# RONNIE LEE'S GENERATOR FOR $L_5(\mathbb{Z}[\mathbb{Z}])$.

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Abstract. We present Ronnie Lee's description of a generator for $L_5(\mathbb{Z}[\mathbb{Z}])$ from the 1970s. This was previously only available as a handwritten letter which is not easily accessible.

## 1. Introduction

The purpose of this note is to provide a typed, freely available description of Ronnie Lee's generator of $L_5(\mathbb{Z}[\mathbb{Z}])$. This was originally presented in a letter [Lee70] addressed to Martin Scharlemann from some time in the 1970s. The main change since then is that we know about Freedman's disc embedding theorem, which simplifies some of the description. This note will assume that the reader understands what surgery is, and what a surgery problem is. For a description of surgery theory there are a wide variety of sources, including [Ran02] and [LM23]. For this note, we recommend [Wal70], which is what Lee cited in the original letter.

One should note the difference in notation between this note and the letter by Lee. We write $L_5(\mathbb{Z}[\mathbb{Z}])$ to mean the set of stable equivalence classes of quadratic formations over stably free $\mathbb{Z}[\mathbb{Z}]$-modules, whereas Lee denotes this as $L_5(\mathbb{Z})$.

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## 2. A generator for $L_5(\mathbb{Z}[\mathbb{Z}])$

2.1. **Constructing any generator.** The aim of this note is to give a 'nice' description for the generator of $L_5(\mathbb{Z}[\mathbb{Z}])$, but first we simply need to construct any generator. Shaneson splitting [Sha69, Theorem 5.1] gives us that

$$L_5(\mathbb{Z}[\mathbb{Z}]) \cong L_5(\mathbb{Z}) \times L_4(\mathbb{Z}).$$

Wall [Wal70, Theorem 13A.1] computes that $L_5(\mathbb{Z}) = 0$ and similarly that $L_4(\mathbb{Z}) \cong L_0(\mathbb{Z}) \cong \mathbb{Z}$, hence $L_5(\mathbb{Z}[\mathbb{Z}]) \cong \mathbb{Z}$, generated by the image of the generator of $L_4(\mathbb{Z})$ in $L_5(\mathbb{Z}[\mathbb{Z}])$. The injection $L_4(\mathbb{Z}) \hookrightarrow L_5(\mathbb{Z}[\mathbb{Z}])$ is given by taking a surgery problem and multiplying it by $S^1$. We have an isomorphism $L_4(\mathbb{Z}) \cong \mathbb{Z}\langle \sigma/8 \rangle$ where $\sigma$ denotes the signature, and so it is generated by the standard surgery problem associated to the $E_8$ manifold. Hence, $L_5(\mathbb{Z}[\mathbb{Z}])$ is generated by the induced surgery problem on $E_8 \times S^1$. We will now be more precise.

Consider the manifold $P$ formed by plumbing along the $E_8$-lattice. The created manifold $P$ is a manifold with boundary $\partial P$ a homology 3-sphere $Y$, and by Freedman [Fre82] $Y$ also bounds a contractible manifold which we will denote by $B$. In a slight abuse of notation, we then define $E_8 := P \cup_Y B$. By construction, $E_8$ has intersection form given by the $E_8$-lattice

$$\lambda_{E_8} = \begin{bmatrix} 2 & 1 & & & & & & \\ 1 & 2 & 1 & & & & & \\ & 1 & 2 & 1 & & & & \\ & & 1 & 2 & 1 & & & \\ & & & 1 & 2 & 1 & & 1 \\ & & & & 1 & 2 & 1 & \\ & & & & & 1 & 2 & 1 \\ & & & & 1 & & 1 & 2 \end{bmatrix} \tag{2.1}$$

and one can diagonalise this matrix to read off its signature (the computation can be found in full detail in [Sco05, Section 3.2]). This gives $\sigma(E_8) = 8$. This means that by Rokhlin's theorem [Rok52] this manifold is non-smoothable. By Wall, this means that the surgery problem corresponding to the standard degree-1 normal map

$$\varphi' \colon E_8 \to S^4$$

has surgery obstruction the generator of $L_4(\mathbb{Z})$ and hence the surgery problem corresponding to the map

$$\varphi := \varphi \times \mathrm{Id}_{S^1} \colon E_8 \times S^1 \to S^4 \times S^1$$

has surgery obstruction the generator of $L_5(\mathbb{Z}[\mathbb{Z}])$.

Now that we have constructed a surgery problem with the required generator as its obstruction, the aim is to find an algebraic description for this element in $L_5(\mathbb{Z}[\mathbb{Z}])$. For this, we follow the method given in [Wal70, §6]. It is not hard to see that our map $\varphi$ is already 2-connected, since it is clearly an isomorphism on $\pi_1$, and $\pi_2(S^4 \times S^1) = 0$. The surgery kernel is therefore $\pi_3(\varphi) \cong H_2(E_8 \times S^1; \Lambda)$, where $\Lambda := \mathbb{Z}[\pi_1(E_8 \times S^1)] = \mathbb{Z}[\mathbb{Z}] = \mathbb{Z}[T^1, T^{-1}]$. Choose a $\Lambda$-basis for this, represented by eight disjoint embeddings $h_i \colon S^2 \times D^3 \hookrightarrow E_8 \times S^1$ corresponding to the basis given for the intersection form in (2.1), and let $U$ denote the union of all of these embeddings $U := \cup_i h_i(S^2 \times D^3)$.

We have now split our manifold $E_8 \times S^1$ into two pieces: $\overline{E_8 \times S^1 \setminus U}$ and $U$. Furthermore, after cellular approximation we can assume that our map $\varphi$ takes the form of a map of triads

$$\varphi : (E_8 \times S^1; (\overline{E_8 \times S^1 \setminus U}), U) \to (S^4 \times S^1; (\overline{S^4 \times S^1 \setminus D^5}), D^5).$$

An element of $L_5(\mathbb{Z}[\mathbb{Z}])$ is a formation. Every formation is equivalent to a formation of the form $(H_\varepsilon(F); F, G)$, but we can see this explicitly in our case. Since $\partial U \cong \sqcup_i S^2 \times S^2$, a disjoint union of embedded copies of $S^2 \times S^2$, $H_2(\partial U; \Lambda)$ is already the standard hyperbolic form over $\Lambda$ with sixteen generators, where $e_i$ corresponds to the $i$th copy of $S^2 \times \{\mathrm{pt}\}$ and $f_i$ corresponds to the $i$th copy of $\{\mathrm{pt}\} \times S^2$.

More specifically, the formation corresponding to this surgery problem is given by $(H; F, G)$ where

$$\begin{aligned} H &:= \ker\big((\varphi\,|_{\partial U})_* \colon H_2(\partial U; \Lambda) \to H_2(\partial D^5; \varphi_*\Lambda)\big),\\ F &:= \ker\big((\varphi\,|_{U})_* \colon H_3(U, \partial U; \Lambda) \to H_3(D^5, \partial D^5; \varphi_*\Lambda)\big),\\ G &:= \ker\big((\varphi\,|_{\overline{E_8 \times S^1 \setminus U}})_* \colon H_3(\overline{E_8 \times S^1 \setminus U}, \partial U; \Lambda) \to H_3(\overline{S^4 \times S^1 \setminus D^5}, \partial D^5; \varphi_*\Lambda)\big). \end{aligned}$$

All of the restriction maps that we take above are the zero maps on their respective homology groups because their targets vanish, hence $H \cong H_2(\partial U; \Lambda)$, $F \cong H_3(U, \partial U; \Lambda)$ and $G \cong H_3(\overline{E_8 \times S^1 \setminus U}, \partial U; \Lambda)$. It is not too hard to see that $F$ is isomorphic to the standard Lagrangian of $H$, generated by the $f_i$ basis elements. Thus, all of the information about the formation as an element in $L_5(\mathbb{Z}[\mathbb{Z}])$ is contained in $G$. We shall describe $G$ in terms of a basis $e'_i$ which can be viewed as the upper half of a $16 \times 16$ matrix. For demonstrative purposes, we can describe $F$ in the same way as the bottom half of the same matrix easily as $f'_i := f_i$ which corresponds to the matrix

$$\begin{bmatrix} \mathbf{0} & \mathrm{Id} \end{bmatrix}$$

where Id denotes the $8 \times 8$ identity matrix.

**Lemma 2.1.** *The matrix corresponding to $G$ by describing a basis $\{e'_i\}$ for $G$ in terms of the elements $e_i$ and $f_i$ is given below (the additional horizontal and vertical rules have been added for readability).*

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $T-1$ | | | | | | | | $T+1$ | | | $1$ | | | | |
| | $T-1$ | | | | | | | | $T+1$ | $1$ | | | | | |
| | | $T-1$ | | | | | | | $T$ | $T+1$ | $1$ | | | | |
| | | | $T-1$ | | | | | $T$ | | $T$ | $T+1$ | $1$ | | | |
| | | | | $T-1$ | | | | | | | $T$ | $T+1$ | $1$ | | |
| | | | | | $T-1$ | | | | | | | $T$ | $T+1$ | $1$ | |
| | | | | | | $T-1$ | | | | | | | $T$ | $T+1$ | $1$ |
| | | | | | | | $T-1$ | | | | | | | $T$ | $T+1$ |

*i.e.* $e'_1 = (T-1)e_1 + (T+1)f_1 + f_4, e'_2 = (T-1)e_2 + (T+1)f_2 + f_3$ *etc.*

*Remark* 2.2. By construction (which will be seen below), the form should die on the $e'_i$, but perhaps it is helpful to explicitly see this. Let $\lambda$ denote the standard $\mathbb{Z}[\mathbb{Z}]$-valued intersection form on $H$. Then

$$\lambda(e'_1, e'_1) = \lambda((T-1)e_1, (T+1)f_1) + \lambda((T+1)f_1, (T-1)e_1) = (T - T^{-1}) + (T^{-1} - T) = 0,$$
$$\lambda(e'_1, e'_4) = \lambda((T-1)e_1, Tf_1) + \lambda(f_4, (T-1)e_4) = (T-1)T^{-1} + T^{-1} - 1 = 0,$$

and all other cases are analogous.

*Proof of Lemma 2.1.* Since we are describing a basis for $G = H_3(\overline{E_8 \times S^1 \setminus U}, \partial U; \Lambda)$ we will work in the universal cover of $E_8 \times S^1$ which is $E_8 \times \mathbb{R}$. First arrange the embeddings $h_i \colon S^2 \times D^3 \hookrightarrow E_8 \times \mathbb{R}$ such that each $h_i(S^2 \times \{\mathrm{pt}\})$ lies in the slice $E_8 \times \{\frac{i-1}{8}\}$ and then write $e_i = h_i(S^2 \times \{\mathrm{pt}\})$ and similarly write $f_i = h_i(\{\mathrm{pt}\} \times S^2)$. Furthermore, we may assume that the projections $p_i$ of $e_i$ onto the $E_8$-factors give a basis for $H_2(E_8)$ corresponding to the basis used for (2.1). We will now describe eight distinct elements in $G$.

Let $A_i := p_i \times [\frac{i-1}{8}, 1 + \frac{i-1}{8}]$ be annuli in $E_8 \times \mathbb{R}$ and note these have $\partial A_i = (T-1)e_i$. The annuli $A_i$ are not disjoint from $U$, but $A_i \cap U$ consists of disjoint 3-balls corresponding to the intersection form given in (2.1). However, note that because of how we chose to make the $e_i$ disjoint in $E_8 \times \mathbb{R}$, if $p_i$ and $p_j$ have non-trivial intersection then $A_i$ intersects $U$ at $e_j$ for $j > i$, at $Te_j$ for $j < i$, and twice at $e_j$ and $Te_j$ for $j = i$. Remove all of these 3-balls from $A_i$ to form the element $A_i'$ which picks up extra boundary components as the boundaries of the removed 3-balls, which can be seen by taking the duals $f_j$ for every $e_j$ that appeared above. We now see that $\partial A_i' = e_i'$ as defined by the matrix in the statement of the lemma.

It remains to be seen that these eight elements generate the whole of $G$. Consider the following diagram, made out of the long exact sequence of the triple $(E_8 \times S^1, \overline{E_8 \times S^1 \setminus U}, \partial U)$ and the pair $(E_8 \times S^1, \partial U)$ (with $\Lambda$-coefficients suppressed).

$$
\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \uparrow & & & & \\
 & & & & H_2(E_8 \times S^1) & & & & \\
 & & & & \uparrow & & & & \\
 & & & & H_2(\partial U) & & & & \\
 & & & & \uparrow{\scriptstyle \partial} & & & & \\
0 & \longrightarrow & H_3(\overline{E_8 \times S^1 \setminus U}, \partial U) & \longrightarrow & H_3(E_8 \times S^1, \partial U) & \longrightarrow & H_3(U, \partial U) & \longrightarrow & 0 \\
 & & & & \uparrow & & & & \\
 & & & & 0 & & & &
\end{array}
$$

Further, the horizontal short exact sequence splits via the map on homology induced by the inclusion of pairs $(U, \partial U) \hookrightarrow (E_8 \times S^1, \partial U)$. Hence $H_3(E_8 \times S^1, \partial U) \cong H_3(\overline{E_8 \times S^1 \setminus U}, \partial U) \oplus H_3(U, \partial U)$.

Let $B \in H_3(\overline{E_8 \times S^1 \setminus U}, \partial U)$. Write $\partial B = \sum_{k=1,\ldots,8} \lambda_k e_k + \mu_k f_k$. Further assume $\lambda_k = (T-1)\overline{\lambda}_k$ for all $k$ (for some $\overline{\lambda}_k \in \Lambda$), then

$$
\begin{aligned}
\partial(B - \sum_k \overline{\lambda}_k \overline{A}_k) &= \sum_k (T-1)\overline{\lambda}_k e_k + \mu_k f_k - \sum_k \overline{\lambda}_k \partial \overline{A}_k \\
&= \sum_k \overline{\mu}_k f_k
\end{aligned}
$$

for some $\overline{\mu}_k \in \Lambda$. Let $C_k$ for $k = 1, \ldots, 8$ denote the basis for $H_3(U, \partial U)$ such that $\partial C_k = f_k$. Then

$$
\partial(B - \sum_k \overline{\lambda}_k \overline{A}_k - \sum_k \overline{\mu}_k C_k) = \sum_k \overline{\mu}_k f_k - \overline{\mu}_k f_k = 0,
$$

and hence the injectivity of $\partial$ implies that $B - \sum_k \overline{\lambda}_k \overline{A}_k - \sum_k \overline{\mu}_k C_k = 0 \in H_3(E_8 \times S^1, \partial U)$. Since this group splits as a direct sum, we see that $B$ can be written as a $\Lambda$-linear combination of the $\overline{A}_i$.

We now claim that the assumption that $\lambda_k = (T-1)\overline{\lambda}_k$ holds for all $B$, which will complete the proof. Consider the following commutative diagram (where we are explicit about the coefficients).

$$
\begin{array}{ccccccc}
H_3(\overline{E_8 \times S^1 \setminus U}, \partial U; \Lambda) & \longrightarrow & H_3(E_8 \times S^1, \partial U; \Lambda) & \longrightarrow & H_2(\partial U; \Lambda) & \longrightarrow & \langle e_i \rangle_\Lambda \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H_3(\overline{E_8 \times S^1 \setminus U}, \partial U; \mathbb{Z}) & \longrightarrow & H_3(E_8 \times S^1, \partial U; \mathbb{Z}) & \longrightarrow & H_2(\partial U; \mathbb{Z}) & \longrightarrow & \langle e_i \rangle_\mathbb{Z}
\end{array}
$$

Here the vertical maps are the augmentation maps (given by setting $T = 1$). To prove the claim, it suffices to show that mapping any element $B \in H_3(\overline{E_8 \times S^1 \setminus U}, \partial U; \Lambda)$ horizontally along the diagram and vertically down to $\langle e_i \rangle_{\mathbb{Z}}$ gives the zero element. By considering the long exact sequence of the pair $(E_8 \times S^1, \partial U)$ with $\mathbb{Z}$-coefficients, one can see that the map $H_3(E_8 \times S^1, \partial U; \mathbb{Z}) \to H_2(\partial U; \mathbb{Z})$ only hits the subgroup generated by the $f_i$, and hence the composition of the final two lower horizontal maps in the diagram is the zero map. This completes the proof of the claim, and hence the proof of the whole lemma. □

2.2. **Constructing the specific generator.** The aim of this subsection is to take the generator for $L_5(\mathbb{Z}[\mathbb{Z}])$ that we constructed in the last section and show that it can be realised algebraically by a much smaller matrix. To do this, we will perform a sequence of row and column operations on the matrix from Lemma 2.1.

Let $\Sigma_i = \frac{1-T^{i+1}}{1-T} = 1 + T + \cdots + T^i$ and let $T' := T - 1$. We then perform the following sequence of row and column operations on the $8 \times 16$ matrix from Lemma 2.1, though we have given the initial matrix again below. Naturally, empty spaces denote zeroes but we have particularly noted zeroes when they have first appeared from the previous stage of row or column operations. We will sometimes use $\star$ to denote an entry that whose value is too lengthy to succinctly state but whose precise value is not important to the calculation. We also give the row and column operations used to go between each step. For example, the notation

$$R_n \to R_n + \Sigma_k R_{n-1}$$

means add $\Sigma_k$ times the $(n-1)$th row to the $n$th row. Similarly, we denote the $n$th column by $C_n$.

We now begin the operations.

$$\left(\begin{array}{cccc|cccc||cccc|cccc}
T' & & & & & & & & T{+}1 & & & 1 & & & & \\
 & T' & & & & & & & & T{+}1 & 1 & & & & & \\
 & & T' & & & & & & & T & T{+}1 & 1 & & & & \\
 & & & T' & & & & & T & & T & T{+}1 & 1 & & & \\
\hline
 & & & & T' & & & & & & & T & T{+}1 & 1 & & \\
 & & & & & T' & & & & & & & T & T{+}1 & 1 & \\
 & & & & & & T' & & & & & & & T & T{+}1 & 1 \\
 & & & & & & & T' & & & & & & & T & T{+}1
\end{array}\right)$$

$$R_8 \to R_8 - \Sigma_1 R_7$$

$$\left(\begin{array}{cccc|cccc||cccc|cccc}
T' & & & & & & & & T{+}1 & & & 1 & & & & \\
 & T' & & & & & & & & T{+}1 & 1 & & & & & \\
 & & T' & & & & & & & T & T{+}1 & 1 & & & & \\
 & & & T' & & & & & T & & T & T{+}1 & 1 & & & \\
\hline
 & & & & T' & & & & & & & T & T{+}1 & 1 & & \\
 & & & & & T' & & & & & & & T & T{+}1 & 1 & \\
 & & & & & & T' & & & & & & & T & T{+}1 & 1 \\
 & & & & & & -\Sigma_1 T' & T' & & & & & & -T\Sigma_1 & -\Sigma_2 & 0
\end{array}\right)$$

$$R_8 \to R_8 + \Sigma_2 R_6 - \Sigma_3 R_5 + \Sigma_4 R_4 - \Sigma_5 R_3 + \Sigma_6 R_2$$

$$\left(\begin{array}{cccc|cccc||cccc|cccc}
T' & & & & & & & & T{+}1 & & & 1 & & & & \\
 & T' & & & & & & & & T{+}1 & 1 & & & & & \\
 & & T' & & & & & & & T & T{+}1 & 1 & & & & \\
 & & & T' & & & & & T & & T & T{+}1 & 1 & & & \\
\hline
 & & & & T' & & & & & & & T & T{+}1 & 1 & & \\
 & & & & & T' & & & & & & & T & T{+}1 & 1 & \\
 & & & & & & T' & & & & & & & T & T{+}1 & 1 \\
 & \Sigma_6 T' & -\Sigma_5 T' & \Sigma_4 T' & -\Sigma_3 T' & \Sigma_2 T' & -\Sigma_1 T' & T' & T\Sigma_4 & \Sigma_7 & 0 & 0 & 0 & 0 & 0 &
\end{array}\right)$$

$$R_3 \to R_3 - \Sigma_1 R_2$$

$$\begin{array}{|cccc|cccc||cccc|cccc|}
\hline
T' & & & & & & & & T+1 & & & 1 & & & & \\
 & T' & & & & & & & & T+1 & 1 & & & & & \\
 & -\Sigma_1 T' & T' & & & & & & & \Sigma_2 & 0 & 1 & & & & \\
 & & & T' & & & & & T & & T & T+1 & 1 & & & \\
\hline
 & & & & T' & & & & & & & T & T+1 & 1 & & \\
 & & & & & T' & & & & & & & T & T+1 & 1 & \\
 & & & & & & T' & & & & & & & T & T+1 & 1 \\
 & \Sigma_6 T' & -\Sigma_5 T' & \Sigma_4 T' & -\Sigma_3 T' & \Sigma_2 T' & -\Sigma_1 T' & T' & T\Sigma_4 & \Sigma_7 & & & & & & \\
\hline
\end{array}$$

$$R_1 \to R_1 - R_3$$

$$\begin{array}{|cccc|cccc||cccc|cccc|}
\hline
T' & \Sigma_1 T' & -T' & & & & & & T+1 & \Sigma_2 & & 0 & & & & \\
 & T' & & & & & & & & T+1 & 1 & & & & & \\
 & -\Sigma_1 T' & T' & & & & & & & -\Sigma_2 & & 1 & & & & \\
 & & & T' & & & & & T & & T & T+1 & 1 & & & \\
\hline
 & & & & T' & & & & & & & T & T+1 & 1 & & \\
 & & & & & T' & & & & & & & T & T+1 & 1 & \\
 & & & & & & T' & & & & & & & T & T+1 & 1 \\
 & \Sigma_6 T' & -\Sigma_5 T' & \Sigma_4 T' & -\Sigma_3 T' & \Sigma_2 T' & -\Sigma_1 T' & T' & T\Sigma_4 & \Sigma_7 & & & & & & \\
\hline
\end{array}$$

$$C_1 \to C_1 + T^{-1} C_2$$

$$C_{10} \to C_{10} - T C_9$$

$$\begin{array}{|cccc|cccc||cccc|cccc|}
\hline
T'(2+T^{-1}) & \Sigma_1 T' & -T' & & & & & & T+1 & 1 & & & & & & \\
T'T^{-1} & T' & & & & & & & & T+1 & 1 & & & & & \\
-T'T^{-1}\Sigma_1 & -\Sigma_1 T' & T' & & & & & & & -\Sigma_2 & & 1 & & & & \\
 & & & T' & & & & & T & -T^2 & T & T+1 & 1 & & & \\
\hline
 & & & & T' & & & & & & & T & T+1 & 1 & & \\
 & & & & & T' & & & & & & & T & T+1 & 1 & \\
 & & & & & & T' & & & & & & & T & T+1 & 1 \\
T'T^{-1}\Sigma_6 & \Sigma_6 T' & -\Sigma_5 T' & \Sigma_4 T' & -\Sigma_3 T' & \Sigma_2 T' & -\Sigma_1 T' & T' & T\Sigma_4 & \Sigma_7 - T^2\Sigma_4 & & & & & & \\
\hline
\end{array}$$

$$R_8 \to R_8 - (\Sigma_7 - T^2\Sigma_4) R_1$$

$$\begin{array}{|cccc|cccc||cccc|cccc|}
\hline
T'(2+T^{-1}) & \Sigma_1 T' & -T' & & & & & & T+1 & 1 & & & & & & \\
T'T^{-1} & T' & & & & & & & & T+1 & 1 & & & & & \\
-T^{-1}\Sigma_1 & -\Sigma_1 T' & T' & & & & & & & -\Sigma_2 & & 1 & & & & \\
 & & & T' & & & & & T & -T^2 & T & T+1 & 1 & & & \\
\hline
 & & & & T' & & & & & & & T & T+1 & 1 & & \\
 & & & & & T' & & & & & & & T & T+1 & 1 & \\
 & & & & & & T' & & & & & & & T & T+1 & 1 \\
\beta(T) & \star & \star & \Sigma_4 T' & -\Sigma_3 T' & \Sigma_2 T' & -\Sigma_1 T' & T' & \alpha(T) & 0 & & & & & & \\
\hline
\end{array}$$

$$\begin{array}{|cccc|cccc||cccc|cccc|}
\hline
0 & 0 & 0 & & & & & & 0 & 1 & & & & & & \\
0 & 0 & & & & & & & & 0 & 1 & & & & & \\
0 & 0 & 0 & & & & & & & 0 & & 1 & & & & \\
 & & & 0 & & & & & 0 & 0 & 0 & 0 & 1 & & & \\
\hline
 & & & & 0 & & & & & & & 0 & 0 & 1 & & \\
 & & & & & 0 & & & & & & & 0 & 0 & 1 & \\
 & & & & & & 0 & & & & & & & 0 & 0 & 1 \\
\beta(T) & \star & \star & \star & \star & \star & \star & \star & \alpha(T) & & & & & & & \\
\hline
\end{array}$$

This concludes the matrix operations. The polynomials given in the last two matrices are defined as[1]

$$\alpha(T) = -1 - T + T^3 + T^4 + T^5 - T^7 - T^8,$$
$$\beta(T) = (T-1)(-2 - T + T^2 + T^3 + T^4 + T^5 - T^6 - 2T^7).$$

We conclude that we can represent the generator of $L_5(\mathbb{Z}[\mathbb{Z}])$ by a 2-dimensional form $H$ over $\Lambda$, and a pair of lagrangians $F'$ and $G'$ given by the matrix

$$M := \begin{array}{|c||c|}\hline \alpha(T) & \beta(T) \\ \hline 0 & 1 \\ \hline \end{array}.$$

Note that although we have used a matrix to encode the information about the lagrangians, this matrix does not correspond to the automorphism of the form which sends $F'$ to $G'$. By Wall [Wal70, Corollary 5.3.1] we know that such an automorphism exists, and we can write it as the following matrix

$$M' := \begin{bmatrix} \gamma(T) & \alpha(T) \\ \delta(T) & \beta(T) \end{bmatrix}.$$

where we know the $\alpha(T)$ and $\beta(T)$ are as above, but $\gamma(T)$ and $\delta(T)$ are unknown. We can, however, say something about the augmentation of this automorphism.

**Lemma 2.3.** *The matrix $M'$ augments to the matrix*

$$M'(1) := \begin{bmatrix} \gamma(1) & \alpha(1) \\ \delta(1) & \beta(1) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

*Proof.* First note that $\alpha(1) = -1$ and $\beta(1) = 0$. Now, since the matrix $M'(1)$ must represent an automorphism of the $\mathbb{Z}$ valued hyperbolic form, the values of $\gamma(1)$ and $\delta(1)$ are already determined. A simple calculation shows that these values are $\gamma(1) = 0$ and $\delta(1) = -1$. □

Presumably it is also possible to compute the exact Laurent polynomials $\gamma(T)$ and $\delta(T)$, but we have not attempted to do so.

2.3. **Interpretation.** Wall realisation [Wal70, Theorem 6.5] and Cappell-Shaneson [CS71, Theorem 3.1] tells us that we can represent the generator of $L_5(\mathbb{Z}[\mathbb{Z}])$ in the following way. Let $W_1$ be the standard cobordism between $S^1 \times S^3$ and $(S^1 \times S^3)\#(S^2 \times S^2)$ and let $W_2$ be the reversed cobordism. Let $(W; \partial_0 W = S^1 \times S^3, \partial_1 W = S^1 \times S^3)$ be the cobordism formed by gluing these via a homeomorphism

$$\sigma \colon S^1 \times S^3 \# S^2 \times S^2 \to S^1 \times S^3 \# S^2 \times S^2$$

whose induced map on $H_2(S^1 \times S^3 \# S^2 \times S^2; \Lambda)$ is exactly the $2 \times 2$-matrix $M'$ above. Such a homeomorphism exists by Stong-Wang [SW00, Theorem 2]. Then the degree-1 normal map

$$f \colon (W; \partial_0 W, \partial_1 W) \to (S^1 \times S^3 \times I; S^1 \times S^3 \times \{0\}, S^1 \times S^3 \times \{1\})$$

has surgery obstruction $\theta(f)$ the generator of $L_5(\mathbb{Z}[\mathbb{Z}])$ defined by $M'$.

## References


[CS71] Sylvain E. Cappell and Julius L. Shaneson. On four dimensional surgery and applications. *Comment. Math. Helv.*, 46:500–528, 1971.

[Fre82] Michael H. Freedman. The topology of four-dimensional manifolds. *J. Differ. Geom.*, 17:357–453, 1982.

[Lee70] Ronnie Lee. A generator for $L_5(\mathbb{Z})$, 1970. Unpublished letter.

[LM23] Wolfgang Lück and Tibor Macko. Surgery theory: Foundations. Ongoing book project, currently available on author's website at https://www.him.uni-bonn.de/lueck/index.php (version uploaded 2nd November 2023), 2023.

[Ran02] Andrew A. Ranicki. *Algebraic and geometric surgery.* Oxford Math. Monogr. Oxford: Oxford University Press, 2002.

[Rok52] Vladimir A. Rokhlin. New results in the theory of four-dimensional manifolds. *Doklady Akad. Nauk SSSR (N.S.)*, 84:221–224, 1952.

[Sco05] Alexandru Scorpan. *The wild world of 4-manifolds.* Providence, RI: American Mathematical Society (AMS), 2005.

[Sha69] Julius L. Shaneson. Wall's surgery obstruction groups for GxZ. *Ann. Math. (2)*, 90:296–334, 1969.


[1] In the original note, Lee's $\alpha(T)$ differs from the $\alpha(T)$ here by multiplication by $-1$. As far as the author can tell, this was a sign error in the original computation in the very last step of the computation, since Lee's computation agrees with the computation here until the very last step.

[SW00] Richard Stong and Zhenghan Wang. Self-homeomorphisms of 4-manifolds with fundamental group $\mathbb{Z}$. *Topology Appl.*, 106(1):49–56, 2000.

[Wal70] C. T C. Wall. *Surgery on compact manifolds*, volume 1 of *Lond. Math. Soc. Monogr.* Academic Press, London, 1970.

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