



Ultradiscrete KdV equation and Box-Ball System (negative soliton in udKdV.eq.)

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Plan of the talk

- Quick review of discrete KdV equation (dKdV eq.), ultradiscrete KdV equation (udKdV eq.) and Box-Ball system (BBS).
[relations to (continuous) KdV equation, Lax form, conserved quantities, periodic case etc.]
- Negative soliton (background solution) in the udKdV eq.
[gauge transformation, conserved quantities and definition of the background solutions, general solutions constructed from soliton solutions of dKdV equation]

Ref.) M.Kanki, J.Mada and T.Tokihiro: “Conserved quantities and generalized solutions of the ultradiscrete KdV equation”, J. Phys. A44, 145202 (15 pages) (2011).

(About correlation functions of the BBS, Mada-T, ‘Two point correlation functions for a periodic box-ball system’, SIGMA 7, 027 (16 pages) (2011).)

Discrete KdV equation

The dKdV equation (Hirota, Hirota-Tsujimoto) :

$$\frac{1}{u_{n+1}^{t+1}} - \frac{1}{u_n^t} + \frac{\delta}{1+\delta} (u_n^{t+1} - u_{n+1}^t) = 0 \quad n, t \in \mathbf{Z}, \delta \in \mathbf{R} : \text{parameter}$$

Relation to continuous dKdV eq.: $u_n^t = 1 + \lambda U(T, X)$

$$T = \varepsilon(t - \frac{1}{2}), X = \delta(n - \frac{1}{2}) - \eta(t - \frac{1}{2}); \quad \lambda = (1 + 2\delta)\delta, \quad \eta = \frac{3}{2}\delta^2, \quad \varepsilon = \frac{1}{3}\delta^3,$$

$$\Downarrow \quad \delta \rightarrow 0 \quad (\lambda, \varepsilon, \eta \rightarrow 0)$$

$$U = U(T, X): \quad U_T + 6UU_X + U_{XXX} = 0 \quad \dots \text{KdVeq.}$$

From dKP eq. to dKdV eq.

- Bilinear form of the dKP eq.: $a, b, c \in \mathbf{C}$; $(l, m, n) \in \mathbf{Z}^3$

$$(a - b)\tau_{l+1, m+1, n} \tau_{l, m, n+1} + (b - c)\tau_{l, m+1, n+1} \tau_{l+1, m, n} \\ + (c - a)\tau_{l+1, m, n+1} \tau_{l, m+1, n} = 0$$

- Reduction to dKdV eq.: $\tau_{l+1, m+1, n} = \tau_{l, m, n}$, $\sigma_n^t := \tau_{t, 0, n}$, $\delta := \frac{b - c}{a - b}$

$$(1 + \delta)\sigma_{n+1}^{t+1} \sigma_n^{t-1} = \delta \sigma_{n+1}^{t-1} \sigma_n^{t+1} + \sigma_{n+1}^t \sigma_n^t$$

$$u_n^t := \frac{\sigma_n^t \sigma_{n+1}^{t-1}}{\sigma_{n+1}^t \sigma_n^{t-1}} \rightarrow \frac{1}{u_{n+1}^{t+1}} - \frac{1}{u_n^t} + \frac{\delta}{1 + \delta} (u_n^{t+1} - u_{n+1}^t) = 0$$

Lax form

(1) Linear equation associated with dKdV eq. :

shift operator $\hat{S} : S\varphi_n^t = \varphi_{n+1}^t$, spectral parameter λ

$$\begin{cases} (\delta' u_{n-1}^{t+1} - \hat{S}^{-1}) \varphi_n^t = \lambda \varphi_n^{t+1} \\ ((u_{n-1}^t)^{-1} - \hat{S}^{-1}) \varphi_n^t = \varphi_n^{t-1} \end{cases} \quad \left(\delta' := \frac{\delta}{1+\delta} \right)$$

(2) Lax equation for the periodic boundary condition: $(u_{n+N}^t = u_n^t, \varphi_{n+N}^t = \eta \varphi_n^t)$

$$R(t-1) := \begin{pmatrix} \delta' u_1^t & & & -\eta \\ -1 & \delta' u_2^t & & \\ & \ddots & \ddots & \\ & & -1 & \delta' u_N^t \end{pmatrix}, \quad M(t) := \begin{pmatrix} (u_1^t)^{-1} & & & -\eta \\ -1 & (u_2^t)^{-1} & & \\ & \ddots & \ddots & \\ & & -1 & (u_N^t)^{-1} \end{pmatrix}$$

$$L(t) := R(t-1)M(t), \Rightarrow \underline{L(t)M(t+1) = M(t+1)L(t+1)}$$

Alternative form of the dKdV eq.

From the bilinear form: $(1 + \delta)\sigma_{n+1}^{t+1}\sigma_n^{t-1} = \delta\sigma_{n+1}^{t-1}\sigma_n^{t+1} + \sigma_{n+1}^t\sigma_n^t$

when we define: $u_n^t := \frac{\sigma_n^t\sigma_{n+1}^{t-1}}{\sigma_{n+1}^t\sigma_n^{t-1}}$, $v_n^t := \frac{\sigma_n^{t+1}\sigma_n^{t-1}}{(\sigma_n^t)^2}$

$$\Rightarrow (1 + \delta)\frac{1}{u_n^{t+1}} = \delta u_n^t + \frac{1}{v_n^t}, \quad u_n^{t+1}v_{n+1}^t = u_n^t v_n^t$$

For the boundary condition: $\lim_{n \rightarrow -\infty} u_n^t = \lim_{n \rightarrow -\infty} v_n^t = 1$

$$\frac{1}{v_n^t} = \frac{u_{n-1}^{t+1}}{u_{n-1}^t} \frac{1}{v_{n-1}^t} \Rightarrow \underline{(1 + \delta)\frac{1}{u_n^{t+1}} = \delta u_n^t + \prod_{k=-\infty}^{n-1} \frac{u_k^{t+1}}{u_k^t}}$$

Soliton solutions

N soliton solution: (determined by $2N$ parameters $\{p_i\}_{i=1}^n, \{\gamma_i\}_{i=1}^n$)

$$\sigma_n^t(N) = \sum_{J \subset [N]} \left(\prod_{i \in J} (-\gamma_i) \left(\frac{1-p_i}{p_i} \right)^t \left(\frac{p_i + \delta}{1 + \delta - p_i} \right)^n \right) \frac{\prod_{i>j, i, j \in J} (p_i - p_j)^2}{\prod_{i, j \in J} (1 - p_i - p_j)}$$

$(p_i \neq p_j (i \neq j), [N] := \{1, 2, \dots, N\})$

Ex.) $\sigma_n^t(1) = 1 + (-\gamma) \left(\frac{1-p}{p} \right)^t \left(\frac{p + \delta}{1 + \delta - p} \right)^n$

$$\begin{aligned} \sigma_n^t(2) = & 1 + (-\gamma_1) \left(\frac{1-p_1}{p_1} \right)^t \left(\frac{p_1 + \delta}{1 + \delta - p_1} \right)^n + (-\gamma_2) \left(\frac{1-p_2}{p_2} \right)^t \left(\frac{p_2 + \delta}{1 + \delta - p_2} \right)^n \\ & + \frac{(\gamma_1 \gamma_2) (p_1 - p_2)^2}{(1 - 2p_1)(1 - 2p_2)(1 - p_1 - p_2)^2} \left(\frac{(1-p_1)(1-p_2)}{p_1 p_2} \right)^t \left(\frac{(p_1 + \delta)(p_2 + \delta)}{(1 + \delta - p_1)(1 + \delta - p_2)} \right)^n \end{aligned}$$

Conserved quantities (periodic case)

- Using Lax form: (Rem.) $L(t) = L(t; \eta)$

$$M(t+1)L(t+1) = L(t)M(t+1) \Rightarrow L(t+1) = M^{-1}(t+1)L(t)M(t+1)$$

$\therefore f(\lambda, \eta) := \det(\lambda I - L(t))$ does not depend on time step t .

\therefore Each coefficient of the monomimal in $f(\lambda, \eta)$ is conserved.

Ex.) *period* = 3

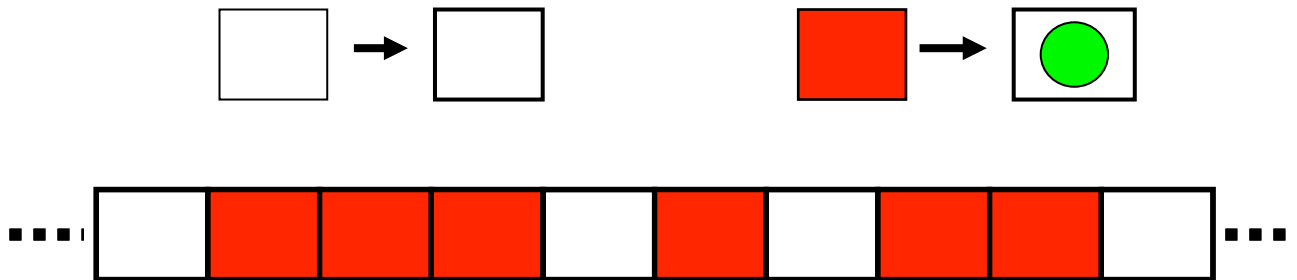
$$L(t) = \begin{pmatrix} \delta' & \eta & -\eta X_3 \\ -X_1 & \delta' & \eta \\ 1 & -X_2 & \delta' \end{pmatrix}, \quad X_i := \frac{1}{u_i^t} + \delta' u_{i+1}^t$$

$$f(\lambda, \eta) = (\delta' - \lambda)^3 + \eta^3 - X_1 X_2 X_3 \eta + (\delta' - \lambda)(X_1 + X_2 + X_3)$$

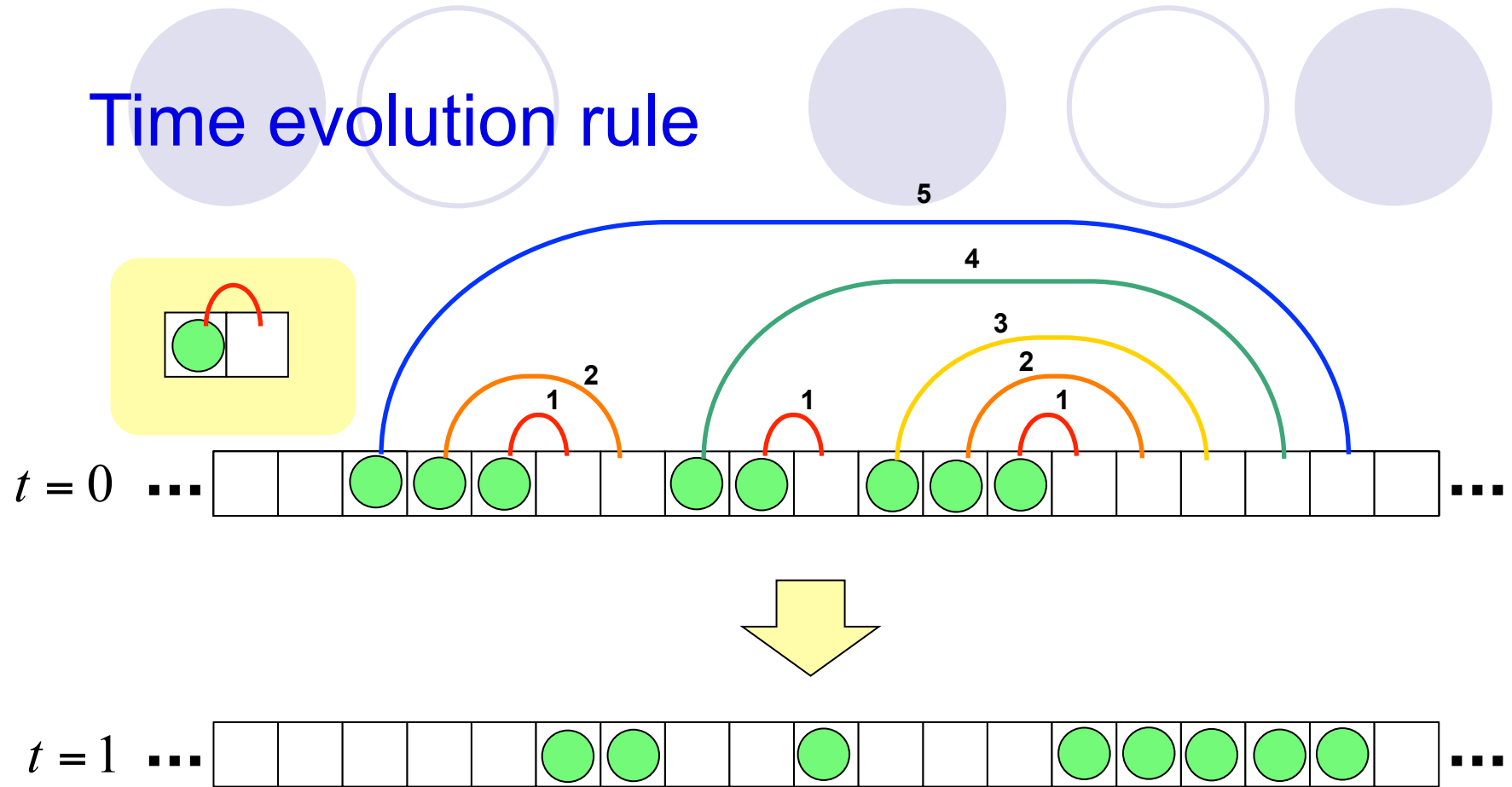
$\therefore X_1 X_2 X_3$ and $X_1 + X_2 + X_3$ are conserved.

Box-ball system (BBS)

- BBS is a reinterpretation of a 2 states soliton CA proposed by Takahashi-Satsuma (1990).
- The number of active cells does not change in the updating process.



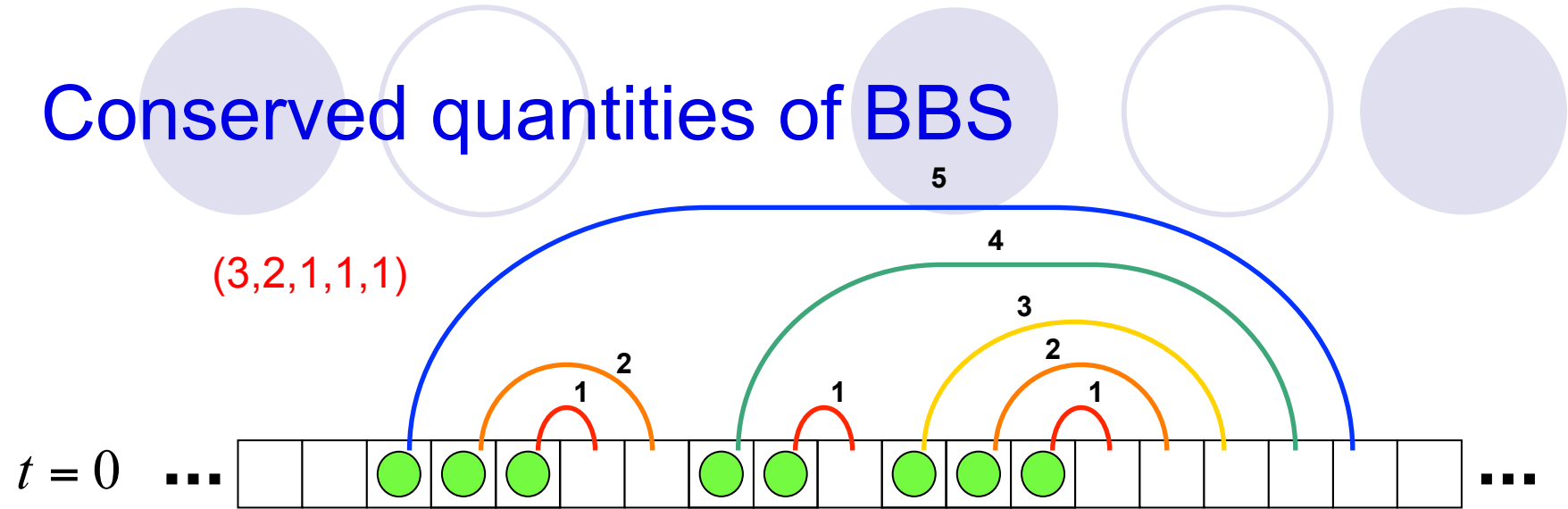
Time evolution rule



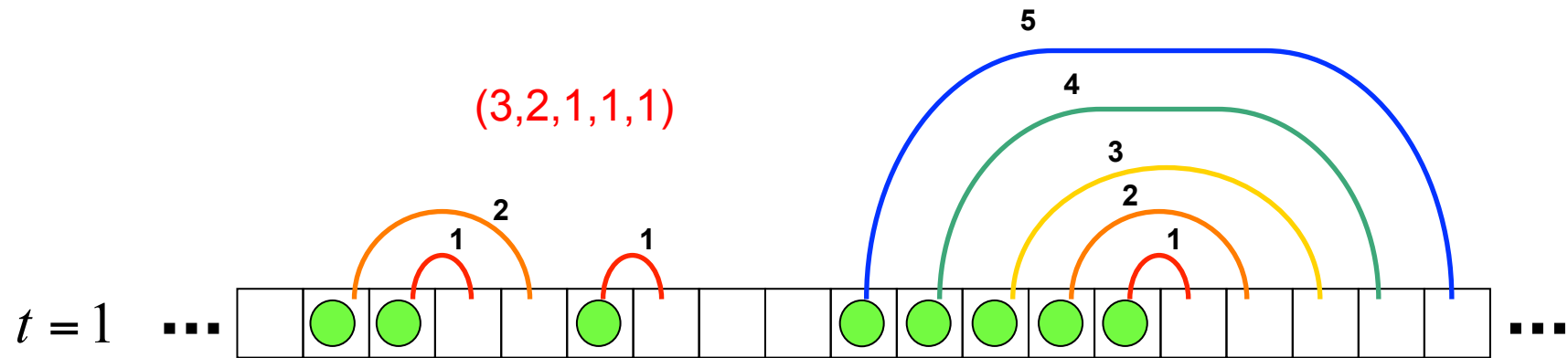
$$U_n^t := (\# \text{ball in } n\text{-box at } t) \in \{0, 1\}$$

$$\lim_{n \rightarrow -\infty} U_n^t = 0 \Rightarrow U_n^{t+1} = \min \left[1 - U_n^t, \sum_{k=-\infty}^{n-1} (U_k^t - U_k^{t+1}) \right]$$

Conserved quantities of BBS



p_j : #lines with index $j \Rightarrow (p_1, p_2, p_3, p_4, \dots)$ is conserved



Ultradiscrete KdV equation

$$\lim_{n \rightarrow -\infty} u_n^t = 1 \Rightarrow (1 + \delta) \frac{1}{u_n^{t+1}} = \delta u_n^t + \prod_{k=-\infty}^{n-1} \frac{u_k^{t+1}}{u_k^t} \quad \dots(1) \quad (\text{dKdV eq.})$$

$$\lim_{n \rightarrow -\infty} U_n^t = 0 \Rightarrow U_n^{t+1} = \min \left[1 - U_n^t, \sum_{k=-\infty}^{n-1} (U_k^t - U_k^{t+1}) \right] \quad \dots(2) \quad (\text{Eq. for BBS})$$

Simple identities:

$$a, b \in \mathbb{R}; \quad \lim_{\varepsilon \rightarrow +0} -\varepsilon \log \left[e^{-a/\varepsilon} \cdot e^{-b/\varepsilon} \right] = a + b,$$

$$\lim_{\varepsilon \rightarrow +0} -\varepsilon \log \left[e^{-a/\varepsilon} + e^{-b/\varepsilon} \right] = \min[a, b]$$

$$\therefore \delta = e^{-1/\varepsilon}, \quad u_n^t \simeq e^{U_n^t/\varepsilon} \Rightarrow \text{Eq.(1) turns to eq.(2).}$$

Eq.(2) is called the ultradiscrete KdV equation.

From KdV eq. to udKdV eq.(summary)

(1) KdV eq. : $w_t + 6ww_x + w_{xxx} = 0$

Discretization of independent variables

(2) dKdV eq.: $\frac{1}{u_{n+1}^{t+1}} - \frac{1}{u_n^t} + \delta (u_n^{t+1} - u_{n+1}^t) = 0$

$\lim_{n \rightarrow -\infty} u_n^t = 1$

$$(1 + \delta) \frac{1}{u_n^{t+1}} = \delta u_n^t + \prod_{k=-\infty}^{n-1} \frac{u_k^{t+1}}{u_k^t}$$

$\delta = e^{-1/\varepsilon},$

$\lim_{\varepsilon \rightarrow +0} \varepsilon \log u_n^t = U_n^t$

Discretization of dependent variables

(3) udKdV eq. : $U_n^{t+1} = \min \left(1 - U_n^t, \sum_{k=-\infty}^{n-1} (U_k^t - U_k^{t+1}) \right)$



Facts on the BBS

- Any state of the BBS is obtained through ultradiscretization from a soliton solution of the dKdV equation.
- Conserved quantities of the BBS are constructed from those of the dKdV eq..
- Initial value problem is solved by various methods.(cf. talk by Iwao).
- BBS is also obtained from integrable lattice model (Crystallization: Hikami-Inoue-Komori 1999).
- Generalisation of the BBS is widely studied—the original BBS corresponds to $A_1^{(1)}$ symmetry lattice (Fukuda et.al, Hatayama et al. 2000).
- Straight forward generalization is the BBS with larger capacity of box (Takahashi1991). BBS with carrier is also well-known (Takahashi-Matsukidira 1997)

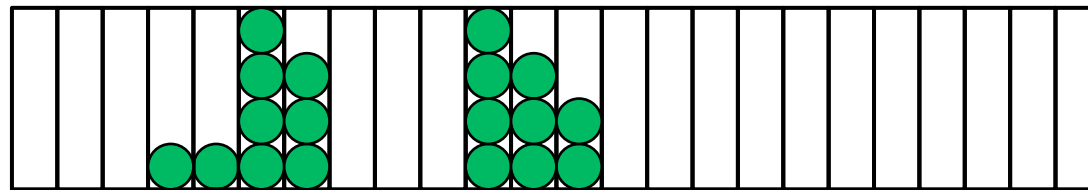
BBS with larger box capacity

Box capacity $1 \rightarrow L$; time evolution rule is essentially the same.

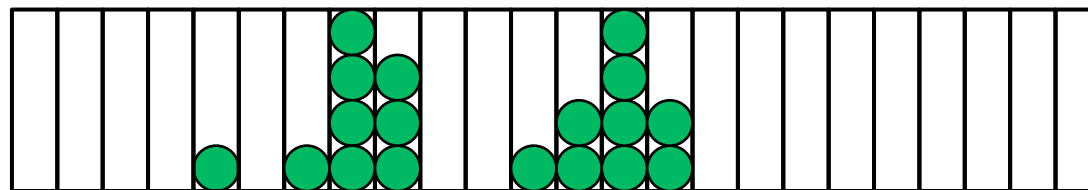
This BBS is also obtained by the udKdV eq.

$$U_n^{t+1} = \min \left(\underline{L} - U_n^t, \sum_{k=-\infty}^{n-1} (U_k^t - U_k^{t+1}) \right)$$

t=0



t=1



Negative soliton

In 2009, Hirota considered the udKdV eq. :

$$(\#) \quad U_n^{t+1} = \min \left(1 - U_n^t, \sum_{k=-\infty}^{n-1} (U_k^t - U_k^{t+1}) \right), \quad \lim_{n \rightarrow -\infty} U_n^t = 0$$

in the case $U_n^t \in \{0, \pm 1, \pm 2, \dots\}$ (instead of $U_n^t \in \{0, 1\}$).

He called **a negative soliton** for an array of negative integers, which move at speed one, and proved that it cannot be obtained by ultradiscretization of any travelling wave solution of the dKdV eq..

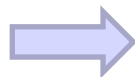
Ex.)

1 := -1, 2 := -2, etc.

$t=0$...00011110000000030000000001212000000000000...
 $t=1$...000011110000000030000000001212000000000000...
 $t=2$...00000111100000000030000000001212000000000000...

Corresponding objects in continuous systems

- Hirota proved that no travelling wave solution of the dKdV eq. gives a negative soliton state by ultradiscretization.
- In the BBS with large box capacity or 2D ultradiscrete KP eq., similar solutions were observed and considered as ‘wiggle’ in the continuous equations. (Solitons corresponds to point spectrum of linear operator, but wiggles correspond to continuous spectrum.)
- Usually the initial value problem for continuous spectrum cannot be solved, but Willox-Nakata-Grammaticos-Ramani-Satsuma have solved initial value problem for the udKdV eq. with negative solitons.



Negative solitons (background solutions) are some special solutions appeared only in ultradiscrete systems?

Preceding studies on negative solitons

- 2009~ : Hirota, Willox et al. started on the studies of the udKdV eq. over negative integer values.
- (1) Hirota gave explicit form of 2 soliton solutions of the udKdV eq. with negative solitons. (They are defined on integer points only.)
He gave the proof that these solutions cannot be constructed by ultradiscretization of the discrete KdV equation. (Hirota[A])
- (2) Solution of the initial value problem (Willox et.al.[B])
- (3) Decomposition of the state into a soliton part and a negative soliton part (Gilson et.al.[C])

A) R. Hirota: “New solutions to the Ultradiscrete Soliton Equations”, Stud. Appl. Math. 122, (2009), 361-376

B) R. Willox, Y. Nakata, J. Satsuma, A. Ramani and B. Grammaticos : “Solving the ultradiscrete KdV equation”, J. Phys. A: Math. Theor. 43, (2010), 482003(7pp)

C) C. Gilson, A. Nagai and J.C.Nimmo: private communication

Known results

	BBS : $U_n^t \in \{0,1\}$	$U_n^t \in \mathbb{Z}$
explicit solutions	✓ Arbitrary solutions	up to 2 solitons (Hirota 2009)
Conserved quantities	✓	✗
Initial value problem	✓ Many methods (geometrical, algebraic, combinatorial methods)	✓ Willox et al. 2010
Relation to integrable lattice model	✓	✗
Periodic boundary conditions	✓	✗

Problems on negative solitons

- How to construct the conserved quantities?
(cf.) For cyclic boundary, we can use Lax form.
However can one find conserved quantities similar to the BBS?)
- What is the relation to an integrable lattice model?
- What is the rigorous definition of a background solution?

t=0...0010000..
t=1...0001000..
t=2...0000100..

1-soliton

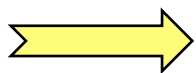
t=0...00**1**0000..
t=1...000**1**000..
t=2...0000**1**00..

negative soliton

t=0...00**111**0000..
t=1...000**111**000..
t=2...0000**111**00..

??

- How to construct explicit formulae of the background solutions and general solutions including both negative and ordinary solitons?



A simple gauge transformation will solve all these problems.

From dKdV eq. to udKdV eq. (revisit)

Bilinear form of dKdV eq.: $(1 + \delta)\sigma_{n+1}^{t+1}\sigma_n^{t-1} = \delta\sigma_{n+1}^{t-1}\sigma_n^{t+1} + \sigma_{n+1}^t\sigma_n^t$

when we define: $u_n^t := \frac{\sigma_n^t\sigma_{n+1}^{t-1}}{\sigma_{n+1}^t\sigma_n^{t-1}}$, $v_n^t := \frac{\sigma_n^{t+1}\sigma_n^{t-1}}{(\sigma_n^t)^2}$

$$\Rightarrow (1 + \delta)\frac{1}{u_n^{t+1}} = \delta u_n^t + \frac{1}{v_n^t}, \quad u_n^{t+1}v_{n+1}^t = u_n^t v_n^t$$

$$\Rightarrow (\%) \begin{cases} U_n^{t+1} = \min(1 - U_n^t, V_n^t) \\ V_{n+1}^t = U_n^t + V_n^t - U_n^{t+1} \end{cases} \quad \dots \text{udKdV eq. (new form)}$$

Note: (%) is a special case of (lattice of the) ultradiscrete version of the Yang-Baxter map (Veselov 2003, Kakei-Nimmo-Wilcox 2009)

Ultradiscrete Yang-Baxter map

Def.: ultradiscrete Yang-Baxter map lattice (cf. Kakei-Nimmo-Wilcox(2009))

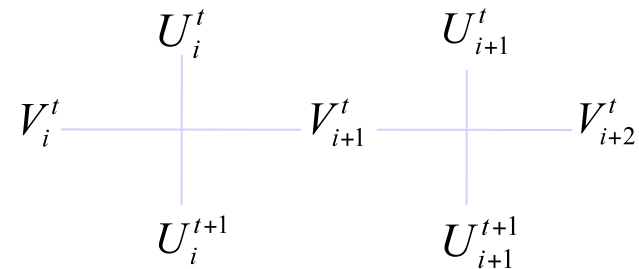
- $(\%')$
$$\begin{cases} U_n^{t+1} = U_n^t - \min(U_n^t, L - V_n^t) + \min(V_n^t, C - U_n^t) \\ V_{n+1}^t = U_n^t + V_n^t - U_n^{t+1} \end{cases}$$

where $L, C \in \mathbb{R}$ are parameters.

(1) When $L, C \in \mathbb{Z}_+$, $(\%')$ is equal to a time evolution equation of the box-ball system with carrier. [Takahashi-Matsukidaira 1997]

(2) It also equals to the action of the R-matrix of $A_1^{(1)}$ integrable lattice model in the crystal limit $q \rightarrow 0$. [Hatayama et al. 2000]

(3) When $L = 1, C \rightarrow \infty, \lim_{n \rightarrow -\infty} V_n^t = 0, \lim_{n \rightarrow -\infty} U_n^t = 0$, $(\%')$ turns into the (ordinary) BBS $(\%)$.



Gauge transformation on the udYB map

- **Key Lemma:**

For a given initial data $\{U_n^{t=0} \in \mathbb{Z}\}_{n=-\infty}^{+\infty}$, variable transformation :

$$\tilde{U}_n^t = U_n^t + m, \quad \tilde{V}_n^t = V_n^t + m, \quad m := \max_n[-U_n^0, U_n^0 - L]$$

changes the udYM map (%)' to

$$(\%)'' \begin{cases} \tilde{U}_n^{t+1} = \tilde{U}_n^t - \min(\tilde{U}_n^t, L + 2m - \tilde{V}_n^t) + \min(\tilde{V}_n^t, C + 2m - \tilde{U}_n^t) \\ \tilde{V}_{n+1}^t = \tilde{U}_n^t + \tilde{V}_n^t - \tilde{U}_n^{t+1} \end{cases}$$

In particular, if $L \in \mathbb{Z}_+$ and $\lim_{n \rightarrow -\infty} V_n^t = 0$, (%)'' is closed under

$$\tilde{U}_n^t \in \{0, 1, \dots, L + 2m\}, \quad \tilde{V}_n^t \in \{0, 1, \dots, C + 2m\}.$$

Corollary: The udKdV eq. with negative integers is equivalent to the BBS with larger box capacity and the boundary condition: $\lim_{n \rightarrow -\infty} U_n^t = m$.

Conserved quantities

- From the corollary, we found:
 - (1) 'udKdV eq. with negative integers' \equiv 'BBS with box capacity $2m+1$ '
 \equiv ' $A_1^{(1)}$ integrable lattice model'
 - (2) Extension to a periodic system (and those with other boundary conditions) is straightforward. [Kanki 2011]
- The conserved quantities of this general BBS are already known.
 - (1) Energy functions of the $A_1^{(1)}$ integrable lattice model [Fukuda-Okado-Yamada 2000]
 - (2) Path representation related to the Weyl group action [Mada-Idzumi-T 2005]
- Can we construct the conserved quantities in a similar way to that of the BBS?.

Construction of conserved quantities

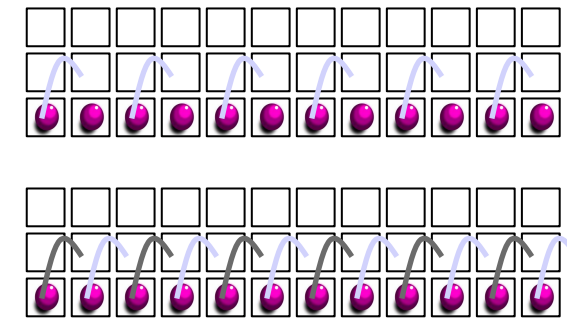
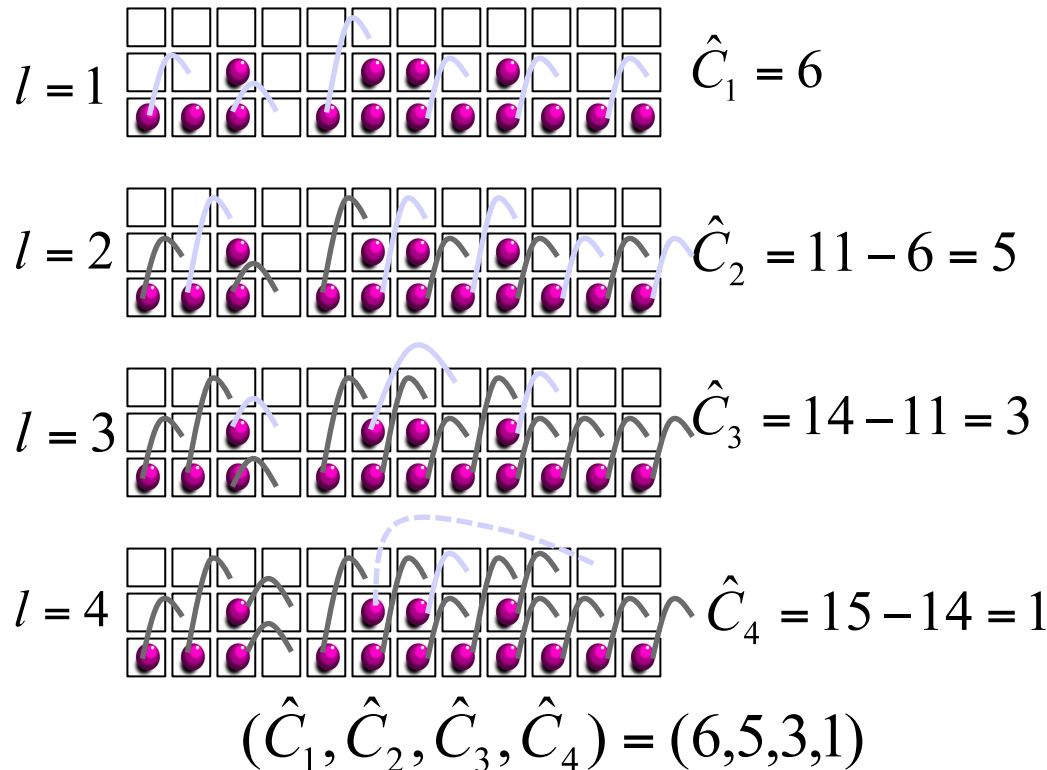
—an example of periodic case —

Ex.) initial state: 001101101000

up-shift ($m=1, L=3$): 112012212111

vacuum: 000000000000

up-shift: 111111111111

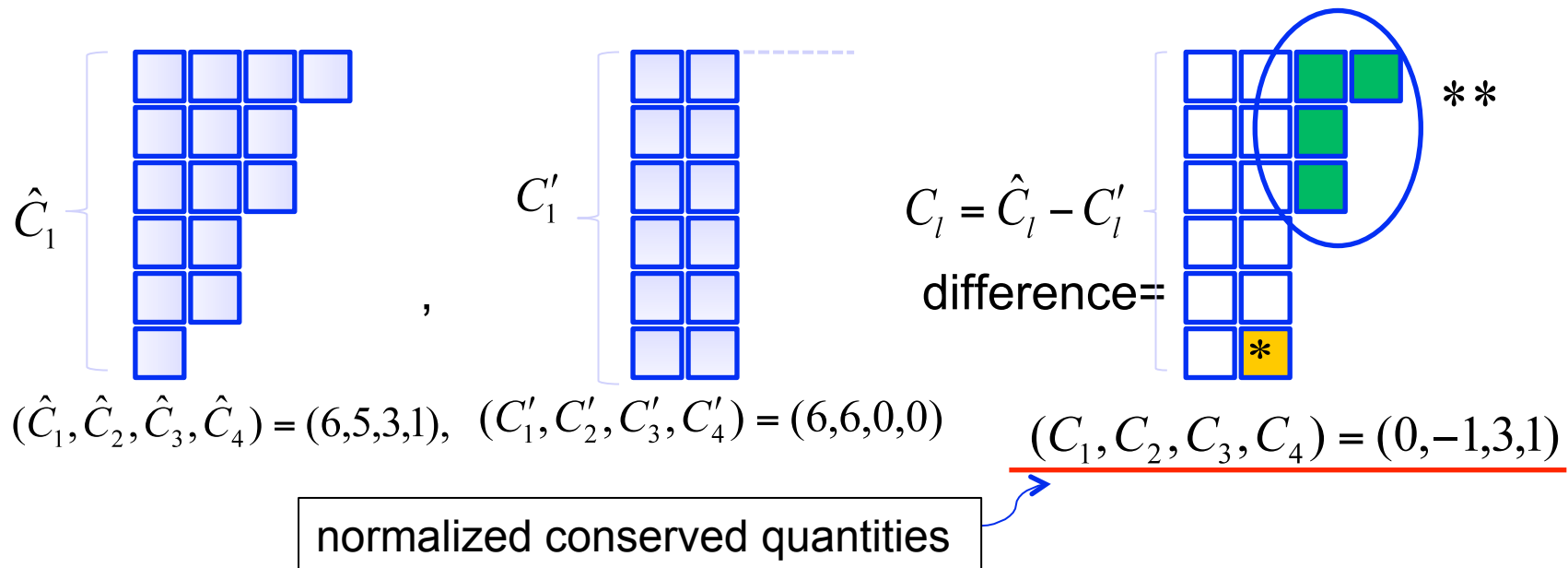


same as above ($l = 2$)

same as above ($l = 2$)

$(C'_1, C'_2, C'_3, C'_4) = (6, 6, 0, 0)$

Conserved quantities and Young diagram



Prop1: (# disappeared squares*) = (total amplitude of negative solitons)

Prop2: squares located at $l \geq 2m + 1$ $(C_{2m+1}, C_{2m+2}, \dots)$ ** constitute a Young diagram corresponding to ordinary solitons. (In this example, there are 3 solitons with amplitudes 1, 1, 2.)

Definition of a background solution

- **Def.**(background solution):

A background solution of the udKdV eq. is a solution corresponding to a state with $C_l = 0$ for $l \geq 2m + L$.

ex.) t=0 ...000**111**00000... (\leftrightarrow ...11102011111...)
t=1 ...0000**111**0000...
t=2 ...00000**111**000... $(C_1, C_2, C_3, C_4, \dots) = (0, -1, 0, 0, \dots)$
 \therefore This is a background solution.

ex.) t=0 ...000**121**00000... (\leftrightarrow ...11103011111...)
t=1 ...0000**121**0000...
t=2 ...00000**121**000... $(C_1, C_2, C_3, C_4, \dots) = (0, -1, 1, 0, \dots)$
 \therefore This is NOT a background solution.

Construction of background solutions

Hirota's argument for nonexistence of corresponding solutions to dKdV equation.

From the bilinear eq.: $(1 + \delta)\sigma_{n+1}^{t+1}\sigma_n^{t-1} = \delta\sigma_{n+1}^{t-1}\sigma_n^{t+1} + \sigma_{n+1}^t\sigma_n^t$,

we have
$$(1 + \delta)\frac{\sigma_{n+1}^{t+1}\sigma_n^{t-1}}{\sigma_{n+1}^t\sigma_n^t} = \delta\frac{\sigma_{n+1}^{t-1}\sigma_n^{t+1}}{\sigma_{n+1}^t\sigma_n^t} + 1 \quad \dots(1)$$

Since negative solitons move at speed 1, $u_n^t := \frac{\sigma_{n+1}^{t-1}\sigma_n^t}{\sigma_{n+1}^t\sigma_n^{t-1}} = f(n-t) \dots(2)$

so that we can essentially put $\sigma_n^t = \sigma(n-t)$, and from (1) and (2)

we have $1 + \delta = \delta u_n^t u_{n-1}^t + 1 \quad \dots(3).$

But, for negative solitons, $u_n^t = e^{U_n/\varepsilon} < 1$, which contradicts (3).

we have to consider solutions:

$$u_n^t(\varepsilon) \neq f_{n-t}(\varepsilon) \wedge \lim_{\varepsilon \rightarrow +0} \varepsilon \log u_n^t(\varepsilon) = U_{n-t}$$

Boundary condition changed by the gauge transformation

A background solution (1 := -1 etc.)

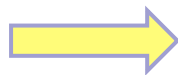
...00000000**11**00**212**0000000000...

$$\lim_{n \rightarrow -\infty} U_n^t = 0 \iff \lim_{n \rightarrow -\infty} u_n^t = 1$$

Gauge transformation (up-shift)

...22222222**1122**0**10**22222222...

$$\lim_{n \rightarrow -\infty} U_n^t = 2 \iff \lim_{n \rightarrow -\infty} u_n^t \simeq e^{2/\varepsilon}$$



We have to find a solution with different b.c.

Cf.) $\vartheta_3(v) := \sum_{n=-\infty}^{\infty} e^{n^2 i \pi \eta} e^{2 n i \pi v}$ ($v = \eta(t - n)$) gives a solution to dKdV eq. with $\lim_{n \rightarrow \pm\infty} u_n^t = a (\neq 1)$. (We do not use this fact here.)

Construction of background solutions from soliton solutions of the discrete KdV eq.

Theorem :

Any background solution can be constructed from N soliton solutions of the

dKdV eq.:
$$\frac{1}{u_{n+1}^{t+1}} - \frac{1}{u_n^t} + \delta(u_n^{t+1} - u_{n+1}^t) = 0$$

through 'scaling' and 'ultradiscrete' limit.

Method of construction :

When we put $u_n^t = \frac{\sigma_n^t \sigma_{n-1}^{t+1}}{\sigma_n^{t+1} \sigma_{n-1}^t}$, N soliton solution of the dKdV eq. is given as

$$\sigma_n^t(N) = \sum_{J \subset [N]} \left(\prod_{i \in J} (-\gamma_i) \left(\frac{1-p_i}{p_i} \right)^t \left(\frac{p_i + \delta}{1 + \delta - p_i} \right)^n \right) \frac{\prod_{i>j, i, j \in J} (p_i - p_j)^2}{\prod_{i, j \in J} (1 - p_i - p_j)}.$$

$$(p_i \neq p_j (i \neq j), [N] = \{1, 2, \dots, N\})$$

Ultradiscrete limit of N soliton solutions

$$\rho_n^t(N) := \lim_{\varepsilon \rightarrow +0} \varepsilon \log \sigma_n^t(N) \quad (\delta = e^{-L/\varepsilon}, p_i = c_i e^{-P_i/\varepsilon}, -\gamma_i = d_i e^{\theta_i/\varepsilon})$$

$$= \max_{J \subset [N]} \left[\left(\sum_{i \in J} P_i \right) t - \left(\sum_{i \in J} \min(L, P_i) \right) n + \sum_{i \in J} \theta_i - \sum_{\substack{i \neq j \\ i, j \in J}} \min(P_i, P_j) \right] \quad \left(\begin{array}{l} P_i \in \mathbf{Z}_+, \theta_i \in \mathbf{Z} \\ c_i, d_i > 0 (c_i \neq c_j) \end{array} \right)$$

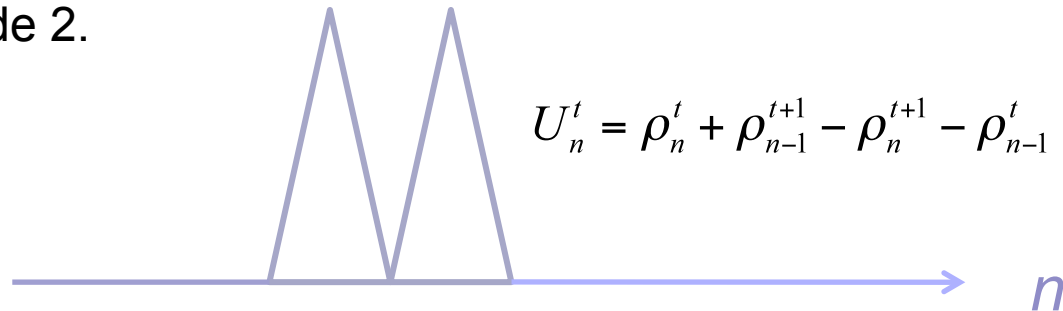
P_i : amplitude of the i th soliton, $P_i = P_j (i \neq j)$ is allowed.

θ_i : phase of the i th soliton

Important fact: Solitons can have the same amplitudes in the udKdV eq.

Ex.) $\rho_n^t = \max[0, 2t - 2n, 4t - 4n - 6]$ gives a 2 soliton solution

both with amplitude 2.



Scaling limit and background solution

If ρ_n^t gives a solution to udKdV eq., $\tilde{\rho}_n^t := \rho_n^t + an + bt$ (a, b : const.) gives the same solution.



By subtracting a linear function $P_{x;N}$ of $x := n - t$ depending on N , the scaling limit $\rho_{bg}(x) = \lim_{N \rightarrow +\infty} (\rho_{n+2N}^t(N) - P_{x,N})$ becomes a function of x and gives the solution to udKdV.eq.

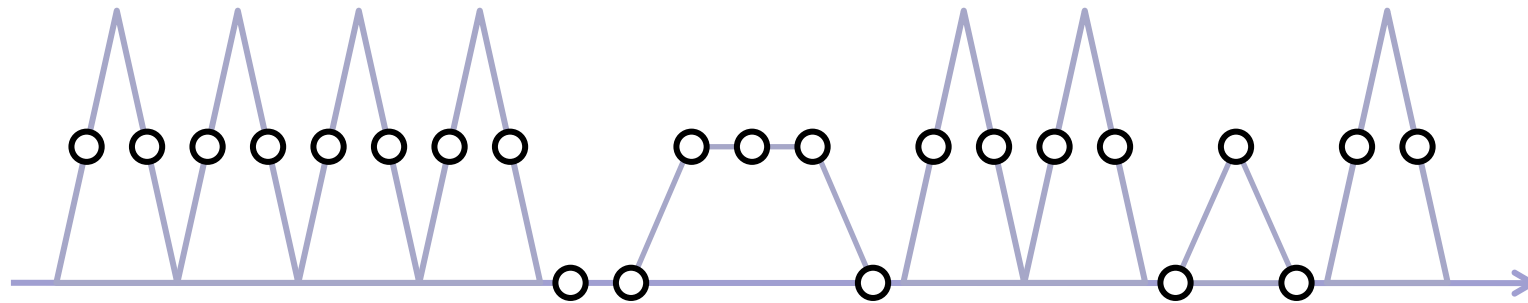
FACT: By proper choice of an N soliton solution,

$$U_{bg}(x) := \rho_{bg}(x) + \rho_{bg}(x+2) - 2\rho_{bg}(x+1)$$

gives an arbitrary background solution.

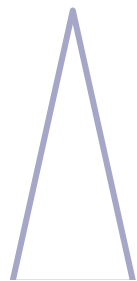
Cf.) For $m=1$, we need solitons with amplitudes only 2 and 1

Explicit form of background solutions



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soliton with amplitude 2

soliton with amplitude 1

General solutions

- **N soliton solution of the udKdV eq.:**

$$\rho_n^t = \max_{J \subset [N]} \left[\left(\sum_{i \in J} P_i \right) t - \left(\sum_{i \in J} \min(L, P_i) \right) n + \sum_{i \in J} \theta_i - \sum_{\substack{i \neq j \\ i, j \in J}} \min(P_i, P_j) \right],$$

($P_i \in \mathbb{Z}_+$: amplitude, $\theta_i \in \mathbb{Z}$: phase)

Theorem:

A solution consisting of a background solution $\tau_{bg}(x)$ and m ordinary solitons with amplitudes and phases: $(\hat{P}_i, \hat{\theta}_i), P_i \geq 3$ ($i = 1, 2, \dots, m$) is given as

$$\rho_n^t = \max_{J \subset [m]} \left[\left(\sum_{i \in J} \hat{P}_i \right) t - 3|J|n + \sum_{i \in J} \hat{\theta}_i - \sum_{\substack{i \neq j \\ i, j \in J}} \min[\hat{P}_i, \hat{P}_j] + \underline{\rho_{bg}(x - 2|J|)} \right]$$

back ground solution

A simple example

Background solution: $\cdots 111101111 \cdots$ + **1 soliton** : $\hat{P}_1 = 4, \hat{\theta}_1 = -6$

$$\begin{cases} \rho_{bg}(x) = \max_{k \in \mathbb{Z}} [2kx - |k| - 2k(k-1)] \\ \rho_n^t = \max [\rho_{bg}(x), 4t - 3n - 6 + \rho_{bg}(x-2)] \end{cases}$$

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Note: The phase shift of negative soliton is -2 when it collides with a soliton.



Concluding remarks

- **Conclusion**

We showed the following results for the udKdV eq. with negative integers.

- 1.) conserved quantities
- 2.) relation to the integrable lattice model ($A_1^{(1)}$ symmetry).
- 3.) explicit formulae for general solutions for $m=1$.

- **Future problems**

- 1.) extension to the more general cases ($A_n^{(1)}$: several kinds of balls etc.)
- 2.) solving initial value problems by this approach and compare it to the previous results



Thank you for your patience