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Verma modules & representations of Lie Algebras

Set-up

- $\mathfrak{g}$: semisimple, Lie algebra / $\mathbb{C}$
- $\mathfrak{h}$: Cartan subalgebra
- Φ : root system
- Δ : fixed choice of simple roots
- $\Phi^{\pm}$: positive/negative roots wrt Δ
- $\mathfrak{n}^{\pm}$: $\bigoplus_{\alpha \in \Phi^{\pm}} \mathfrak{g}_\alpha$ where $\mathfrak{g}_\alpha$ is the root space of α
- $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$ Borel subalgebra
- $W = \langle s_\alpha \mid \alpha \in \Phi \rangle$ Weyl group

2 decompositions

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$ Root space decomposition

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$$ Triangular decomposition

Universal enveloping algebra

functor $U : \mathrm{Lie}_k \to \mathrm{Alg}_k$ left adjoint to

$$\mathrm{Alg}_k \longrightarrow \mathrm{Lie}_k$$

$$A \longmapsto (A, [\,,\,]_A)$$

multiplication in A

where $[a,b]_A = a \cdot b - b \cdot a$

determined by universal property: $\forall A \in \mathrm{Alg}_k$

$$\mathrm{Hom}_{\mathrm{Alg}_k}(U\mathfrak{g}, A) \cong \mathrm{Hom}_{\mathrm{Lie}_k}(\mathfrak{g}, A_{\mathrm{Lie}})$$

Hopf algebra structure on $U\mathfrak{g}$:

counit: $\mathfrak{g} \to 0$ Lie homomorphism $\to$ apply U

$\varepsilon : U\mathfrak{g} \to U0 = k$ $\quad x \mapsto 0 \quad \forall x \in \mathfrak{g}$

comultiplication: $\mathfrak{g} \to \mathfrak{g} \times \mathfrak{g}$, $x \mapsto (x,x)$ Lie homomorphism

$\Delta : U\mathfrak{g} \to U(\mathfrak{g} \times \mathfrak{g}) \cong U\mathfrak{g} \otimes U\mathfrak{g}$, $\quad x \mapsto x \otimes 1 + 1 \otimes x \quad x \in \mathfrak{g}$

comultiplication is cocommutative: $\Delta = \tau \circ \Delta$ where $\tau \in \mathrm{Aut}_{\mathrm{Alg}_k}(U\mathfrak{g} \otimes U\mathfrak{g})$

$\tau(a \otimes b) = b \otimes a$

$\Delta(x) = x \otimes 1 + 1 \otimes x = \tau(x \otimes 1 + 1 \otimes x) = \tau\Delta(x)$ for $x \in \mathfrak{g}$

Coassociativity:

$$\begin{tikzcd} U\mathfrak{g} \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & U\mathfrak{g}\otimes U\mathfrak{g} \arrow[d, "\Delta\otimes id"] \\ U\mathfrak{g}\otimes U\mathfrak{g} \arrow[r, "id\otimes\Delta"] & U\mathfrak{g}\otimes U\mathfrak{g}\otimes U\mathfrak{g} \end{tikzcd}$$

Counit:

$$\begin{tikzcd} & U\mathfrak{g} \arrow[dl, "\sim"'] \arrow[dr, "\sim"] \arrow[d, "\Delta"] & \\ U\mathfrak{g}\otimes k & U\mathfrak{g}\otimes U\mathfrak{g} \arrow[l, "\varepsilon\otimes id"'] \arrow[r, "id\otimes\varepsilon"] & U\mathfrak{g}\otimes k \end{tikzcd}$$

Antipode: $\mathfrak{g} \longrightarrow \mathfrak{g}^{op}$ $\quad x \longmapsto -x^{op}$ $\quad \mathfrak{g}^{op} = \mathfrak{g}$ with $[x,y]^{op} = [y,x] = -[x,y]$

Lie homomorphism

$$\chi : U\mathfrak{g} \longrightarrow U(\mathfrak{g}^{op}) \cong U\mathfrak{g}^{op}, \quad x \longmapsto -x^{op} \quad \text{for } x \in \mathfrak{g}$$

Representations

rep of $\mathfrak{g}$: homomorphism $\mathfrak{g} \to \mathfrak{gl}(V)$ for $V \in \mathrm{Vect}_k$

↳ in Lie_k

$$\underbrace{\mathrm{Hom}_{\mathrm{Lie}_k}(\mathfrak{g}, \mathfrak{gl}(V))}_{\substack{\text{representations} \\ \text{of } \mathfrak{g}}} \cong \underbrace{\mathrm{Hom}_{\mathrm{Alg}_k}(U\mathfrak{g}, \mathrm{End}_k(V))}_{\substack{\text{representations} \\ \text{of } U\mathfrak{g}}}$$

Equivalence of categories : $\mathrm{Rep}\,\mathfrak{g} \cong \mathrm{Rep}\,U\mathfrak{g}$

$M \in \operatorname{Rep}\mathfrak{h} \rightsquigarrow \operatorname{soc} M = \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda$, $\quad M_\lambda = \{ m \in M \mid h \cdot m = \lambda(h) m \ \forall h \in \mathfrak{h}^* \}$

$V \in \operatorname{Rep} \mathfrak{g} \rightsquigarrow \operatorname{soc}(V|_{\mathfrak{h}}) = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda \hookleftarrow$ weight space if $\neq \{0\}$

$\lambda \in \mathfrak{h}^*$ weight of V

$v \in V_\lambda$: weight vectors

an arbitrary $V \in \operatorname{Rep} \mathfrak{g}$ need not have any weights

<u>Prop:</u> $V \in \mathrm{Rep}_{fin}\, \mathfrak{g}$, finite-dimensional

- $V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$

- all weights of V belong to the weight lattice

$$\Lambda = \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, \alpha \rangle \in \mathbb{Z} \ \forall \alpha \in \Phi\} = \bigoplus_{i=1}^{n} \mathbb{Z}.\lambda_i$$

fundamental weights

- if λ is a weight of V, then so are all elements of its Weyl orbit $W.\lambda$.

Highest weight representations:

Def: a non-zero $V \in \mathrm{Rep}\,\mathfrak{g}$ that is generated by $v \in V_\lambda$ with
$x \cdot v = 0 \;\; \forall x \in \mathfrak{n}_+$

v - highest weight vector, λ - highest weight

partial order on Λ: $\mu \leq \lambda \Leftrightarrow \lambda - \mu \in \bigoplus_{\alpha \in \Delta} \mathbb{Z}_+ \alpha$

Prop: Any highest weight rep. decomposes into weight spaces and $\mu \leq \lambda \;\; \forall$ weights of V.

Thm: Every irreducible fin-dim rep of $\mathfrak{g}$ is a highest weight representation.

Verma modulen:

given $\lambda \in \mathfrak{h}^*$, construct universal highest weight rep. of highest weight λ

- $h \cdot v = \lambda(h) v \quad \forall h \in \mathfrak{h}$
- $x \cdot v = 0 \quad \forall x \in \mathfrak{n}_+$

Verma module of highest weight λ:

$$M_\lambda = U\mathfrak{g} / I_\lambda \qquad I_\lambda = \langle x \in \mathfrak{n}_+, \, h - \lambda(h) \ \forall h \in \mathfrak{h} \rangle$$

Alternative construction:

relations $\cdot\ h\cdot v = \lambda(h)\,v \quad \forall h\in\mathfrak{h}$ give a rep $\mathbb{C}_\lambda$ of $\mathfrak{b} = \mathfrak{h}\oplus\mathfrak{n}_+$

$\cdot\ X\cdot v = 0 \quad \forall X\in\mathfrak{n}_+$

(and hence of $U\mathfrak{b}$)

$$\rho : \mathfrak{b} \longrightarrow \mathrm{End}(\mathbb{C}), \quad \rho(h) = \lambda(h) \ \forall h\in\mathfrak{h}, \ \rho(x) = 0 \ \forall x\in\mathfrak{n}_+$$

$$M_\lambda \cong \mathrm{Ind}_{U\mathfrak{b}}^{U\mathfrak{g}} \mathbb{C}_\lambda = U\mathfrak{g} \otimes_{U\mathfrak{b}} \mathbb{C}_\lambda$$

Universality:

$$\mathrm{Hom}_{\mathfrak{g}}(M_\lambda, V) = \mathrm{Hom}_{\mathfrak{b}}(\mathbb{C}_\lambda, V|_{\mathfrak{b}}) \cong k \quad \text{(Frobenius reciprocity)}$$

for any highest weight module V

Lem: V highest weight rep of highest weight λ

$$V \cong M_\lambda / W \quad \text{for } W \subseteq M_\lambda \text{ submodule}$$

Verma modules are weight modules

<u>Thm:</u> $M_\lambda = \bigoplus_\mu M_\lambda[\mu]$ where the $M_\lambda[\mu]$ are finite-dimensional

Structure of M_λ:

<u>Prop:</u> every $v \in M_\lambda$ can be written uniquely $v = u v_\lambda$ for $u \in U\mathfrak{n}_-$

i.e.
$$\begin{aligned} U\mathfrak{n}_- &\xrightarrow{\sim} M_\lambda \\ u &\longmapsto u v_\lambda \end{aligned}$$
is an isomorphism of vector spaces

Theorem of the highest weight:

$V_\lambda = \text{head } M_\lambda$ is the unique simple rep. of highest weight λ

Thm: V_λ is fin-dim $\iff \lambda \in \Lambda_+ = \{ \lambda \in \Lambda \mid \langle \lambda, \alpha \rangle \in \mathbb{Z}_+ \ \forall \alpha \in \Phi_+ \}$
In this case, the weights of V_λ is a union of Weyl orbits $W\lambda'$ with $\lambda' \in \Lambda_+$, $\lambda' \leq \lambda$.

Λ_+ parametrises the fin-dim. irreps of $\mathfrak{g}$ (up to iso).

$$\Lambda_+ \xrightarrow{\sim} \mathrm{Irr}_{fin}\,\mathfrak{g} \quad \text{is a bijection}$$
$$\lambda \longmapsto V_\lambda$$