Hopf Algebras & Feynman Graphs

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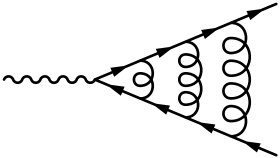
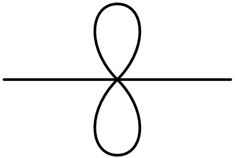
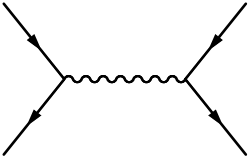






PART I: COMBINATORICS

- Rooted Trees
- Connes–Kreimer Hopf Algebra of Rooted Trees

PART II: COMBINATORIAL PHYSICS

- a hint of Physics
- Feynman Graphs
- Combinatorial Physical Theories
- Renormalization Hopf Algebras
- Rooted Trees & Feynman Graphs

Combinatorial Classes

Definition

A **combinatorial class** is a set C equipped with a size function $||\cdot|| : C \longrightarrow \mathbb{Z}_{n\geq 0}$, such that $C_n \coloneqq \{c \in C \mid |c| = n\}$ is finite $\forall n$.

- C : *configuration space* (e.g., finite graphs)
- C_n : *macrostate* (e.g., finite graphs with 4 vertices)
- $c \in C_n$: *microstate* (e.g., chain graph of 4 vertices)

Important: combinatorial class T of *planar rooted trees*

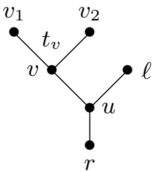
|tree|=#(vertices)

Rooted Trees (T^*) & Planar Rooted Trees (T)

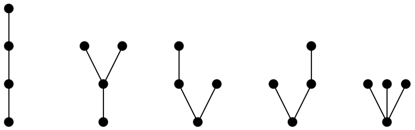
Definition

A **rooted** n**-tree** t is a connected finite graph with n vertices, no cycles, and a distinguished vertex r called the **root**. It is called **planar** if it is endowed with an ordering on vertices (to the children at each vertex).

- **parent of** v**:** unique vertex u adjacent to v, closer to the root.
- **children of** u**:** all vertices (if any) with u as parent.
- **leaf:** a vertex ℓ with no children.
- **subtree** t_v **rooted at** v**:** tree consisting of v and its descendants.
- $\mathbb{I}$**:** empty tree (size 0)



(a) rooted tree $t \in T^*$



(b) class T_4 of planar rooted 4-trees $t \in T$

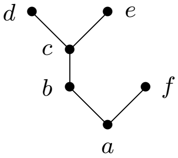
Antichain in the Set $V(t)$ of Vertices

Definition

An **antichain** in the set $V(t)$ of vertices of a rooted tree t is any subset $C \subseteq V(t)$ consisting of vertices that cannot be compared.

i.e., no two vertices in C have an ancestor-descendant relationship.

Examples: siblings *or* the set of all children in the same generation.



Antichains: $\{d, e, f\}$

$\{d, e\}$, $\{b, f\}$, $\{c, f\}$, $\{d, f\}$, $\{e, f\}$

$\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$, $\{e\}$, $\{f\}$

$\varnothing$

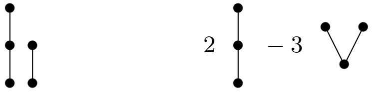
k : field of characteristic 0

Definition

The **algebra of planar rooted trees** is the polynomial algebra $k[T]$ generated by the elements of T, with formal addition as addition and disjoint union as multiplication. The empty tree is identified with $1 \in k$.

Definition

forest (n, m) **of size** $n = \sum_{i=1}^{m} n_i$ **(vertices) with** m **trees:**
Multiplication of m n_i-trees $(i = 1, \ldots, m)$.



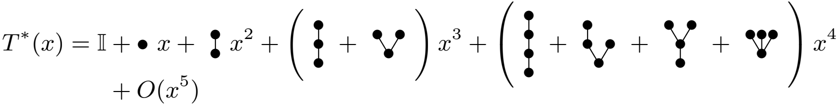
(a) forest $(5, 2)$ (b) element in $k[T]$

Augmented Generating Function

Definition

Given a combinatorial class C, the **augmented generating function** of C is the formal power series

$$C(x) = \sum_{c \in C} c x^{|c|} \in (k[C])[[x]]$$



Denote: GF=generating function

AGF=augmented generating function

Definition

An **evaluation map** is a map $\phi : k[C] \longrightarrow \mathbb{Z}_{n\geq 0}$.

- **ordinary:** $\mathrm{or}(c) = 1$
- **exponential:** $\mathrm{ex}(c) = \frac{1}{|c|!}$

Note: defined on elements $c \in C$, extended to $k[C]$ as an algebra homomorphism.

- **ordinary GF:** $\mathrm{or}(C(x)) = \sum_{c \in C} x^{|c|}$

$$\mathrm{or}(T^*(x)) = 1 + x + x^2 + 2x^3 + 4x^4 + O(x^5) \quad (\mathrm{coef}(x^n) = \#\ n\text{-trees})$$

- **exponential GF:** $\mathrm{ex}(C(x)) = \sum_{c \in C} \frac{x^{|c|}}{|c|!}$

$$\mathrm{ex}(T^*(x)) = 1 + x + \frac{1}{2!}x^2 + \frac{2}{3!}x^3 + \frac{4}{4!}x^4 + O(x^5) \quad \left(\mathrm{coef}(x^n) = \frac{\#\ n\text{-trees}}{\#\ n\text{-labels}}\right)$$

- Combinatorial objects
- Combine objects $\longrightarrow$ product
- Decompose objects $\longrightarrow$ coproduct
- *"inverse"* transformation $\longrightarrow$ antipode

Connes-Kreimer Hopf Algebra of Rooted Trees

$H = k[T^*]$: the space of forests of rooted trees

$\mathbb{I}$: the empty forest (empty monomial)

Definition

The **Connes-Kreimer Hopf algebra of rooted trees** is the following combinatorial (commutative) Hopf algebra:

- **grading on H:** number of vertices of a forest
- **connected:** since $\mathrm{span}(H_0) = k\mathbb{I} \cong k$
- **unit:** $\mathbb{I}$
- **counit:** $\varepsilon(t) = 0$, $\varepsilon(\mathbb{I}) = 1$
- **product:** disjoint union of rooted trees
- **coproduct:** $\Delta(t) = \sum_{\substack{C \subseteq V(t) \\ C \text{ antichain}}} \left(\prod_{v \in C} t_v \right) \otimes \left(t - \prod_{v \in C} t_v \right) \begin{pmatrix} \text{splitting:} \\ \text{“pruned”} \\ \& \\ \text{“stump”} \end{pmatrix}$
- **antipode:** $S(t) = -t - \sum_{\substack{\varnothing \subsetneq C \subsetneq V(t) \\ C \text{ antichain}}} S\left(\prod_{v \in C} t_v \right) \left(t - \prod_{v \in C} t_v \right) \begin{pmatrix} \text{recursive} \\ \text{relation} \end{pmatrix}$

Hopf Structure Interpretation I

• **Grading of forests:** measures how ecologically matured/grown/complex the forest is;

low grading: newly planted forest, young and sparse
high grading: thriving ecosystem of well-established, sprawling woodland

• H **connected:** empty forest has meaning; land that can be planted

• **Product:** plants trees one next to the other in a forest

• **Unit:** empty land

• **Counit:** probability P of absence of tree structure;

empty forest: $P(\mathbb{I}) = 1 = \varepsilon(\mathbb{I})$
any other tree: $P(t) = 0 = \varepsilon(t)$

answers the question: *"Can I plant here?"* $1 = \text{YES},\ 0 = \text{NO}$.

Hopf Structure Interpretation II

Coproduct Δ:
- encodes all different ways to decompose trees
- cuts edges under vertices in antichain
- results in a “pruned” part and the remaining “stump”

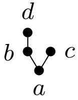
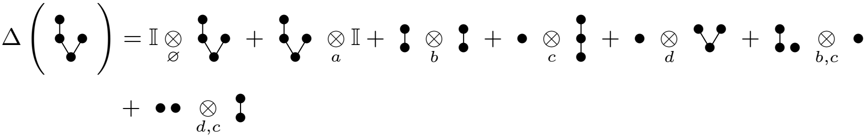
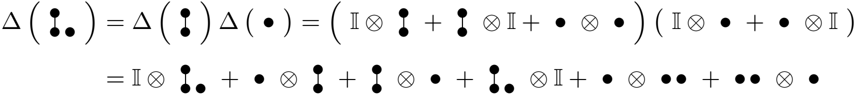


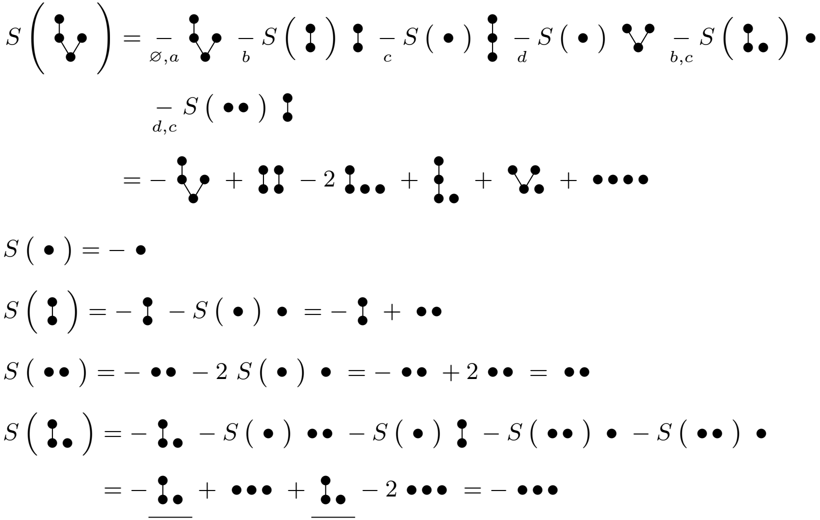
Figure: Antichains: $\varnothing$, $\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$, $\{b, c\}$, $\{d, c\}$.





Hopf Structure Interpretation III

Antipode S:
- recursively dismantles trees; flips sign at each step
- coproduct splits tree; antipode compensates for splitting



Hide & Seek: A Tale of QFT

scattering amplitude A = **probability of this outcome** $\left(|A|^2 \right)$

$\uparrow$

particles sent in $\longrightarrow$ **collide & interact** $\longrightarrow$ **detect what comes out**

What happened? $\downarrow$ NO IDEA

perturbative expansion = take *weighted sum* over all possibilities

$\downarrow$

Particular scenario: graph in spacetime
edges = propagating particles & *vertices* = interactions

forget spacetime embedding (*when* & *where*) $\downarrow$ combine all possibilities reduced to same abstract graph

1 possibility = 1 Feynman Graph
(*which* particles interacted & *how*)

$\downarrow$

weight = Feynman Integral

Cake interpretation of previous diagram:

- scattering amplitude = finished cake
- perturbative expansion = recipe
- sum = mixing ingredients & cooking
- graphs in spacetime = packaged ingredients (we know label & position)
- forgetting spacetime embedding = unpacking the ingredients
- possibilities = ingredients (= Feynman graphs)
- weight = quantity of ingredient (= Feynman Integral)

Feynman Integrals:

- **Coupling Constants:**
  - ⋆ capture strength of each interaction
  - ⋆ contribute to Feynman integral
  - ⋆ can be interpreted as counting variables
- **Feynman Rules:**
  - ⋆ how to read Feynman integrand expression off the graph
  - ⋆ each edge & each vertex contribute a factor
- **Renormalization:**
  - ⋆ extracting physically meaningful quantities from divergent integrals
- **Sums of Feynman Integrals:**
  - ⋆ expected to be divergent for all interesting cases.

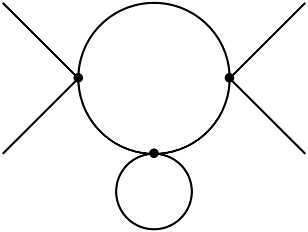
Half-Edge Feynman Graphs

Definition

A **half-edge Feynman graph** G is a set H of half edges along with

- **a set** $V(G)$ **of vertices:** vertex=disjoint subset of half edges
- **a set** $E(G)$ **of internal edges:** internal edge=disjoint pair of half edges.

External edges=half edges not in internal edges.

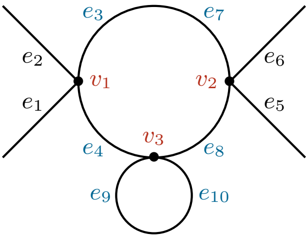


Half-Edge Labelled Feynman Graphs

Definition

A **half-edge labelled Feynman graph** G_L is a half-edge graph along with

- **a labelling** L**:** bijection $H \longrightarrow \{1, 2, \dots, |H|\}$.



$$V(G) = \{v_1, v_2, v_3\}, \quad E(G) = \{i_1, i_2, i_3\}$$
$$i_1 = \{e_3, e_7\}, \quad i_2 = \{e_4, e_8\}, \quad i_3 = \{e_9, e_{10}\}$$
$$v_1 = \{e_1, e_2, e_3, e_4\}, \quad v_2 = \{e_5, e_6, e_7, e_8\}, \quad v_3 = \{e_4, e_8, e_9, e_{10}\}$$

Theorem

Let G be a connected graph with n half edges. Let m be the number of half-edge labelled graphs giving G upon forgetting the labelling. Then:

$$\frac{m}{n!} = \frac{1}{|\operatorname{Aut}(G)|} \quad \textit{(symmetry factor)}$$

Why do we care?

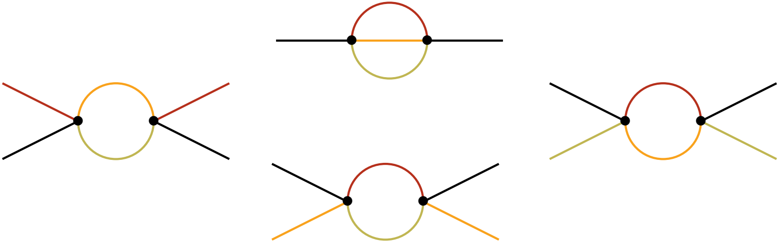
- $L(x) = \text{aug}\,(\,\text{labelled}\,) \quad \& \quad U(x) = \text{aug}\,(\,\text{unlabelled}\,)$
- $\phi =$ "forgetful" evaluation map, $\quad \phi(\text{labelled}) = \phi(\text{unlabelled})$
- $\phi_L(\cdot) = \frac{1}{n!}\phi(\cdot) \quad \& \quad \phi_U(\cdot) = \frac{1}{|\operatorname{Aut}(\cdot)|}\phi(\cdot)$
- $\phi_L(L(x)) = \phi_U(U(x))$
- $\text{or}\,(\,\text{unlabelled; weighted}\,) = \text{ex}\,(\,\text{labelled}\,)$
- **FACT:** $\text{ex}\,(\,\text{labelled}\,) = \exp(\,\text{ex}\,(\,\text{connected}\,)\,)$
- $\mathbf{A}_{\sum \text{all}} = \exp\left(\mathbf{A}_{\sum \text{connected}}\right)$

1PI Feynman Graphs

Definition

One-particle irreducible *or* **1PI** Feynman graph:

- connected
- connected after removing any one internal edge.



Analogy

$$\mathbf{1PI} \xleftarrow[\text{Transform}]{\text{Legendre}} \textbf{connected graphs} \xrightarrow{\exp} \textbf{graphs}$$

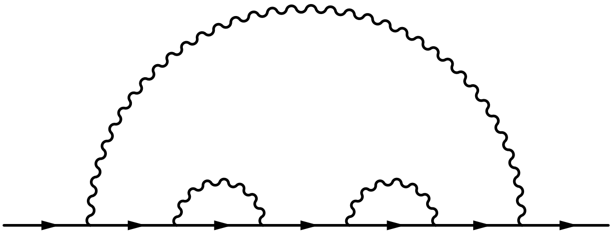
Analogy:

- connected graphs = walls
- 1PI graphs = bricks
- non-1PI graphs = part of wall with multiple bricks

Feynman Subgraphs

Definition

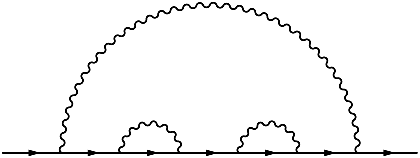
subgraph γ **of graph** G**:** subset of vertices of G



Divergent & Primitive Feynman Graphs

Definition

A Feynman graph is **divergent** if it has a divergent Feynman integral. A divergent Feynman graph with no divergent subgraphs is called **primitive**.



Definition

A subgraph γ is called **subdivergence** if its connected components are 1PI and divergent.
Overlapping subdivergences: they share at least 1 vertex.

Definition

A **combinatorial physical theory** T is a set of of half edge types with

- **permissible edge types:** set of types of edges (pairs of half edges)
- **permissible vertex types:** set of types of vertices
- **power counting weight** w**:** an integer for each type (edge & vertex)
- **dimension** D **of spacetime:** a nonnegative integer

Note: edges here are *internal* !

Definition

superficial degree of divergence of G in T:

$$\omega = D\ell - \sum_e w(e) - \sum_v w(v)$$

- $\omega \geq 0$: G **divergent**
- $\omega = 0$: G **logarithmically divergent**

Definition

external leg structure $L(G)$**:** multiset of half edge types of external edges of G

Definition

A combinatorial physical theory T is **renormalizable** if:
$\omega(G, T) = f(L(G)), \ \forall G \in T$

Scalar Field Theories ϕ^k

$$\phi^k : \begin{cases} 1 \text{ half-edge type} \\ 1 \text{ edge type, } w(e) = 2 \\ 1\ k\text{-valent vertex, } w(v) = 0 \end{cases}$$

- $\phi^4 : D = 4$
- $\phi^3 : D = 6$

Denote: $\ell = \#$ loops, $v = \#$ vertices, $e = \#$ edges, $x = \#$ external legs

Euler characteristic: $\ell = e - v + 1$

Regularity equation: $kv = x + 2e$

$$\begin{aligned} \text{Then:} \quad \omega &= D\ell - \sum_e w(e) - \sum_v w(v) \\ &= D\ell - 2e \\ &= D + \left(1 - \frac{D}{2}\right)x + \underline{\left(\frac{Dk}{2} - D - k\right)}v \end{aligned}$$

$$\implies D = \frac{2k}{k-2}$$

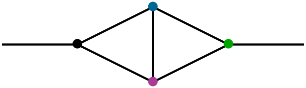
Definition

Let G be a Feynman graph in a theory T. Let γ be a subgraph with each connected component 1PI and divergent. The **contraction** G/γ of γ is the Feynman graph in T constructed as follows:

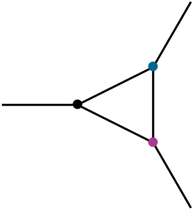
- start from G
- identify subgraph γ
- collapse vertices and internal edges into a single vertex
- keep external edges attached to external vertices
- this is G/γ.

Contraction of γ: Example

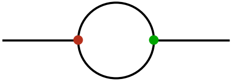
$G =$



$\gamma =$



$G/\gamma =$



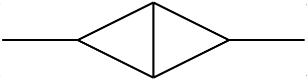
Definition

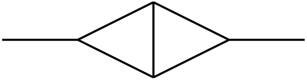
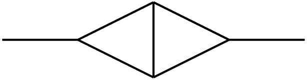
Fix a renormalizable theory T. Let $\mathscr{G}$ be the set of connected 1PI graphs in T. The **renormalization Hopf algebra** $\mathscr{H}$ associated to T is the polynomial algebra $k[\mathscr{G}]$ with:

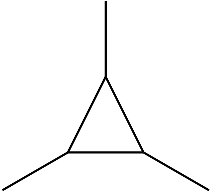
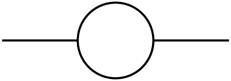
- **generators:** divergent 1PI graphs of T
- **grading:** loop order
- **unit:** empty graph $\mathbb{I}$
- **counit:** $\epsilon(\mathbb{I}) = 1$, $\epsilon(G) = 0$ if G nonempty
- **product:** disjoint union of graphs
- **coproduct:** $\Delta(G) = \sum\limits_{\substack{\gamma \subseteq G \\ \gamma = \gamma_p}} \gamma \otimes G/\gamma$
- **antipode:** $S(G) = -G - \sum\limits_{\substack{\varnothing \subsetneq \gamma \subsetneq G \\ \gamma = \gamma_p}} S(\gamma)\, G/\gamma$

γ_p : product of divergent 1PI subgraphs

Coproduct: Example (in ϕ^3)

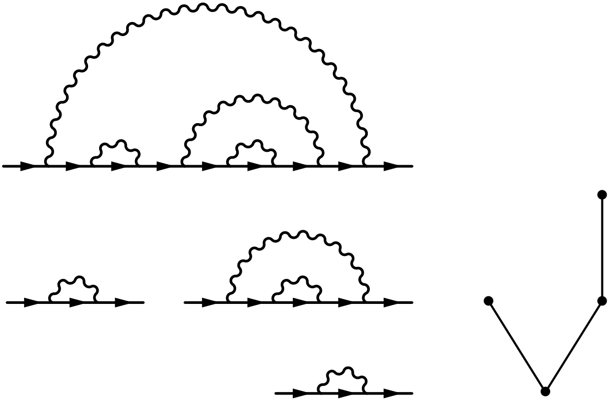
$\Delta\Big($  $\Big)$

$= \mathbb{I} \otimes$  $+$  $\otimes \mathbb{I}$

$+ \ 2$  $\otimes$ 

Note: γ with 1 or 2 vertices is NOT 1PI, so not taken into account!

Insertion Trees: Rooted Trees Take The Stage!



The End

...or maybe Not

The slides are full of mathematical *Easter Eggs* ! Have Fun !!

Bibliography

[1] Yeats, K. *A Combinatorial Perspective on Quantum Field Theory*. Springer, 2017.

[2] Cartier, P., and Patras, F. *Classical Hopf Algebras and Their Applications*. Springer, 2021.

[3] Panzer, E. *Hopf-algebraic Renormalization of Kreimer's Toy Model*. Master's thesis, Humboldt University of Berlin, 2012.

Appendix A: Feynman Diagrams in Mathematics & Physics

- Algebraic Geometry:
  - Moduli Spaces
  - Intersection Theory
- Geometric Topology:
  - Knot Invariants
  - 3-Manifolds
- Representation Theory:
  - of Quantum Groups
  - of Lie Algebras
- Number Theory:
  - Prime Numbers
  - Riemann Zeta Function
- Probability and Statistics:
  - Stochastic Processes
  - Random Walks
- Combinatorics:
  - Graphical Representations
  - Generating & Partition Functions
- Graph Theory:
  - Connectivity, cycles, paths
  - Graph Algorithms
- Mathematical Physics:
  - Statistical Mechanics
  - Perturbative Expansions
- Quantum Computing:
  - Quantum Circuits
  - Quantum Algorithms
- Artificial Intelligence:
  - Neural Networks
  - AI Systems/Models

$$
\begin{aligned}
\omega &= D\ell - \sum_e w(e) - \sum_v w(v) \\
&= D\ell - 2e \\
&= D(e - v + 1) - kv + x \\
&= D\left(\frac{kv}{2} - \frac{x}{2} - v + 1\right) - kv + x \\
&= D + \left(\frac{Dk}{2} - D - k\right) v + \left(1 - \frac{D}{2}\right) x
\end{aligned}
$$

Euler characteristic: $\ell = e - v + 1$

regularity equation: $kv = x + 2e$