Faà di Bruno Hopf algebras

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Bialgebra

Let $\mathbb{K}$ be a field and A a $\mathbb{K}$-vector space. We call $(A, m, i, \Delta, \epsilon)$ a biagebra if:

$\rightarrow$ (A, m, i) is an associative unital algebra where:

- m is the multiplication: $m : A \otimes A \rightarrow A$;
- i is the unit: $i : \mathbb{K} \rightarrow A$.

$\rightarrow$ (A, Δ, ϵ) is a coassociative counital coalgebra where:

- Δ is the comultiplication: $\Delta : B \rightarrow B \otimes B$;
- ϵ is the counit: $\epsilon : B \rightarrow \mathbb{K}$.

Remark

1. (A, Δ, ϵ) *coalgebra* $\xrightarrow{\text{dual}}$ $(A^*, \Delta^*, \epsilon^*)$ *algebra;*
2. *if* (A, m, i) *is* ***finitely-dimensional****,* (A, m, i) *algebra* $\xrightarrow{\text{dual}}$ (A^*, m^*, i^*) *coalgebra;* *otherwise,* (A, m, i) *algebra* $\xrightarrow{\text{restricted dual}}$ $(A^\circ, m^\circ, i^\circ)$ *coalgebra;*

Definition (Graded bialgebra)

A is graded if $A = \bigoplus_{n \geq 0} A_n$ such that A is graded both as an algebra and a coalgebra.

Definition (Connected bialgebra)

A is connected if $A_0 = \mathbb{K}$.

Hopf algebra

Definition (Hopf algebra)

A Hopf algebra $H(m, i, \Delta, \epsilon, S)$ is a structure with the following properties:

1. 1. $H(m, i, \Delta, \epsilon)$ is a bialgebra;
2. 2. antipode S : H $\rightarrow$ H such that $m(S \otimes id)\Delta = m(id \otimes S)\Delta$.

Remark

The dual of any Hopf algebra H is a Hopf algebra.

Example

The universal enveloping algebra of a Lie algebra $\mathfrak{g}$ is:

$$U(\mathfrak{g}) = \frac{T(\mathfrak{g})}{\langle x \otimes y - y \otimes x = [x,y] \rangle}$$

for $x, y \in \mathfrak{g}$.

Hopf algebra structure:

- This is an associative algebra (multiplication and unit included);
- comultiplication: $\Delta : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g}), \Delta(x) = 1 \otimes x + x \otimes 1$;
- counit: $\epsilon : U(\mathfrak{g}) \to \mathbb{K}, \epsilon(x) = 0$;
- antipode: $S : U(\mathfrak{g}) \to U(\mathfrak{g}), S(x) = -x$.

Graded and connected Hopf algebras

Let H be a Hopf algebra.

Definition (Graded Hopf algebra)

H is graded if it is graded as a bialgebra and the antipode preserves the given grading.

Definition (Connected Hopf algebra)

H is connected if $H_0 = \mathbb{K}$.

Hopf algebra structure 1

On the group G of formal exponential power series

$$f(t) = \sum_{n=1}^{\infty} \frac{f_n}{n!} t^n, f_1 > 0$$

we take the coordinate functions

$$a_n(f) := f_n = f^{(n)}(0), n \geq 1.$$

and form the associative unital algebra $\mathbb{K}[a_1, a_2, ...]$

Faà di Bruno's formula

Let f, g be two formal exponential power series and we construct their composition:

$$h^{(n)}(t) = (f \circ g)^{(n)}(t) = \sum_{k=1}^{n} \sum_{\lambda} \frac{n!}{\lambda_1!\lambda_2!...\lambda_n!} f^{(k)}(g(t))(\frac{g^{(1)}(t)}{1!})^{\lambda_1}(\frac{g^{(2)}(t)}{2!})^{\lambda_2}...(\frac{g^{(n)}(t)}{n!})^{\lambda_n},$$

where $\lambda_1 + 2\lambda_2 + ... + n\lambda_n = n$ and $\lambda_1 + ... + \lambda_n = k$.

Hopf algebra structure 2

Let $F = \mathbb{K}[a_1, a_2, ...]$ and we describe the coalgebra and antipode structures:

- commutative algebra (multiplication and unit included);
- comultiplication (not cocommutative):
  $\Delta : H \to H \otimes H, \Delta a_n = \sum_{k=1}^{n} \sum_{\lambda} \frac{n!}{\lambda_1!\lambda_2!...\lambda_n!(1!)^{\lambda_1}(2!)^{\lambda_2}...(n!)^{\lambda_n}} a_1^{\lambda_1} a_2^{\lambda_2} ... a_n^{\lambda_n} \otimes a_k;$
- counit: $\epsilon : H \to \mathbb{K}$, $\epsilon(a_n) = \begin{cases} 1, \text{if } n = 1 \\ 0 \text{ otherwise}; \end{cases}$
- antipode: $S : H \to H$

$$S(a_n) = -a_n + \sum_{j=2}^{n-1} (-1)^j \sum_{1<k_{j-1}<...<k_1,n} B_{n,k_1} ... B_{k_{j-2},k_{j-1}} a_{k_{j-1}} \text{ with } a_1 = 1.$$

Examples for comultiplication

Using the comultiplication formula, we have the following:

$$\Delta a_1 = a_1 \otimes a_1;$$
$$\Delta a_2 = a_2 \otimes a_1 + a_1^2 \otimes a_2;$$
$$\Delta a_3 = a_3 \otimes a_1 + a_1^3 \otimes a_3 + 3a_2a_1 \otimes a_2;$$
$$\Delta a_4 = a_4 \otimes a_1 + a_1^4 \otimes a_4 + 6a_2a_1^2 \otimes a_3 + (3a_2^2 + 4a_3a_1) \otimes a_2.$$

Graded dual

Definition (Primitive elements)

We say that $P(H)$ is the set of primitives if

$$P(H) = \{h \in H \text{ such that } \Delta(h) = 1 \otimes h + h \otimes 1\}.$$

Let $F' = \mathbb{K}[a'_2, a'_3, ...]$ be the graded dual of F with the following relations

$$< a'_n, a_m >= \delta_{nm},$$

for a'_n primitive elements $\in F'$ and $a_m \in F$.

Remark

F' non-commutative but cocommutative.

Explicit formula

We evaluate now

$$< a'_n, a_p a_q > \text{ and } < a'_n a'_m, a_q >;$$

$$\begin{aligned} < a'_n, a_p a_q > = < \Delta a'_n, a_p \otimes a_q > &= < a'_n \otimes 1 + 1 \otimes a'_n, a_p \otimes a_q > \\ &= < a'_n \otimes 1, a_p \otimes a_q > + < 1 \otimes a'_n, a_p \otimes a_q > \\ &= < a'_n, a_p >< 1, a_q > + < 1, a_q >< a'_n, a_p > = 0 \end{aligned}$$

$$\begin{aligned} < a'_n a'_m, a_q > = < a'_n \otimes a'_m, \Delta a_q > &= \sum_{k=1}^{q} < a'_n \otimes a'_m, B_{q,k}(1, a_2, ..., a_{q+1-k}) \otimes a_k > \\ &= \sum_{k=1}^{q} < a'_n \otimes B_{q,k}(1, a_2, ..., a_{q+1-k}) >< a'_m \otimes a_k >, \\ &= < a'_n, B_{q,m}(1, a_2, ..., a_{q+1-m}) >, \end{aligned}$$

$$< a'_n a'_m, a_q >= \begin{cases} \binom{m+n-1}{n}, \text{ if} q = m+n-1 \\ 0, \text{ otherwise.} \end{cases}$$

At the same time, Δ is a homomorphism,
$\Delta(a_q a_r) = \Delta(a_q)\Delta(a_r) = a_q a_r \otimes 1 + 1 \otimes a_q a_r + a_q \otimes a_r + a_r \otimes a_q + \ldots$, therefore

$$< a'_n a'_m, a_q a_r >= \begin{cases} 1, \text{ if } n = q \neq m = r \text{ or } n = q \neq m = r, \\ 2 \text{ if } n = r = q = r, \\ 0, \text{ otherwise.} \end{cases}$$

Final expression

Finally, by using both these expressions we get

$$a'_n a'_m = \binom{m+n-1}{n} a'_{n+m-1} + (1+\delta_{nm})(a_n a_m)'.$$

Example

1. $a'_2 a'_3 = \binom{4}{2} a'_4;$
2. $a'_3 a'_2 = \binom{4}{3} a'_4;$

$$[a'_n, a'_m] = (m-n)\frac{(n+m-1)!}{n!m!} a'_{n+m-1}.$$

Intermission: Witt algebra

By taking $b'_n = (n+1)!a'_{n+1}$, we get

$$\begin{aligned}[b'_n, b'_m] &= [(n+1)!a'_{n+1}, (m+1)!a'_{m+1})] = (n+1)!(m+1)![a'_{n+1}, a'_{m+1}] \\ &= (n+1)!(m+1)!(m-n)\frac{(n+m+1)!}{(n+1)!(m+1)!}a'_{n+m+1} \\ &= (m-n)(m+n+1)!a'_{m+n+1} = (m-n)b'_{n+m}.\end{aligned}$$

Let $\{e_n, n \in \mathbb{N} \setminus \{0\}\}$ be the basis of W_+(the Witt algebra) where $e_n = x^{n+1}\partial \in W_+$ and the bracket is

$$[e_n, e_m] = (m-n)e_{n+m},$$

for $e_m \in W_+$.

Theorem

Let $\mathbb{K}$ be a field of characteristic 0 and H a graded connected cocommutative Hopf algebra over $\mathbb{K}$, then the inclusion $P(H) \subset H$ induces an isomorphism of Hopf algebras

$$U(P(H)) \cong H.$$

$\rightarrow$ For our case:
$H = F', P(F') = W_+$ therefore we get $U(P(F')) \cong F'$, where $P(F') = W_+$.