ISLAND 3

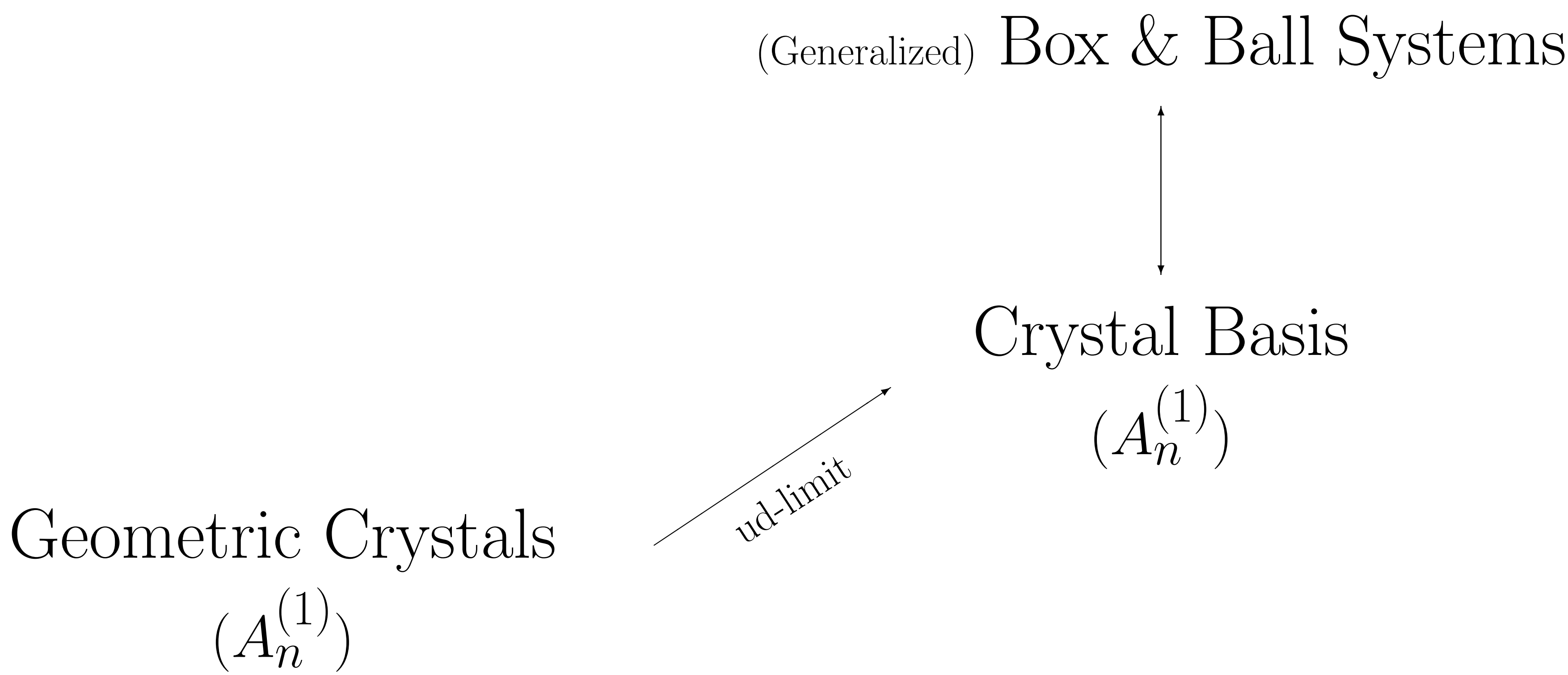
Local Darboux transformations and geometric crystals (I) :

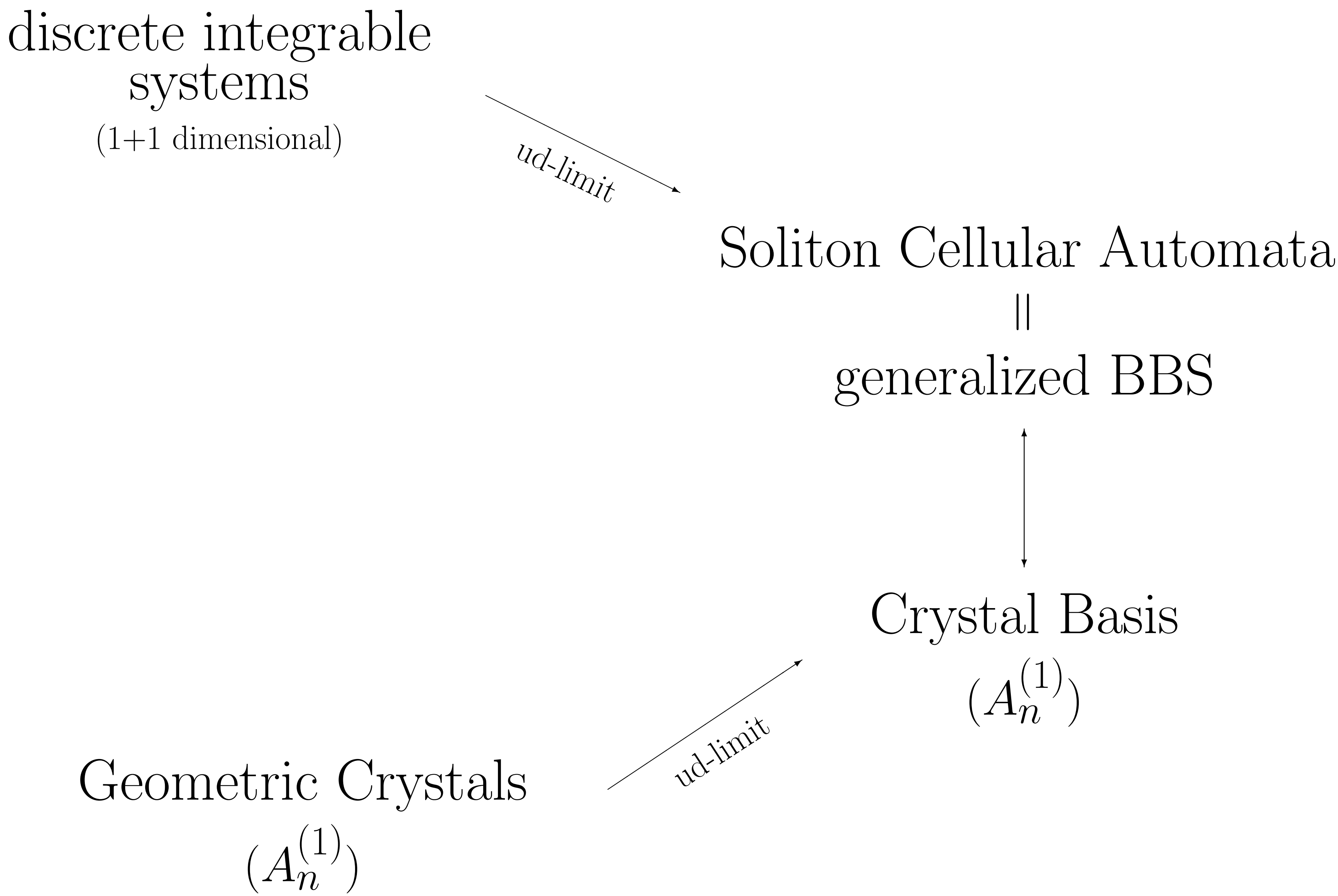
the dKP hierarchy and its reductions

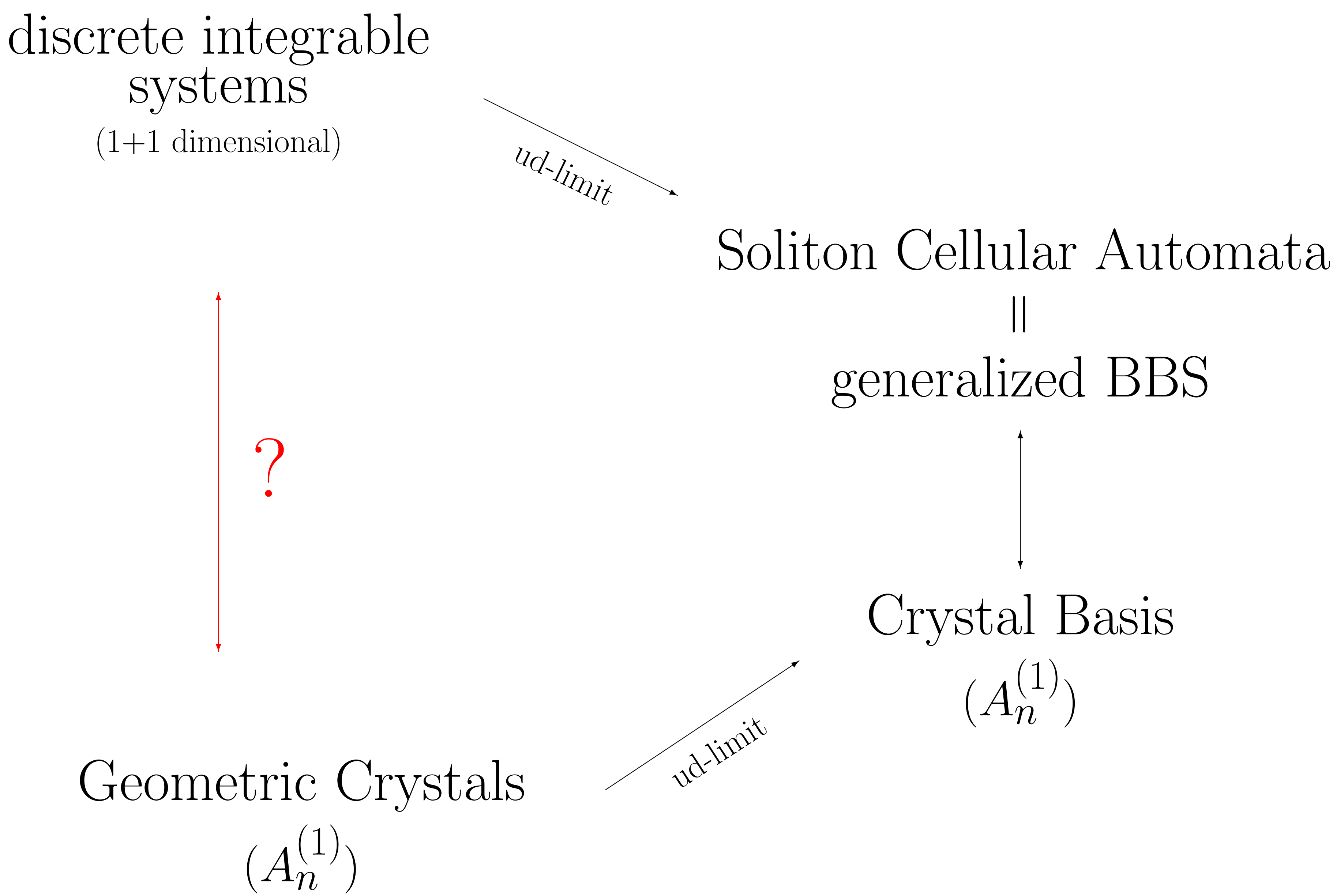
S. Kakei, J.J.C. Nimmo, R.W.

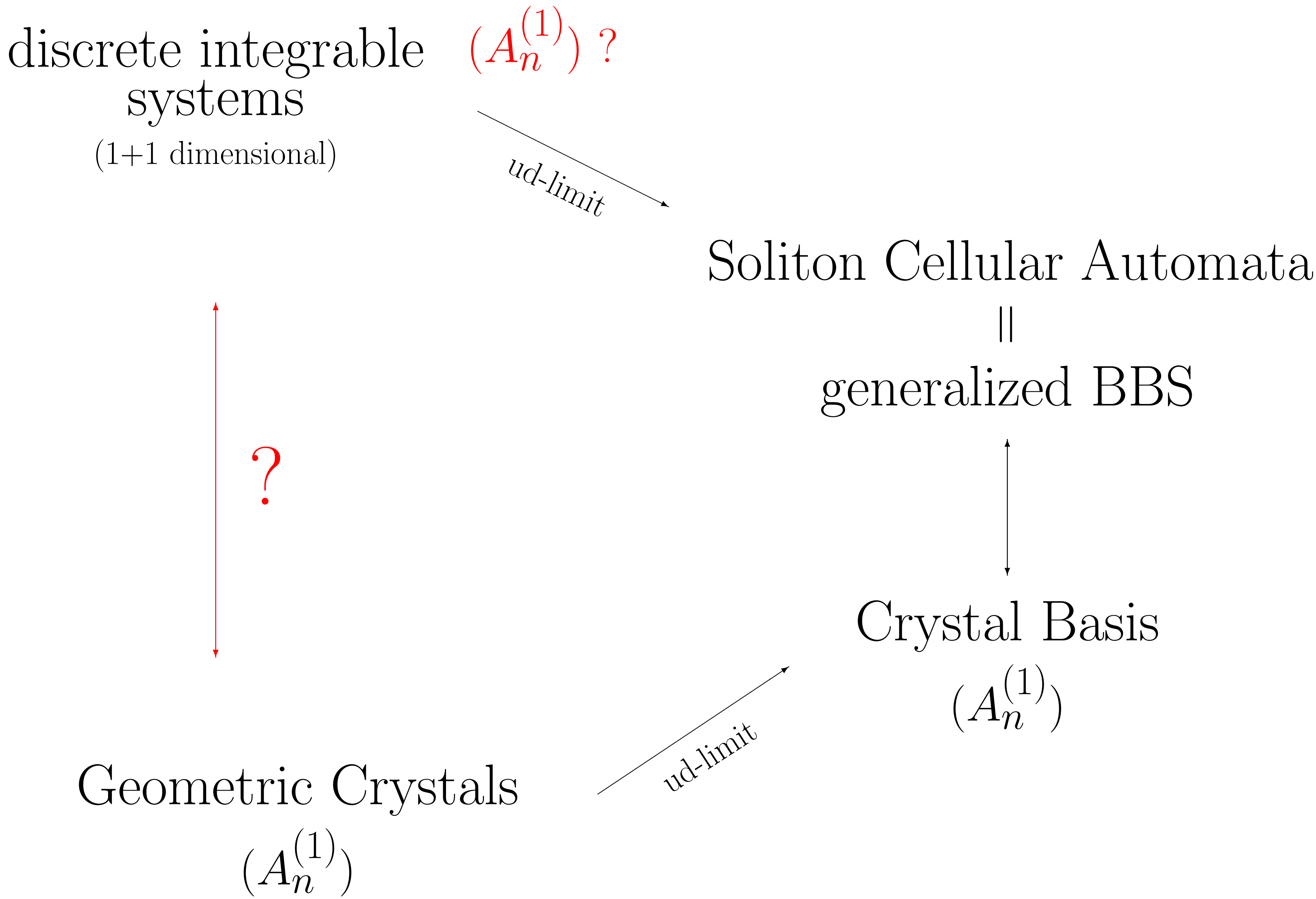
Motivation

- clarify the relationship between integrable systems and geometric crystals
- in particular : to clarify this relationship in terms of the fundamental symmetries of these (discrete) integrable systems
- preferably : to do so in a general framework


(Generalized) Box & Ball Systems
Crystal Basis
$(A_n^{(1)})$
ud-limit
Geometric Crystals
$(A_n^{(1)})$


discrete integrable systems
(1+1 dimensional)
ud-limit
Soliton Cellular Automata
||
generalized BBS
Crystal Basis
$(A_n^{(1)})$
ud-limit
Geometric Crystals
$(A_n^{(1)})$


discrete integrable systems
(1+1 dimensional)
ud-limit
Soliton Cellular Automata
||
generalized BBS
?
Crystal Basis
$(A_n^{(1)})$
ud-limit
Geometric Crystals
$(A_n^{(1)})$


discrete integrable systems
(1+1 dimensional)
$(A_n^{(1)})$?
ud-limit
Soliton Cellular Automata
||
generalized BBS
?
Crystal Basis
$(A_n^{(1)})$
Geometric Crystals
$(A_n^{(1)})$
ud-limit

KP hierarchy ($gl(\infty)$) $\xrightarrow{\text{reduction}}$ 1+1 dim. integrable systems ($A_n^{(1)}$)

KP hierarchy
$(gl(\infty))$

reduction →

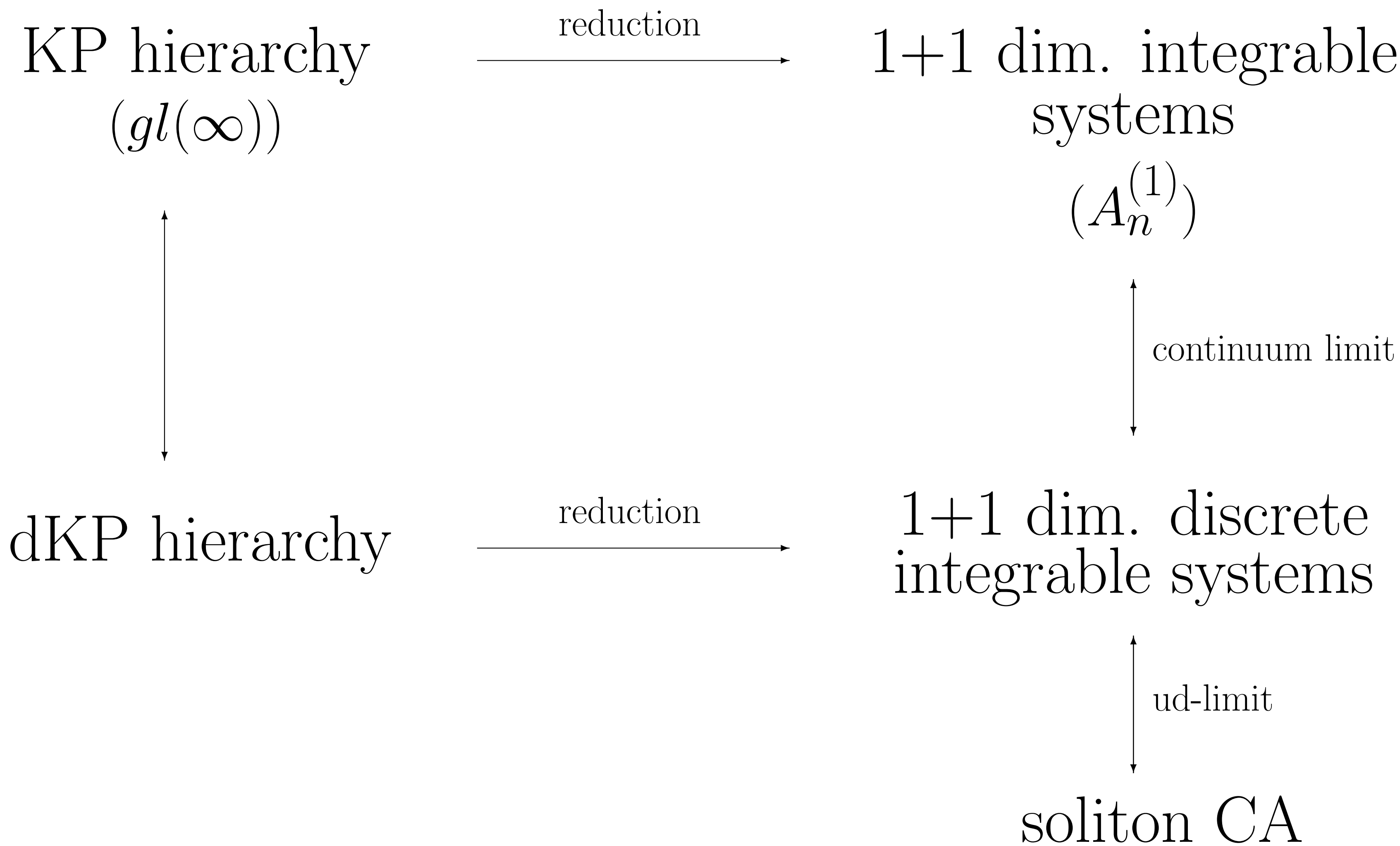
1+1 dim. integrable systems
$(A_n^{(1)})$

↕

dKP hierarchy

reduction →

1+1 dim. discrete integrable systems


KP hierarchy
$(gl(\infty))$
reduction
1+1 dim. integrable systems
$(A_n^{(1)})$
continuum limit
dKP hierarchy
reduction
1+1 dim. discrete integrable systems
ud-limit
soliton CA

KP hierarchy
$(gl(\infty))$

reduction →

1+1 dim. integrable systems
$(A_n^{(1)})$

↕ continuum limit

dKP hierarchy

reduction →

1+1 dim. discrete integrable systems

↕ ud-limit

soliton CA

G. Hatayama et al., J. Math. Phys. 42 (2001), 274–308

nonautonomous dKP

↓

generalized BBS, corresponding to $A_n^{(1)}$ crystal basis for symmetric tensors

dKP equation : $\tau(\ell, m, n)\ :\ \mathbb{Z}^3 \to \mathbb{C} \qquad a, b, c \in \mathbb{C}$

$$\boxed{(b-c)\tau_\ell \tau_{mn} + (c-a)\tau_m \tau_{\ell n} + (a-b)\tau_n \tau_{\ell m} = 0}$$

$$(\tau_\ell = \tau(\ell+1, m, n),\ \ \cdots)$$

- has $gl(\infty)$ symmetry $\leftrightarrow$ free-fermionic description of $\tau(\ell, m, n)$
  [E. Date et al., J. Phys. Soc. Jpn. 51 (1982) 4125–4131.]

- has Wronskian and Grammian type solutions, the entries of which are expressed in terms of

$$\varphi_{p,q} = \left(\frac{a-q}{a-p}\right)^\ell \left(\frac{b-q}{b-p}\right)^m \left(\frac{c-q}{c-p}\right)^n \qquad p, q \in \mathbb{C}$$

e.g.: $\quad \tau^{(1)} = 1 + \varphi_{p,q}, \quad \tau^{(2)} = 1 + \varphi_{p_1,q_1} + \varphi_{p_2,q_2} + \dfrac{(p_1-p_2)(q_1-q_2)}{(p_1-q_2)(q_1-p_2)}\varphi_{p_1,q_1}\varphi_{p_2,q_2}$

- has the KP equation as its continuum limit for $a, b, c \to \infty$:

$$\tau_\ell = \tau(x + \frac{1}{a}, y + \frac{1}{2a^2}, t + \frac{1}{3a^3}), \quad \tau_m = \tau(x + \frac{1}{b}, y + \frac{1}{2b^2}, t + \frac{1}{3b^3}), \quad \cdots$$

Problems

1) Reduction : $\tau_{mn} = \tau$

$$\tau\tau_\ell + \delta\tau_m\tau_{\ell m'} - (1+\delta)\tau_{\ell m}\tau_{m'} \ = \ 0 \qquad \text{(dKdV)}$$

$$(\tau_{m'} = \tau(\ell, m-1, n))$$

$$\delta = \frac{c-a}{b-c}\ , \quad \delta \to 0 : \text{Lotka-Volterra} \xrightarrow[limit]{2nd\ \ cont.} \text{KdV}$$

Problems

1) Reduction : $\tau_{mn} = \tau$

$$\left(\varphi_{p,q}\right)_{mn} = \varphi_{p,q} \quad \Rightarrow \quad p + q = b + c$$

$$\leftrightarrow \quad A_1^{(1)} : \; p + q = 0 \; ?$$

continuum limit : $p + q = \mathrm{c}^{\underline{\mathrm{t}}}$ ("pseudo-reduction")

cont. case : symmetry algebra related to $A_1^{(1)}$ [G. Post, J. Phys. A 20 (1987) 6591–6592]

Problems

1) Reduction : $\tau_{mn} = \tau$

$$(\varphi_{p,q})_{mn} = \varphi_{p,q} \quad \Rightarrow \quad p + q = b + c$$

$$\leftrightarrow \quad A_1^{(1)} : \; p + q = 0 \; ?$$

continuum limit : $p + q = \text{c}^{\underline{\text{t}}}$ ("pseudo-reduction")

cont. case : symmetry algebra related to $A_1^{(1)}$ [G. Post, J. Phys. A 20 (1987) 6591–6592]

discrete case ?

Problems

1) Reduction : $\tau_{mn} = \tau$

$$\left(\varphi_{p,q}\right)_{mn} = \varphi_{p,q} \quad \Rightarrow \quad p + q = b + c$$

$$\leftrightarrow \quad A_1^{(1)} : \; p + q = 0 \; ?$$

continuum limit : $p + q = c^{\underline{t}}$ (“pseudo-reduction”)

cont. case : symmetry algebra isomorphic to $A_1^{(1)}$ [G. Post, J. Phys. A 20 (1987) 6591–6592]

2) dKP Hierarchy :

– [E. Date et al., JPSJ 51 (1982) 4125–4131] : higher order lattice eqns. from bilinear identity

– [Y. Ohta et al., JPSJ 62 (1993) 1872–1886] : explicit determinant form of hierarchy

– [F. Nijhoff et al., PLA 105 (1984) 267–272] : formulation in terms of linear integral equation

– [S. Tsujimoto, Publ. RIMS 38 (2002) 113–133] : “Sato-like” construction of dToda hierarchy

– [M. Białecki (unpublished)] : “Sato-like” construction of dKP over a finite field

Sato construction of the dKP hierarchy

For $f(\ell, \mathbf{m}) = f(\ell, m_1, \ldots, m_M)$:

$$T_\ell f(\ell, \mathbf{m}) = f(\ell+1, \mathbf{m}), \quad T_{m_j} f(\ell, \mathbf{m}) = f(\ell, m_1, \ldots, m_j + 1, \ldots, m_M) \quad (j = 1..M)$$

Sato-Wilson operators : Ueno-Takasaki [Adv. Stud. Pure Math. 4 (1984) 1–95]

$$\begin{aligned} W(\ell, \mathbf{m}) &:= \; I + w_1 T_\ell^{-1} + w_2 T_\ell^{-2} + \cdots \\ \overline{W}(\ell, \mathbf{m}) &:= \; \overline{w}_0 + \overline{w}_1 T_\ell + \overline{w}_2 T_\ell^2 + \cdots \end{aligned}$$

for $w_i(\ell, \mathbf{m}),\ \overline{w}_i(\ell, \mathbf{m}),\ \overline{w_0} \neq 0$

projection to non-negative part : $A(T_\ell) = \sum_{n \in \mathbb{Z}} A_n T_\ell^n$, $[A]_{\geq 0} = \sum_{n \geq 0} A_n T_\ell^n$

discrete Sato-equations :

$$\begin{aligned} \left(T_{m_j} W\right)(1 - \alpha_j + \alpha_j T_\ell) &= B_j W \qquad ({}^\forall j = 1 \cdots M) \\ \left(T_{m_j} \overline{W}\right)(1 - \alpha_j + \alpha_j T_\ell) &= B_j \overline{W} \end{aligned}$$

$$B_j := \left[\left(T_{m_j} W\right)(1 - \alpha_j + \alpha_j T_\ell) W^{-1}\right]_{\geq 0} \equiv \; \alpha_j T_\ell + (1 - \alpha_j) u_j$$

$$u_j = \frac{T_{m_j} \overline{w}_0}{\overline{w}_0}$$

Construction : $\Delta_\ell := a(T_\ell - 1)\ ,\quad \Delta_{m_j} := b_j(T_{m_j} - 1)$ $\boxed{\alpha_j \equiv \frac{a}{b_j}}$

consider solutions of $\widetilde{W}_n f^{(k)} = 0\ (k = 1..n)$ that satisfy $\boxed{\Delta_\ell f^{(k)} = \Delta_{m_j} f^{(k)}}$ for :

$$\widetilde{W}_n := T_\ell^n + \omega_{n-1} T_\ell^{n-1} + \cdots + \omega_0$$

$$\Rightarrow \qquad \omega_0 = \frac{T_\ell\, \tau}{\tau}\ , \qquad \tau(\ell, \mathbf{m}) = \begin{vmatrix} f^{(1)} & \cdots & \Delta_\ell^{n-1} f^{(1)} \\ \vdots & & \vdots \\ f^{(n)} & \cdots & \Delta_\ell^{n-1} f^{(n)} \end{vmatrix}$$

$\longrightarrow$ two Sato-Wilson operators are needed to construct B_{m_j} when $n \longrightarrow +\infty$

$$\boxed{\omega_0 \longrightarrow \overline{w}_0 = \frac{\tau_\ell}{\tau}}$$

as opposed to the continuous case : $\widetilde{W}_n = \partial_x^n + a_{n-1}\partial_x^{n-1} + \cdots + a_0\ ,\quad \widetilde{W}_n f^{(k)} = 0$

$$\Rightarrow \qquad a_{n-1} = -\frac{\tau_x}{\tau}$$

Linear equations

$$\varphi_\lambda := \left(1-\frac{\lambda}{a}\right)^{\ell} \prod_{j=1}^{M}\left(1-\frac{\lambda}{b_j}\right)^{m_j}$$

$$\Psi_\lambda(\ell,\mathbf{m}) := W(\ell,\mathbf{m})\varphi_\lambda = \big(1+\sum_{k=1}^{+\infty} w_k (1-\lambda/a)^{-k}\big)\varphi_\lambda$$

$$\overline{\Psi}_\lambda(\ell,\mathbf{m}) := \overline{W}(\ell,\mathbf{m})\varphi_\lambda = \big(\overline{w}_0+\sum_{k=1}^{+\infty} \overline{w}_k (1-\lambda/a)^{k}\big)\varphi_\lambda$$

These satisfy : $T_{m_j}\Phi = B_j\Phi \equiv \alpha_j\big(T_\ell\,\Phi\big) + (1-\alpha_j)u_j\Phi$

$$\Leftrightarrow \quad \boxed{\Phi = \frac{1}{a-b_j}\,\frac{1}{u_j}\,\big(aT_\ell\,\Phi - b_j T_{m_j}\Phi\big) \qquad {}^{\forall} j = 1\cdots M}$$

with $\boxed{u_j = \dfrac{\tau\;\tau_{\ell m_j}}{\tau_\ell\;\tau_{m_j}}}$ if one takes $\overline{w}_0 = \dfrac{\tau_\ell}{\tau}$

Compatibility condition : $\left(T_{m_j} B_k\right) B_j = \left(T_{m_k} B_j\right) B_k$

$$\begin{cases} u_j(T_{m_j} u_k) = u_k(T_{m_k} u_j) \\ \alpha_j(1-\alpha_k)(T_{m_j} u_k) + \alpha_k(1-\alpha_j)(T_\ell\, u_j) = \alpha_k(1-\alpha_j)(T_{m_k} u_j) + \alpha_j(1-\alpha_k)(T_\ell\, u_k) \end{cases}$$

In particular, for $M = 2,\ m_1 = m,\ m_2 = n,\ b_1 = b,\ b_2 = c$:

$$u_1 = \frac{\tau \tau_{\ell m}}{\tau_\ell \tau_m}\ , \qquad u_2 = \frac{\tau \tau_{\ell n}}{\tau_\ell \tau_n}$$

$$\Leftrightarrow \qquad (b-c)\tau_\ell\, \tau_{mn} + (c-a)\tau_m \tau_{\ell n} + (a-b)\tau_n \tau_{\ell m} = 0 \qquad (*)$$

The linear problem can be symmetrized :

$$\begin{aligned} \Phi &= \frac{1}{a-b}\, \frac{\tau_\ell\, \tau_m}{\tau\, \tau_{\ell m}} \left(a\, T_\ell\, \Phi - b\, T_m \Phi\right) \\ &= \frac{1}{a-c}\, \frac{\tau_\ell\, \tau_n}{\tau\, \tau_{\ell n}} \left(a\, T_\ell\, \Phi - c\, T_n \Phi\right) \end{aligned}$$

$$(*)\ \frac{\tau\, \Phi}{\tau_\ell\, \tau_m \tau_n} \qquad \Leftrightarrow \qquad \Phi = \frac{1}{b-c}\, \frac{\tau_m\, \tau_n}{\tau\, \tau_{mn}} \left(b\, T_m\, \Phi - c\, T_n \Phi\right)$$

Denote the dKP equation involving (ℓ, m_j, m_k) as $\mathrm{E}(\ell, m_j, m_k)$

i) $\mathbf{E} = \{\ \mathrm{E}(\ell, m_j, m_k) \big|\ m_j, m_k \in \{m_1, \ldots, m_M\},\ m_j \neq m_k\}$ generates all bilinear equations in $\overline{\mathbf{E}} = \{\mathrm{E}(\ell_1, \ell_2, \ell_3) \big|\ \ell_1, \ell_2, \ell_3 \in \{\ell, \mathbf{m}\},\ \ell_1 \neq \ell_2 \neq \ell_3\}$.

E (or $\overline{\mathbf{E}}$) for $M \to +\infty$: dKP hierarchy

ii) The equations in **E** generate the bilinear equations proposed by Ohta et al. [Y. Ohta et al., J. Phys. Soc. Jpn. 62 (1993) 1872–1886] as the dKP hierarchy

$$
{}^{\forall}N = 3, \ldots, M+1\ \ : \qquad \begin{vmatrix} \tau_{\ell_1}\tau_{\widehat{\ell_1}} & \tau_{\ell_2}\tau_{\widehat{\ell_2}} & \cdots & \tau_{\ell_N}\tau_{\widehat{\ell_N}} \\ 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_N \\ \vdots & \vdots & & \vdots \\ a_1^{N-2} & a_2^{N-2} & \cdots & a_N^{N-2} \end{vmatrix} \ = \ 0
$$

$$
\tau_{\widehat{\ell_k}} = \tau(\ell_1 + 1, \ldots, \ell_{k-1} + 1,\ \ell_k\ , \ell_{k+1} + 1, \ldots, l_N + 1)
$$

for N-tuples $\quad (\ell_1, \ldots, \ell_n) = (\ell, m_1, \ldots, m_{N-1})\ , \quad a_j = \begin{cases} a & (j = 1) \\ b_{j-1} & (j \geq 2) \end{cases}$

Bilinear identity

Lattice equation of order $M+1$: $$\sum_{n=1}^{M+1} \frac{\tau_{\ell_n}\,\tau_{\widehat{\ell_n}}}{\prod_{k\neq n}(a_n - a_k)} = 0$$

$$\Leftrightarrow \quad \sum_{n=1}^{M+1} \operatorname*{Res}_{\lambda=a_n} \left[\, \tau(\mathbf{x} - \boldsymbol{\varepsilon}[\lambda^{-1}])\ \tau(\mathbf{x}' + \boldsymbol{\varepsilon}[\lambda^{-1}])\ \prod_{k=1}^{M+1} (a_k - \lambda)^{-1} \right] = 0$$

$$\mathbf{x} = (x_1, x_2, \ldots, x_{M+1}) := \sum_{n=1}^{M+1} \ell_n\, \boldsymbol{\varepsilon}[a_n^{-1}]\ , \qquad \boldsymbol{\varepsilon}[\zeta] = (\zeta, \frac{\zeta^2}{2}, \ldots, \frac{\zeta^{M+1}}{M+1})$$

$$\mathbf{x}' = \mathbf{x} - \sum_{n=1}^{M+1} \boldsymbol{\varepsilon}[a_n^{-1}]$$

$$\Leftrightarrow \quad \oint_{\lambda=\infty} \frac{\mathrm{d}\lambda}{2\pi i}\ \tau(\mathbf{x} - \boldsymbol{\varepsilon}[\lambda^{-1}])\tau(\mathbf{x}' + \boldsymbol{\varepsilon}[\lambda^{-1}])\ e^{\sum_{n=1}^{+\infty}(x_n - x'_n)\lambda^n} = 0$$

KP bilinear identity as $M \to +\infty$

Darboux transformation

– **Fundamental symmetry of the dKP hierarchy :** $\tau \;\longrightarrow\; \widetilde{\tau} = \tau\,\Phi$

Φ : solution to the linear problem associated to τ ; the operator $G_\Phi := a(T_\ell - \dfrac{\Phi_\ell}{\Phi})$ maps solutions of the linear problem for τ to that for $\widetilde{\tau}$.

– **Darboux transformations for τ functions are equivalent to shifts.**

Particularly : $$\tau \;\longrightarrow\; \tau\,\overline{\Psi}_\lambda\;, \qquad \overline{\Psi}_\lambda = \Big(\;\tau_\ell/\tau + \sum_{k=1}^{+\infty} \overline{w}_k (1-\lambda/a)^k\;\Big)\,\varphi_\lambda$$

Remark that the dKP equation and the potentials u_j are invariant under linear gauge transformations $\tau \longrightarrow \tau\varphi_\lambda$

Hence, $$\widetilde{\tau} = \;\tau\,\overline{\Psi}_\lambda \varphi_\lambda^{-1} \;=\; \tau_\ell + \tau \sum_{k=1}^{+\infty} \overline{w}_k (1-\lambda/a)^k$$

is a Darboux transformed tau function, and so is $\widetilde{\tau}\big|_{\lambda=a} \equiv \tau_\ell$.

Comment :

F. Nijhoff et al. [PLA 97 (1993) 125–128 ; LMP 9 (1985) 235–241] have shown that linear integral equations such as

$$\varphi(k) = e^{\theta(k)} - e^{\theta(k)} \iint_{\mathcal{C}} \mathrm{d}\eta(\lambda,\mu)\, \frac{e^{-\theta(\mu)}}{k-\mu}\varphi(\lambda)$$

can be used to generate linear problems and Bäcklund transformations for integrable lattice equations, notably for the dKP lattice.

In fact, one can show that any $\tau(\mathbf{x})$ satisfying

$$\tau(\mathbf{x}-\boldsymbol{\varepsilon}[k^{-1}]) = \tau(\mathbf{x}) - \int_{\mathcal{C}_\lambda} \frac{\mathrm{d}\lambda}{2\pi i} \int_{\mathcal{C}_\mu} \frac{\mathrm{d}\mu}{2\pi i}\, h(\lambda,\mu) \frac{e^{\sum_{n=1}^{+\infty} x_n(\lambda^n-\mu^n)}}{k-\mu}\, \tau(\mathbf{x}-\boldsymbol{\varepsilon}[\lambda^{-1}])$$

for a given kernel $h(\lambda,\mu)$ satisfies the KP bilinear identity. Conversely, a KP tau function satisfies the integral equation for the kernel

$$h(\lambda,\mu) = \frac{1}{\mu-\lambda}\Big[\frac{\tau(\mathbf{x}'-\boldsymbol{\varepsilon}[\mu^{-1}]+\boldsymbol{\varepsilon}[\lambda^{-1}])}{\tau(\mathbf{x}')} e^{\sum_{n=1}^{+\infty} x'_n(\mu^n-\lambda^n)} - 1\Big]$$

for arbitrary $\mathbf{x}'$. [R.W., RIMS kōkyūroku 1170 (2000) 111-118]

Reductions of the dKP hierarchy

i) reduction to dKdV : $\tau_{mn} = \tau$

What does this imply for solutions of the dKP equation, in the continuum limit ?

$$\text{Set} \quad x = \frac{\ell}{a} + \frac{m}{b} + \frac{n}{c} \;, \quad y = \frac{\ell}{2a^2} + \frac{m}{2b^2} + \frac{n}{2c^2} \;, \quad t = \frac{\ell}{3a^3} + \frac{m}{3b^3} + \frac{n}{3c^3}$$

$$\Rightarrow \quad \tau_{mn} = \tau + (\frac{1}{b} + \frac{1}{c})\ \tau_x + \frac{1}{2}(\frac{1}{b^2} + \frac{1}{c^2})\ \tau_y + \frac{1}{2}(\frac{1}{b} + \frac{1}{c})^2\ \tau_{2x} + \mathcal{O}(b^{-3}, \cdots)$$

To obtain nontrivial solutions one needs : $\boxed{b + c = 0}$ in which case :

$$\tau_y = \mathcal{O}(b^{-1}) \;\; \Rightarrow \;\; \tau_y \stackrel{b\to\infty}{\longrightarrow} 0 \qquad \text{(KdV reduction)}$$

and for the soliton solutions : $p + q = 0 \quad (\leftrightarrow \; A_1^{(1)}$ symmetry)

Soliton solutions :

$$\varphi_{p,q} \xrightarrow[p+q=0]{b+c=0} \left(\frac{a+p}{a-p}\right)^{\ell}\left(\frac{b+p}{b-p}\right)^{m-n}$$

$$= \exp\left[2p\ (\frac{\ell}{a}+\frac{m-n}{b}) + \frac{2p^3}{3}(\frac{\ell}{3a^3}+\frac{m-n}{3b^3}) + \mathcal{O}(a^{-5}, b^{-5}, ...)\ \right]$$

$$\xrightarrow{a,b\to\infty} \exp[2px + 2p^3 t]$$

Bilinear equation : $$\boxed{2b\,\tau\,\tau_\ell - (b+a)\tau_m\,\tau_{\ell m'} + (a-b)\tau_{\ell m}\,\tau_{m'} = 0}$$

or : $$\boxed{u_{\ell m} - u = \frac{b+a}{b-a}(\frac{1}{u_m} - \frac{1}{u_\ell})} \qquad (u = \frac{\tau\,\tau_{\ell m}}{\tau_\ell\,\tau_m})$$

Lax pair : $$(T_m T_n)\psi_\lambda = (1-\frac{\lambda^2}{b^2})\,\psi_\lambda$$

$$\begin{cases} L\psi_\lambda \equiv \left[T_\ell^2 + [(\frac{b}{a}-1)u_\ell - (\frac{b}{a}+1)\frac{1}{u}]T_\ell + (1-\frac{b^2}{a^2})\right]\psi_\lambda = \frac{\lambda^2 - b^2}{a^2}\,\psi_\lambda \\ T_m\psi_\lambda = B\psi_\lambda \equiv \left[\frac{a}{b}T_\ell + (1-\frac{a}{b})u\right]\psi_\lambda \end{cases}$$

ultra-discretization

Introduce $\delta = -\dfrac{a+b}{2b}$: $\qquad (1+\delta)\,\tau_{m'}\,\tau_{\ell m} = \tau\,\tau_{\ell} + \delta\,\tau_m\,\tau_{\ell m'}$

and define $\delta =: e^{-D/\varepsilon},\ D > 0\ ,\qquad T := \lim_{\varepsilon \to 0^+} \varepsilon \log \tau(\varepsilon)$

$$\xrightarrow{ud-limit} \qquad \boxed{T_{\ell m} + T_{m'} = \max\big(T + T_{\ell}\ ,\ T_m + T_{\ell m'} - D\big)} \qquad \text{(udKdV)}$$

[T. Tokihiro et al., PRL 76 (1996) 3247-3250]

To ultra-discretize the soliton solutions one introduces

$$\frac{b+p}{b-p} =: e^{-\Omega/\varepsilon}, \qquad \frac{a+p}{a-p} \equiv \frac{1 + e^{\Omega/\varepsilon}(1 + e^{D/\varepsilon})}{1 + e^{\Omega/\varepsilon} + e^{D/\varepsilon}} =: e^{\kappa/\varepsilon}$$

$$\Rightarrow \lim_{\varepsilon \to 0^+} \varepsilon \log\big(1 + \exp[\frac{\kappa\,\ell - \Omega\,m}{\varepsilon}]\big) \ = \ \boxed{\max(0, \kappa\,\ell - \Omega\,m)}$$

$$\kappa = \max(0, \Omega + D) - \max(\Omega, D) \equiv \mathrm{sgn}(\Omega)\min(D, |\Omega|)$$

(if $D = 1$: Takahashi-Satsuma CA)

ii) reduction to discrete Boussinesq : $\boxed{\tau_{\ell mn} = \tau}$

$$\tau_{\ell mn} = \tau + (\frac{1}{a} + \frac{1}{b} + \frac{1}{c})\tau_x + \frac{1}{2}(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2})\tau_y + \frac{1}{2}(\frac{1}{a} + \frac{1}{b} + \frac{1}{c})^2\,\tau_{2x} + \frac{1}{3}(\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3})\tau_t$$
$$\frac{1}{2}(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2})(\frac{1}{a} + \frac{1}{b} + \frac{1}{c})\tau_{xy} + \frac{1}{6}(\frac{1}{a} + \frac{1}{b} + \frac{1}{c})^3\,\tau_{3x} + \mathcal{O}(a^{-4}, \ldots)$$

Impose $\boxed{b = \omega a,\ c = \omega^2 a, \quad 1 + \omega + \omega^2 = 0}$

$$\varphi_{p,q} = \left(\frac{a-q}{a-p}\right)^{\ell} \left(\frac{\omega a-q}{\omega a-p}\right)^{m} \left(\frac{\omega^2 a-q}{\omega^2 a-p}\right)^{n}$$

$$\left(\varphi_{p,q}\right)_{\ell mn} = \varphi \ \Leftrightarrow\ \boxed{q^3 = p^3} \quad (q \neq p) \quad : \ A_2^{(1)} \text{ symmetry}$$

set $q = jp \quad (1 + j + j^2 = 0)$:

$$\varphi_{p,jp} \ \simeq \ \exp\left[(1-j)\,p\ \frac{\ell + \omega^2 m + \omega n}{a} + (1-j^2)\,p^2\ \frac{\ell + \omega m + \omega^2 n}{2a^2} + \mathcal{O}(a^{-3})\right]$$
$$\xrightarrow[a\to\infty]{} \ \exp[kx \pm i\sqrt{3}k^2 y] \qquad (k = (1-j)p)$$

dBSq :

$$\boxed{\tau_\ell\,\tau_{\ell'} \;+\; \omega\,\tau_m\,\tau_{m'} \;+\; \omega^2\,\tau_{\ell m}\,\tau_{\ell' m'} \;=\; 0}$$

$$\boxed{v_m \;= v\,\frac{u_\ell \;+\; \omega^2\,v_\ell}{u \;+\; \omega^2\,v}\;, \qquad u\,v_m = v\,u_{\ell' m'}}$$

$$\left(u = \frac{\tau\,\tau_{\ell m}}{\tau_\ell\,\tau_m},\;\; v = \frac{\tau\,\tau_{m'}}{\tau_\ell\,\tau_{\ell' m'}}\;\right)$$

Lax pair :

$$\begin{cases} L\psi \equiv \left[T_\ell^3 + [(\omega-1)u_{2\ell} + (\omega^2-1)v_{\ell m}]T_\ell^2 + 3u_\ell v_{\ell m} T_\ell\right]\psi = \eta\psi \\ T_m\psi = B\psi \equiv \left[\omega^2 T_\ell + (1-\omega^2)u\right]\psi \end{cases}$$

gauge transformation : $\tau \longrightarrow \omega^{\ell m}\,\tau \;\;\Rightarrow\;\; u \longrightarrow \omega\,u,\;\; v \longrightarrow \omega^2\,v$

$$\Rightarrow\;\; \boxed{v_m \;= v\,\frac{u_\ell \;+\; v_\ell}{u \;+\; v}\;, \qquad u\,v_m = v\,u_{\ell' m'}}$$

and : $$\boxed{\tau_\ell\,\tau_{\ell'} \;= -\omega\left[\tau_m\,\tau_{m'} \;+\; \tau_{\ell m}\,\tau_{\ell' m'}\right]}$$

(which is ultra-discretizable)

Conclusions:

- Sato construction of the dKP hierarchy
- One should not forget about the solutions when “discretizing” integrable systems
- interesting form of a dBsq in a equation