greater generality (of Lemma 8) in Subsection 4.2.

Lemma 7 *Let $U, Z \in (\mathbf{b} \cup \mathbf{b}^{-1})^*$ with $[U] = [Z]$, and suppose that $U = U_1WU_2$ with $W \in \mathbf{w}$. Then*

$$\mathbb{Q}_{Z,U} \simeq_{\mathbf{d}\cup\{\mathbb{Q}_W\}} \mathbb{Q}_{Z,U_1U_2}.$$

Proof. Modulo $\mathbf{d}$ we can take $\mathbb{P}_{Z,U}$ to be $\mathbb{P}_{Z,U_1U_2}(U_1 \cdot \mathbb{P}_W \cdot U_2)$. Then the boundary of the diagram in Figure 5 is labelled by $\mathbb{Q}_{Z,U}$. The two lower rectangles are bounded by null-homopotic paths in $\mathcal{D}$, and the boundary label of the lower triangle is null-homotopic in $\mathcal{D}^{\{\mathbb{Q}_W\}}$. Thus, the boundary label $\mathbb{Q}_{Z,U_1U_2}$ of the upper pentagon is homotopic in $\mathcal{D}^{\{\mathbb{Q}_W\}}$ to $\mathbb{Q}_{Z,U}$. □

If in Lemma 7 we take W of the form $b^\varepsilon b^{-\varepsilon}$ ($b \in \mathbf{b}, \varepsilon = \pm 1$), then $\mathbb{Q}_W \in \mathbf{q}_0$. Thus, we deduce the following.

Lemma 8 *If Z, U, U' are words on $\mathbf{b} \cup \mathbf{b}^{-1}$ with $[Z] = [U]$ and U, U' freely equivalent, then*

$$\mathbb{Q}_{Z,U} \simeq_{\mathbf{d}\cup\mathbf{q}_0} \mathbb{Q}_{Z,U'}.$$

Figure 4: $\mathbb{Q}_W\mathbb{Q}_{W'} \simeq_{\mathbf{d}} \mathbb{Q}_{WW'}$

Figure 5: $\mathbb{Q}_{Z,U_1WU_2}$.

Let F be the free group on $\mathbf{b}$. We will denote the free equivalence class of a word Z on $\mathbf{b}\cup\mathbf{b}^{-1}$ by $\langle Z\rangle$. For any subset $\mathbf{v}$ of $\mathbf{w}$, let $N(\mathbf{v})$ denote the normal closure of $\{\langle V\rangle : V \in \mathbf{v}\}$ in F. We let $\mathbf{q}_\mathbf{v} = \{\mathbb{Q}_V : V \in \mathbf{v}\}$, and we denote the 2-complex $\mathcal{D}(\mathcal{M})^{\mathbf{d}\cup\mathbf{k}\cup\mathbf{q}_0\cup\mathbf{q}_\mathbf{v}}$ by $\overline{\mathcal{D}}(\mathbf{v})$. We will be interested in the component $\overline{\mathcal{D}}(\mathbf{v})_t$ of $\overline{\mathcal{D}}(\mathbf{v})$ containing the vertex t. For a closed path $\mathbb{X}$ on $\mathcal{D}$ we will denote its homotopy class in $\overline{\mathcal{D}}(\mathbf{v})$ by $(\mathbb{X})_\mathbf{v}$.

When $\mathbf{v} = \mathbf{w}$ we simply write N instead of $N(\mathbf{w})$. Note that $F/N \cong H$.

Lemma 9 *The fundamental group $\pi_1(\overline{\mathcal{D}}(\mathbf{v})_t)$ of $\overline{\mathcal{D}}(\mathbf{v})_t$ (at the vertex t) is isomorphic to $N/N(\mathbf{v})$.*

Proof. We have a mapping

$$\mathbf{w} \to \pi_1(\overline{\mathcal{D}}(\mathbf{v})_t), \quad W \mapsto (\mathbb{Q}_W)_\mathbf{v} \quad (W \in \mathbf{w}).$$

By Lemma 8 (with $Z = 1$), this gives rise to a well-defined mapping

$$\phi : N \to \pi_1(\overline{\mathcal{D}}(\mathbf{v})_t), \quad \langle W\rangle \mapsto (\mathbb{Q}_W)_\mathbf{v},$$

and by Lemma 6, ϕ is a homomorphism.

The map ϕ is surjective. For if in the proof of Lemma 5 we take $\mathbb{X}$ with $\iota(\mathbb{X}) = \tau(\mathbb{X}) = t$ (so that, in the notation of that proof, $\#_t(\mathbb{X}) = 1$ and Z, V are empty), then the argument given there shows that

$$\mathbb{X} \simeq_{\mathbf{d}\cup\mathbf{k}} \mathbb{Q}_W$$

for some $W \in \mathbf{w}$.

Now $N(\mathbf{v})$ is generated by elements of the form $\langle C\rangle\langle V\rangle\langle C\rangle^{-1}$ ($V \in \mathbf{v}$, C a word on $\mathbf{b} \cup \mathbf{b}^{-1}$). Taking $U_1 = C, W = V, U_2 = C^{-1}$ and $Z = 1$ in Lemma 7 we see that $\mathbb{Q}_{CVC^{-1}}$ is homotopic modulo $\mathbf{d}\cup\mathbf{q}_\mathbf{v}$ to $\mathbb{Q}_{CC^{-1}}$, and by Lemma 8, $\mathbb{Q}_{CC^{-1}}$ is null-homotopic modulo $\mathbf{d} \cup \mathbf{q}_0$. Thus, $\phi(\langle C\rangle\langle V\rangle\langle C\rangle^{-1})$ is trivial. Hence $N(\mathbf{v}) \subseteq \mathrm{Ker}\ \phi$, so we get an induced surjective homomorphism

$$\phi_* : \frac{N}{N(\mathbf{v})} \to \pi_1(\overline{\mathcal{D}}(\mathbf{v})_t).$$

It remains to show that ϕ_* is injective. Define a function λ from the edges of $\overline{\mathcal{D}}(\mathbf{v})$ to $A = F/N(\mathbf{v})$ as follows: all non-$\mathbf{t}$-edges are sent to the identity, and for a $\mathbf{t}$-edge $\mathbb{T} = (U, T_{b,\varepsilon}, \delta, U')$ $(U, U' \in (\mathbf{a}\cup\mathbf{a}^{-1}\cup\{t\})^*, b \in \mathbf{b}, \varepsilon, \delta \in \{+1, -1\})$, $\lambda_\mathbb{T} = \langle b^{\varepsilon\delta}\rangle N(\mathbf{v})$.

This *does not* in general give rise to a mapping (as described in Subsection 2.1) from $\overline{\mathcal{D}}(\mathbf{v})$ to A. Certainly for all defining paths

$$\mathbb{P} \in (\mathbf{a}\cup\mathbf{a}^{-1}\cup\{t\})^* \cdot (\mathbf{d}\cup\mathbf{k}\cup\mathbf{q}_0\cup\mathbf{q}_\mathbf{v}) \cdot (\mathbf{a}\cup\mathbf{a}^{-1}\cup\{t\})^*$$

we have $\lambda_\mathbb{P} = 1$. Also, for a defining path $[\mathbb{E}, \mathbb{E}']$, we have $\lambda_{[\mathbb{E},\mathbb{E}']} = 1$ *unless* $\mathbb{E}$, $\mathbb{E}'$ are both $\mathbf{t}$-edges. If $\mathbb{E}$, $\mathbb{E}'$ are both $\mathbf{t}$-edges, then $\lambda_{[\mathbb{E},\mathbb{E}']}$ will in general just be an element of the derived subgroup of A. However, if we restrict attention to the component $\overline{\mathcal{D}}(\mathbf{v})_t$, then the endpoints of any edge have just one occurrence of t, so no path $[\mathbb{E}, \mathbb{E}']$ with $\mathbb{E}$, $\mathbb{E}'$ both $\mathbf{t}$-edges can lie in $\overline{\mathcal{D}}(\mathbf{v})_t$. Thus, we *do* obtain a mapping from $\overline{\mathcal{D}}(\mathbf{v})_t$ to A, giving rise to a group homomorphism

$$\lambda_* : \pi_1(\overline{\mathcal{D}}(\mathbf{v})_t) \longrightarrow A.$$

The injectivity of ϕ_* now follows, since $\lambda_*((\mathbb{Q}_W)_\mathbf{v}) = \langle W\rangle N(\mathbf{v})$. □

We can now prove part (i) of our Main Theorem.

If G is not FDT, then neither is M, since the map that kills t is a retraction of M onto G. So we can assume that G is FDT, which means that we can choose $\mathbf{d}$ to be finite. Since the set $\mathbf{d} \cup \mathbf{q} \cup \mathbf{k}$ is a homotopy trivializer for $\mathcal{D}(\mathcal{M})$, we then have the following sequence of equivalent statements:

$$\begin{aligned} M \text{ is FDT} \;&\Leftrightarrow\; \text{some finite subset of } \mathbf{d} \cup \mathbf{k} \cup \mathbf{q} \text{ is a homotopy trivializer for } \mathcal{D}(\mathcal{M}) \\ &\Leftrightarrow\; \text{there is a finite subset } \mathbf{v} \subset \mathbf{w} \text{ such that } \mathbf{d} \cup \mathbf{k} \cup \mathbf{q}_0 \cup \mathbf{q}_\mathbf{v} \text{ is a homotopy trivializer for } \mathcal{D}(\mathcal{M}) \\ &\Leftrightarrow\; \text{there is a finite subset } \mathbf{v} \subset \mathbf{w} \text{ such that each closed path in } \mathbf{q} \text{ is null-homotopic mod } \mathbf{d} \cup \mathbf{k} \cup \mathbf{q}_0 \cup \mathbf{q}_\mathbf{v} \\ &\Leftrightarrow\; \text{there is a finite subset } \mathbf{v} \subset \mathbf{w} \text{ such that } N = N(\mathbf{v}), \end{aligned}$$

where the last implication follows by Lemma 9. Hence, M is FDT if and only if N is the normal closure of a finite set of elements of F, that is, if and only if H is finitely presented.

4 Homology of a rewriting system for M

Throughout this section it will be convenient to regard H as lying inside M. Thus, the elements of H will be taken to be congruence classes $[U]$ of words $U \in (\mathbf{b} \cup \mathbf{b}^{-1})^*$.

In addition we will let $\mathcal{Q} = \langle \mathbf{b}; \mathbf{v} \rangle$ be a *group* presentation for H on the generators $\mathbf{b}$. As previously, we will denote the free group on $\mathbf{b}$ by F, and we will denote the free equivalence class of a word $U \in (\mathbf{b} \cup \mathbf{b}^{-1})^*$ by $\langle U \rangle$. We have the natural epimorphism

$$F \longrightarrow H, \quad \langle U \rangle \to [U] \quad (U \in (\mathbf{b} \cup \mathbf{b}^{-1})^*).$$

The kernel of this epimorphism is then $N(\mathbf{v})$.

It follows from Lemmas 5 and 9 that the set $\mathbf{d} \cup \mathbf{k} \cup \mathbf{q}_0 \cup \mathbf{q}_\mathbf{v}$ is a homotopy trivializer for $\mathcal{D}(\mathcal{M})$, and so

$$\{ \zeta_\mathbb{P} \mid \mathbb{P} \in \mathbf{d} \cup \mathbf{k} \cup \mathbf{q}_0 \cup \mathbf{q}_\mathbf{v} \}$$

is a set of bi-module generators for $\pi^{(b)}(\mathcal{M})$. We let B denote the submodule of $\pi^{(b)}(\mathcal{M})$ generated by the set

$$\{ \zeta_\mathbb{P} \mid \mathbb{P} \in \mathbf{d} \cup \mathbf{k} \cup \mathbf{q}_0 \}.$$

For $\zeta \in \pi^{(b)}(\mathcal{M})$ we denote its image in $\pi^{(b)}(\mathcal{M})/B$ by $\overline{\zeta}$. Note that the bi-module $\pi^{(b)}(\mathcal{M})/B$ is generated by the elements $\overline{\zeta}_{\mathbb{Q}_V}$ $(V \in \mathbf{v})$. Note also that $\pi^{(b)}(\mathcal{M})/B$ is isomorphic to $H_1(\mathcal{D}^{\mathbf{d} \cup \mathbf{k} \cup \mathbf{q}_0})$ (see the end of Subsection 2.2).

4.1 The relation (bi)module

(For further general information on relation bi-modules, see [Iv] and the references cited there.)

Let J denote the kernel of the ring homomorphism

$$\mathbb{Z}F \longrightarrow \mathbb{Z}H$$

induced by the group epimorphism $F \to H$ above. Then J is a $(\mathbb{Z}F, \mathbb{Z}F)$-bi-module and J^2 is a submodule of J. Thus, J/J^2 is a $(\mathbb{Z}F, \mathbb{Z}F)$-bi-module, and J annihilates J/J^2 on both the left and the right. Consequently J/J^2 inherits a $(\mathbb{Z}H, \mathbb{Z}H)$-bi-module structure with action given by

$$[U_1] \cdot (\xi + J^2) \cdot [U_2] = \langle U_1 \rangle \xi \langle U_2 \rangle + J^2 \quad (U_1, U_2 \in (\mathbf{b} \cup \mathbf{b}^{-1})^*, \xi \in J).$$

This bi-module is the *relation bi-module* of $\mathcal{Q}$ [Iv], which we denote by $\mathrm{Rel}^{(b)}(\mathcal{Q})$ (or simply $\mathrm{Rel}^{(b)}$). The *left* relation module of $\mathcal{Q}$ is

$$\mathrm{Rel}^{(l)}(\mathcal{Q}) = \mathrm{Rel}^{(b)}(\mathcal{Q}) \otimes_{\mathbb{Z}H} \mathbb{Z}$$

(where H acts trivially on $\mathbb{Z}$). Similarly there is the right relation module

$$\mathrm{Rel}^{(r)}(\mathcal{Q}) = \mathbb{Z} \otimes_{\mathbb{Z}H} \mathrm{Rel}^{(b)}(\mathcal{Q}),$$

but we will not need to consider this.

As an abelian group, J is generated by the elements

$$\langle U_1 \rangle - \langle U_2 \rangle \quad (U_1, U_2 \in (\mathbf{b} \cup \mathbf{b}^{-1})^*, [U_1] = [U_2]).$$

For $h \in H$, we let J_h be the subgroup generated by the elements of the above form with $[U_1] = [U_2] = h$. Then, as an abelian group,

$$J = \bigoplus_{h \in H} J_h.$$

Clearly, $J_{h_1} J_{h_2} \subseteq J_{h_1 h_2}$ for all $h_1, h_2 \in H$. Hence

$$J^2 = \bigoplus_{h \in H} K_h,$$

where

$$K_h = \sum_{\substack{h_1, h_2 \in H \\ h_1 h_2 = h}} J_{h_1} J_{h_2} \leq J_h.$$

Letting

$$\mathrm{Rel}_h^{(b)} = \frac{J_h}{K_h} \quad (h \in H),$$

we then have the abelian group decomposition

$$\mathrm{Rel}^{(b)} = \bigoplus_{h \in H} \mathrm{Rel}_h^{(b)}. \tag{7}$$

Moreover, for all $h, h_1, h_2 \in H$, we have $h_1 \cdot \mathrm{Rel}_h^{(b)} \cdot h_2 = \mathrm{Rel}_{h_1 h h_2}^{(b)}$.

It is clear that if $\mathrm{Rel}^{(b)}(\mathcal{Q})$ is finitely generated as a bi-module, then $\mathrm{Rel}^{(l)}(\mathcal{Q})$ is finitely generated as a left module. We will need the converse of this. The converse follows from the decomposition (7), by making use of a general result of McGlashan [McG01], which we now describe.

Let Γ be a group, and let A be a $(\mathbb{Z}\Gamma, \mathbb{Z}\Gamma)$-bi-module with an abelian group decomposition

$$A = \bigoplus_{\gamma \in \Gamma} A_\gamma$$

such that $\gamma_1 \cdot A_\gamma \cdot \gamma_2 = A_{\gamma_1 \gamma \gamma_2}$ for all $\gamma, \gamma_1, \gamma_2 \in \Gamma$. Then A_1 has a left $\mathbb{Z}\Gamma$-module structure with Γ-action

$$\gamma * a = \gamma \cdot a \cdot \gamma^{-1} \quad (\gamma \in \Gamma, a \in A_1).$$

Lemma 10 [McG01]

(i) *$A \otimes_{\mathbb{Z}\Gamma} \mathbb{Z}$ and A_1 are isomorphic as left $\mathbb{Z}\Gamma$-modules.*

(ii) *If A_1 is finitely generated as a left $\mathbb{Z}\Gamma$-module, then A is finitely generated as a bi-module.*

As a bi-module, $\mathrm{Rel}^{(b)}(\mathcal{Q})$ is generated by the elements

$$(\langle V\rangle - 1) + J^2 \quad (V \in \mathbf{v}).$$

There is an embedding [BD], [Le] (see also [Iv])

$$\begin{array}{rccc} \mu : & \mathrm{Rel}^{(b)}(\mathcal{Q}) & \longrightarrow & \bigoplus_{b\in\mathbf{b}} \mathbb{Z}H \cdot e_b \cdot \mathbb{Z}H \\ & (\langle V\rangle - 1) + J^2 & \mapsto & \frac{\partial V}{\partial \mathbf{b}} \quad (V \in \mathbf{v}) \end{array} \tag{8}$$

into the free $(\mathbb{Z}H, \mathbb{Z}H)$-bi-module with basis e_b $(b \in \mathbf{b})$. Here $\frac{\partial}{\partial \mathbf{b}}$ is the *Fox bi-derivative*, where for any word $Z = b_1^{\varepsilon_1} b_2^{\varepsilon_2} \dots b_r^{\varepsilon_r}$ $(b_i \in \mathbf{b}, \varepsilon_i = \pm 1, i = 1, \dots, r)$

$$\frac{\partial Z}{\partial \mathbf{b}} = \sum_{i=1}^{r} \varepsilon_i [b_1^{\varepsilon_1} \dots b_{i-1}^{\varepsilon_{i-1}} b_i^{\frac{\varepsilon_i - 1}{2}}] \cdot e_{b_i} \cdot [b_i^{\frac{\varepsilon_i - 1}{2}} b_{i+1}^{\varepsilon_{i+1}} \dots b_r^{\varepsilon_r}].$$

We then get an induced mapping

$$\mu^{(l)} : \mathrm{Rel}^{(l)}(\mathcal{Q}) \xrightarrow{\mu\otimes 1} \left(\bigoplus_{b\in\mathbf{b}} \mathbf{Z}H \cdot e_b \cdot \mathbb{Z}H \right) \otimes_{\mathbf{Z}H} \mathbb{Z} \;\cong\; \bigoplus_{b\in\mathbf{b}} \mathbb{Z}H \cdot e_b$$

into the free left $\mathbb{Z}H$-module with basis e_b $(b \in \mathbf{b})$ given by the left Fox derivation. It turns out that $\mu^{(l)}$ is also an embedding [BD], [Le] (see [Iv] §1 or [Br], p. 43, taking account of Prop. 0.1 of [Iv]). Moreover, there is an exact sequence

$$0 \longrightarrow \mathrm{Rel}^{(l)}(\mathcal{Q}) \xrightarrow{\mu^{(l)}} \bigoplus_{b\in\mathbf{b}} \mathbb{Z}H \cdot e_b \longrightarrow \mathbb{Z}H \longrightarrow \mathbb{Z} \longrightarrow 0.$$

It follows from this and the generalized Schanuel Lemma [Br] that, if H is finitely generated, then H is of type FP_2 if and only if $\mathrm{Rel}^{(l)}(\mathcal{Q})$ is finitely generated.

Summarizing the above, we have the following.

Lemma 11 *Let H be a finitely generated group, and let $\mathcal{Q}$ be a group presentation for H on a finite set of generating symbols. Then the following are equivalent:*

(i) *The group H is of type FP_2;*

(ii) $\mathrm{Rel}^{(l)}(\mathcal{Q})$ *is finitely generated as a left $\mathbb{Z}H$-module;*

(iii) $\mathrm{Rel}^{(b)}(\mathcal{Q})$ *is finitely generated as a $(\mathbb{Z}H, \mathbb{Z}H)$-bi-module.*

4.2 The FHT property for M

From (4) and (8) we obtain the embedding

$$\overline{\mu} = id \otimes \mu \otimes id : \mathbb{Z}M \otimes_{\mathbb{Z}H} \mathrm{Rel}^{(b)}(\mathcal{Q}) \otimes_{\mathbb{Z}H} \mathbb{Z}M \longrightarrow \bigoplus_{b \in \mathbf{b}} \mathbb{Z}M \cdot e_b \cdot \mathbb{Z}M.$$

Also, we have the embedding (see (1))

$$\rho : \pi^{(b)}(\mathcal{M}) \to \Phi \oplus \left(\bigoplus_{b \in \mathbf{b}} \mathbb{Z}M \cdot e_b \cdot \mathbb{Z}M \right) \oplus \left(\bigoplus_{b \in \mathbf{b}} \mathbb{Z}M \cdot e_{b^{-1}} \cdot \mathbb{Z}M \right),$$

where Φ is the free $(\mathbb{Z}M, \mathbb{Z}M)$-bi-module with basis in one-to-one correspondence with the rewriting rules of $\hat{\mathcal{P}}$ and e_{b^ε} corresponds to the rewriting rule $T_{b,\varepsilon}$ of $\mathcal{M}$ $(b \in \mathbf{b}, \varepsilon = \pm 1)$. Then

$$\rho(\zeta_{\mathbb{D}}) \in \Phi \quad (\mathbb{D} \in \mathbf{d}),$$

and for $b \in \mathbf{b}$, $\varepsilon = \pm 1$,

$$\begin{aligned} \rho(\zeta_{\mathbb{K}_{b,\varepsilon}}) &= e_{b^\varepsilon} + [b^\varepsilon] \cdot e_{b^{-\varepsilon}} \cdot [b^\varepsilon] + \kappa_{b,\varepsilon}, \\ \rho(\zeta_{\mathbb{Q}_{b^\varepsilon b^{-\varepsilon}}}) &= e_{b^\varepsilon} \cdot [b^{-\varepsilon}] + [b^\varepsilon] \cdot e_{b^{-\varepsilon}} + \kappa'_{b,\varepsilon}, \end{aligned}$$

where $\kappa_{b,\varepsilon}, \kappa'_{b,\varepsilon} \in \Phi$. Also for $V \in \mathbf{v}$, say $V = b_1^{\varepsilon_1} b_2^{\varepsilon_2} \dots b_r^{\varepsilon_r}$ $(b_i \in \mathbf{b}, \varepsilon_i = \pm 1, i = 1, \dots, r)$,

$$\rho(\zeta_{\mathbb{Q}_V}) = \alpha_V + \sum_{i=1}^{r} [b_1^{\varepsilon_1} \dots b_{i-1}^{\varepsilon_{i-1}}] \cdot e_{b_i^{\varepsilon_i}} \cdot [b_{i+1}^{\varepsilon_{i+1}} \dots b_r^{\varepsilon_r}],$$

where $\alpha_V \in \Phi$.

The homomorphism

$$\Phi \oplus \left(\bigoplus_{b \in \mathbf{b}} \mathbb{Z}M \cdot e_b \cdot \mathbb{Z}M \right) \oplus \left(\bigoplus_{b \in \mathbf{b}} \mathbb{Z}M \cdot e_{b^{-1}} \cdot \mathbb{Z}M \right) \longrightarrow \bigoplus_{b \in \mathbf{b}} \mathbb{Z}M \cdot e_b \cdot \mathbb{Z}M$$

$$\Phi \to 0, \quad e_b \mapsto e_b, \quad e_{b^{-1}} \mapsto -[b^{-1}] \cdot e_b \cdot [b^{-1}] \quad (b \in \mathbf{b}),$$

maps $\rho(B)$ to 0 and maps $\rho(\zeta_{\mathbb{Q}_V})$ to $\frac{\partial V}{\partial \mathbf{b}}$ $(V \in \mathbf{v})$, so we obtain an induced bi-module homomorphism

$$\rho_* : \frac{\pi^{(b)}(\mathcal{M})}{B} \longrightarrow \bigoplus_{b \in \mathbf{b}} \mathbb{Z}M \cdot e_b \cdot \mathbb{Z}M, \quad \overline{\zeta}_{\mathbb{Q}_V} \mapsto \frac{\partial V}{\partial \mathbf{b}} \quad (V \in \mathbf{v}).$$

This mapping, and the embedding $\overline{\mu}$, have the same image. Thus,

$$\begin{aligned} \overline{\mu}^{-1}\rho_* : \quad \tfrac{\pi^{(b)}(\mathcal{M})}{B} &\longrightarrow \mathbb{Z}M \otimes_{\mathbb{Z}H} \mathrm{Rel}^{(b)}(\mathcal{Q}) \otimes_{\mathbb{Z}H} \mathbb{Z}M \\ \overline{\zeta}_{\mathbb{Q}_V} &\mapsto 1 \otimes ((\langle V \rangle - 1) + J^2) \otimes 1 \quad (V \in \mathbf{v}) \end{aligned} \qquad (9)$$

is a surjective bi-module homomorphism. Below we will derive the following result.

Proposition 12 *The mapping $\overline{\mu}^{-1}\rho_*$ is an isomorphism.*

For this we will need the following technical results.

Lemma 13

(i) *If* $[U_0] = [U_1] = [U_2]$ *for some* $U_0, U_1, U_2 \in (\mathbf{b} \cup \mathbf{b}^{-1})^*$, *then* $\overline{\zeta}_{\mathbb{Q}_{U_0,U_1}} - \overline{\zeta}_{\mathbb{Q}_{U_0,U_2}} = \overline{\zeta}_{\mathbb{Q}_{U_2,U_1}}$.

(ii) *If* $Z, U, U' \in (\mathbf{b} \cup \mathbf{b}^{-1})^*$ *and* $[U] = [U']$, *then*

$$[Z] \cdot \overline{\zeta}_{\mathbb{Q}_{U,U'}} = \overline{\zeta}_{\mathbb{Q}_{ZU,ZU'}} \quad \textit{and} \quad \overline{\zeta}_{\mathbb{Q}_{U,U'}} \cdot [Z] = \overline{\zeta}_{\mathbb{Q}_{UZ,U'Z}}.$$

Proof. (i) We may take $\mathbb{P}_{U_2,U_1}$ to be $\mathbb{P}^{-1}_{U_0,U_2}\mathbb{P}_{U_0,U_1}$. Then the 1-cycle arising from the outer circuit in the diagram in Figure 6 represents $\overline{\zeta}_{\mathbb{Q}_{U_2,U_1}}$. The diagram shows that this 1-cycle is the sum of two 1-cycles arising from the smaller circuits representing $\overline{\zeta}_{\mathbb{Q}_{U_0,U_1}}$ and $-\overline{\zeta}_{\mathbb{Q}_{U_0,U_2}}$, respectively.

Figure 6: $\mathbb{Q}_{U_2,U_1}$

(ii) We only prove the first equality, as the second follows by symmetry. By definition, $[Z] \cdot \overline{\zeta}_{\mathbb{Q}_{U,U'}} = \overline{\zeta}_{Z \cdot \mathbb{Q}_{U,U'}}$. Now we may take $\mathbb{P}_{ZU,ZU'}$ to be $Z \cdot \mathbb{P}_{U,U'}$. Then in the diagram in Figure 7 the 1-cycle arising from the outer circuit represents $\overline{\zeta}_{\mathbb{Q}_{ZU,ZU'}}$, and the smaller righthand circuit represents $[Z] \cdot \overline{\zeta}_{\mathbb{Q}_{U,U'}}$. The smaller lefthand circuit is $[\mathbb{B}_Z, \mathbb{P}_{U,U'}]$, which is null-homotopic (and therefore null-homologous) in $\mathcal{D}(\mathcal{M})$ (see Subsection 2.2).

□

Figure 7: $\mathbb{Q}_{ZU,ZU'}$

Proof of Proposition 12. The elements of H are congruence classes of words on $\mathbf{b} \cup \mathbf{b}^{-1}$. We choose a fixed but arbitrary representative from each congruence class, subject to the proviso that we choose the empty word as the representative of the identity of H. For $U \in (\mathbf{b} \cup \mathbf{b}^{-1})^*$, the representative of $[U]$ will be denoted by $\overline{U}$. Then, as an abelian group, J is *free* abelian on the set of elements

$$\langle U \rangle - \langle \overline{U} \rangle \quad (U \in (\mathbf{b} \cup \mathbf{b}^{-1})^*, \langle U \rangle \neq \langle \overline{U} \rangle).$$

We can therefore define an abelian group homomorphism

$$\psi : J \to \frac{\pi^{(b)}(\mathcal{M})}{B}$$

by

$$\langle U \rangle - \langle \overline{U} \rangle \mapsto \overline{\zeta}_{\mathbb{Q}_{\overline{U},U}}.$$

This is well-defined (that is, it does not depend on the choice of representative of the free equivalence class $\langle U \rangle$) by Lemma 8. If $[U_1] = [U_2]$, then $\overline{U}_1 = \overline{U}_2 = U_0$, say, and we have

$$\begin{aligned} \psi(\langle U_1 \rangle - \langle U_2 \rangle) &= \psi(\langle U_1 \rangle - \langle U_0 \rangle) - \psi(\langle U_2 \rangle - \langle U_0 \rangle) \\ &= \overline{\zeta}_{\mathbb{Q}_{U_0,U_1}} - \overline{\zeta}_{\mathbb{Q}_{U_0,U_2}} \quad \text{(by definition of } \psi) \\ &= \overline{\zeta}_{\mathbb{Q}_{U_2,U_1}} \quad \text{(by Lemma 13 (i))}. \end{aligned}$$

Now $J^2 \subseteq \operatorname{Ker}\psi$. For, as an abelian group, J^2 is generated by the elements

$$(\langle Z_1\rangle - \langle Z_2\rangle)(\langle U_1\rangle - \langle U_2\rangle), \text{where } [Z_1] = [Z_2] \text{ and } [U_1] = [U_2],$$

and we have

$$\begin{aligned}
\psi((\langle Z_1\rangle - \langle Z_2\rangle)(\langle U_1\rangle - \langle U_2\rangle)) &= \psi(\langle Z_1U_1\rangle - \langle Z_1U_2\rangle) - \psi(\langle Z_2U_1\rangle - \langle Z_2U_2\rangle)\\
&= \overline{\zeta}_{\mathbb{Q}_{Z_1U_2,Z_1U_1}} - \overline{\zeta}_{\mathbb{Q}_{Z_2U_2,Z_2U_1}} \text{ (from above)}\\
&= [Z_1]\cdot\overline{\zeta}_{\mathbb{Q}_{U_2,U_1}} - [Z_2]\cdot\overline{\zeta}_{\mathbb{Q}_{U_2,U_1}} \text{ (by Lemma 13 (ii))}\\
&= 0.
\end{aligned}$$

We therefore get an induced mapping

$$\psi_* : \operatorname{Rel}^{(b)}(\mathcal{Q}) \to \frac{\pi^{(b)}(\mathcal{M})}{B},$$

which by Lemma 13 (ii) is actually a $(\mathbb{Z}H, \mathbb{Z}H)$-bi-module homomorphism. This in turn gives us an induced $(\mathbb{Z}M, \mathbb{Z}M)$-bi-module homomorphism

$$\overline{\psi}_* := id \otimes \psi_* \otimes id : \mathbb{Z}M \otimes_{\mathbb{Z}H} \operatorname{Rel}^{(b)}(\mathcal{Q}) \otimes_{\mathbb{Z}H} \mathbb{Z}M \longrightarrow \mathbb{Z}M \otimes_{\mathbb{Z}H} \frac{\pi^{(b)}(\mathcal{M})}{B} \otimes_{\mathbb{Z}H} \mathbb{Z}M \cong \frac{\pi^{(b)}(\mathcal{M})}{B}.$$

Applying this to the righthand side of (9) gives $\overline{\zeta}_{\mathbb{Q}_{1,V}} = \overline{\zeta}_{\mathbb{Q}_V}$. So $\overline{\psi}_*$ is the inverse of $\overline{\mu}^{-1}\rho_*$. This completes the proof of Proposition 12. □

We now prove part (ii) of our Main Theorem.

If G is not FHT ($\Leftrightarrow$ FDT), then neither is M, since G is a retract of M. So we can assume that G is FHT, and therefore FDT, which means we can choose $\mathbf{d}$ to be finite. Then the submodule B of $\pi^{(b)}(\mathcal{M})$ is finitely generated. Thus, we have the following sequence of equivalent statements:

$\pi^{(b)}(\mathcal{M})$ is finitely generated, that is, M is FHT

$\Leftrightarrow$ $\pi^{(b)}(\mathcal{M})/B$ is finitely generated (as B is finitely generated)

$\Leftrightarrow$ $\mathbb{Z}M \otimes_{\mathbb{Z}H} \operatorname{Rel}^{(b)}(\mathcal{Q}) \otimes_{\mathbb{Z}H} \mathbb{Z}M$ is finitely generated (using Proposition 12)

$\Leftrightarrow$ H is of type FP_2 (by Lemma 11).

5 The word problem for M

The *generalized word problem* for the subgroup H of G is the problem of deciding, given a word $W \in (\mathbf{a} \cup \mathbf{a}^{-1})^*$, whether $[W]$ belongs to the subgroup H. In the monoid M, we have $[tW] = [Wt]$ if and only if $[W]$ belongs to the subgroup H. Thus, the word problem for M will be undecidable, if the generalized word problem for H is undecidable. If G is the direct product of two copies of a free group of rank at least 2, then the word problem for G is easily solved, but G contains a finitely generated subgroup H such that the generalized word problem for H in G is undecidable [Mih] (see, e.g., [Mil]). Thus, in general the word problem for the monoid M will be undecidable, even if the word problem for the group G is decidable.

Suppose now that H is the kernel of an epimorphism from the group G onto the additive group $\mathbb{Z}$ of integers. Choose an element $g \in G$ such that the image of g in $\mathbb{Z}$ is

1. Then $\{ g^r \mid r \in \mathbb{Z} \}$ is a transversal for H in G, and the corresponding unique normal forms (2) for the elements of M are

$$g^{r_0} t g^{r_1} t \ldots t g^{r_n} h \quad (n \geq 0, r_0, r_1, \ldots, r_n \in \mathbb{Z}, h \in H).$$

If G is given by means of a presentation $\langle \mathbf{a}; \mathbf{u} \rangle$, then an epimorphism onto $\mathbb{Z}$ will arise from a function

$$\phi : \mathbf{a} \cup \mathbf{a}^{-1} \to \mathbb{Z} \ \text{ satisfying } \ \phi(a^{-1}) = -\phi(a) \quad (a \in \mathbf{a}),$$

such that the induced monoid homomorphism (also denoted ϕ)

$$(\mathbf{a} \cup \mathbf{a}^{-1})^* \to \mathbb{Z}$$

is surjective, and maps each $U \in \mathbf{u}$ to 0. Then if

$$W = W_0 t W_1 t \ldots t W_n \quad (n \geq 0, W_0, W_1, \ldots, W_n \in (\mathbf{a} \cup \mathbf{a}^{-1})^*)$$

is a word in the alphabet of M, the corresponding normal form is

$$g^{\phi(W_0)} t g^{\phi(W_1)} t \ldots t g^{\phi(W_n)} (g^{-(\phi(W_0) + \ldots + \phi(W_n))} [W_0 W_1 \ldots W_n]).$$

It follows that if

$$W' = W'_0 t W'_1 t \ldots t W'_m$$

is another such word, then $[W] = [W']$ in M if and only if

(a) $m = n$,

(b) $\phi(W'_i) = \phi(W_i)$ $(0 \leq i \leq n)$,

(c) $[W'_0 W'_1 \ldots W'_n] = [W_0 W_1 \ldots W_n]$ in G.

Since (a) and (b) can be checked in linear time, it follows that the time complexity of the word problem for M is the same as that for G (up to linear time). Thus, in general the word problem for M will still be undecidable.

In the case of the Bestvina and Brady example, however, G is a right-angled Coxeter group, and therefore has word problem solvable in quadratic time [Va, ECH$^+$], and H is the kernel of the epimorphism from G onto $\mathbb{Z}$ that maps each generator $a \in \mathbf{a}$ to 1. Thus, from the above discussion we see that the word problem for our example monoid M is also decidable in quadratic time.

6 The properties FDT_2, FHT_2

We briefly discuss work of McGlashan [McG01], [McG02] on higher dimensional properties FDT_2, FHT_2 of rewriting systems.

Given a rewriting system $\mathcal{R} = [\mathbf{x}; \mathbf{r}]$ and a set $\mathbf{p}$ of closed paths in $\mathcal{D} = \mathcal{D}(\mathcal{R})$, McGlashan constructs a 3-complex $(\mathcal{D}, \mathbf{p})$ as follows[1]. The 2-skeleton is (the geometric realization of) $\mathcal{D}^{\mathbf{p}}$, and for each 1-cell $\mathbb{E}$ in $\mathcal{D}$ and each 2-cell $\mathbb{C}$ in $\mathcal{D}^{\mathbf{p}}$, there is a 3-cell $[\mathbb{E}, \mathbb{C}]$. This 3-cell is attached to the 2-skeleton by a "drum," where the top is mapped to $\iota\mathbb{E} \cdot \mathbb{C}$, the bottom to $\tau\mathbb{E} \cdot \mathbb{C}$, and the side panels are mapped to 2-cells arising from defining paths $[\mathbb{E}, \mathbb{E}_i]$ $(1 \leq i \leq n)$, where $\mathbb{E}_1 \mathbb{E}_2 \ldots \mathbb{E}_n$ is the attaching path of $\mathbb{C}$ (see Figure 8). The free monoid $\mathbf{x}^*$ acts on these 3-cells by $U \cdot [\mathbb{E}, \mathbb{C}] \cdot V = [U \cdot \mathbb{E}, \mathbb{C} \cdot V]$ $(U, V \in \mathbf{x}^*)$. Analogously, 3-cells $[\mathbb{C}, \mathbb{E}]$ are attached.

[1] Our treatment is slightly different from McGlashan's, but equivalent to it.

Figure 8: The 3-cell $[\mathbb{E}, \mathbb{C}]$

Roughly speaking, the rewriting system is said to be FDT_2 (respectively, FHT_2) if it is FDT (respectively, FHT) and for some finite homotopy (respectively, homology) trivializer $\mathbf{p}$, the 3-complex $(\mathcal{D}, \mathbf{p})$ has finitely based second homotopy (respectively, second homology). It is shown in [McG01] that these properties are monoid invariants, and that they are invariant under retractions. Moreover, the two properties are equivalent for groups.

It turns out in fact (due to the Hurewicz Isomorphism Theorem) that, if a particular monoid is FDT, then for this monoid the properties FDT_2, FHT_2 are equivalent. Consequently, the fact that we have shown that in general FDT and FHT are not equivalent is important for McGlashan's theory, because it means that the higher dimensional properties are not necessarily equivalent.

There are examples known of monoids (in fact, groups) which are FDT but not FDT_2 (a group of type FP_3 but not of type FP_4 is such an example). However, there are no known examples of monoids which are FHT (but not FDT!) which fail to be FHT_2. It is reasonable to suppose that taking H in our construction to be of type FP_2 but not of type F_2 and also not of type FP_3, would yield such an example.

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