SMSTC Hopf Algebras

Frobenius algebras vs 2-d TQFTs

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Frobenius Algebras

Recall (Def^{n} 3.13) that a Frobenius form on a finite-dim^{l} k-algebra A is an element $\varepsilon \in A^*$ whose kernel contains no nontrivial left ideals. These data together specify a Frobenius algebra (A, ε).

Recall also

Th^{m} 5.11 (Larson-Sweedler): Every finite-dim^{l} Hopf algebra is a Frobenius algebra.

Our main punchline is going to be the following (canonical) equivalence of categories:

$$\underline{2TQFT}_k \simeq \underline{cFA}_k$$

2-dim$^{\underline{l}}$ topological quantum field theories over k

commutative Frobenius k-algebras

To explore this, the following alternative def$^{\underline{n}}$ of a Frobenius algebra will be more helpful.

Note that $\varepsilon : A \longrightarrow k$ (Frobenius or not) canonically determines an associative pairing:

$$\langle\cdot,\cdot\rangle : A \otimes A \longrightarrow k$$
$$(x,y) \longmapsto \varepsilon(xy) =: \langle x, y\rangle.$$

It's basic linear algebra to prove that in our setting, TFAE:

(i) $\mathrm{Ker}(\varepsilon)$ contains no nontrivial left ideals;

(ii) " " " " right " ;

(iii) the associated pairing $\langle\cdot,\cdot\rangle$ is nondegenerate.

$\exists$ a copairing $\gamma : k \longrightarrow A \otimes A$ s.t. the following is id:

$$\begin{array}{ccccc}
A & & (A \otimes A) \otimes A & \xrightarrow{\varepsilon \otimes \mathrm{id}_A} & k \otimes A \\
\| & & \| & & \| \\
A \otimes k & \xrightarrow[\mathrm{id}_A \otimes \gamma]{} & A \otimes (A \otimes A) & & A
\end{array}$$

($\not\exists$ some starting with $A = k \otimes A$).

So now

Defn: A **Frobenius algebra** is a finite-dim^{l} k-algebra endowed with **Frobenius pairing**: that is, with an associative nondegenerate pairing $\beta : A \otimes A \longrightarrow k$.

Eg. • A f.d. division ring$/k$ w/ any nonzero linear form $A \longrightarrow k$.

• $(Mat_n(k), tr)$

• For G finite gp. $\varepsilon : kG \longrightarrow k$ (c.f. Eg. 5.18)

$$\begin{aligned} e &\longmapsto 1 \\ g \neq 1 &\longmapsto 0 \end{aligned}$$

• X compact oriented n-mfd. Then integration w.r.t. a chosen volume form provides a linear map $H^*_{dR}(X) \longrightarrow \mathbb{R}$, & Poincaré duality gives that the corresponding bilinear pairing $H^*_{dR}(X) \otimes H^*_{dR}(X) \longrightarrow \mathbb{R}$ is nondegenerate.

We'll revisit Frobenius algebras shortly. Δ

Cobordisms & TQFTs

"Manifold" will always mean smooth, compact, oriented (but not nec. connected).

Note that we let submflds have their own orientation – not necessarily just the one induced from the ambient mfld. So:

Let Σ be a connected cmpt of ∂M. At a pt $x \in \Sigma$, let $\{v_1, \dots, v_{n-1}\}$ be a +vely oriented basis for $T_x\Sigma$. Σ is called an **in-boundary** if a vector w pointing in to the mfld M yields a +vely oriented basis $\{v_1, \dots, v_{n-1}, w\}$ for T_xM. Define **out-boundary** analogously.

Def$^{\underline{n}}$: Let Σ_0 & Σ_1 be closed $(n-1)$-mflds (ie (compact &) w/o boundary).
An oriented cobordism from Σ_0 to Σ_1 is an oriented n-mfld M together with maps

$$\underbrace{\Sigma_0 \longrightarrow}_{\text{diffeo}^{\underline{m}} \text{ onto in-boundary of } M} M \underbrace{\longleftarrow \Sigma_1}_{\text{diffeo}^{\underline{m}} \text{ onto out-boundary of } M}.$$

Write $\Sigma_0 \overset{M}{\Longrightarrow} \Sigma_1$ or $M : \Sigma_0 \Longrightarrow \Sigma_1$.

Eg. Here are two cobordisms from a pair of circles to another pair of circles:

Are M & M' the "same"? Surely not! $\triangle$

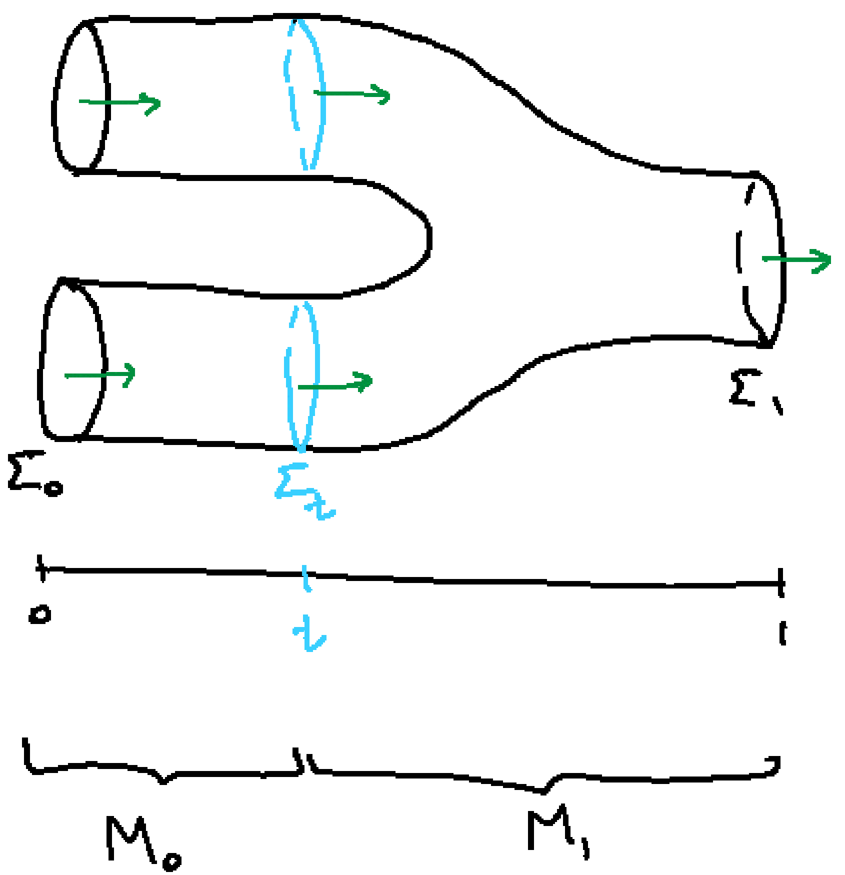
Def$^{\underline{n}}$: Two cobordisms M, M' from Σ_0 to Σ_1 are **equivalent** if $\exists$ diffeo$^{\underline{m}}$ $M \xrightarrow{\cong} M'$ making diagram on right commute.

$$\begin{tikzcd} & M' & \\ \Sigma_0 \arrow[ur] \arrow[dr] & & \Sigma_1 \arrow[ul] \arrow[dl] \\ & M \arrow[uu, "\cong"'] & \end{tikzcd}$$

Note Σ_0 & Σ_1 are completely fixed, not just up to diffeo$^{\underline{m}}$.

Equivalence classes of cobordisms are called cobordism classes (but we'll probably just be lazy & say 'cobordism').

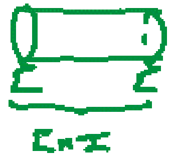
You can decompose a cobordism M: introduce a new codim-1 submfld which splits M into 2 parts: one with all the in-∂ & one w/ all the out-∂.

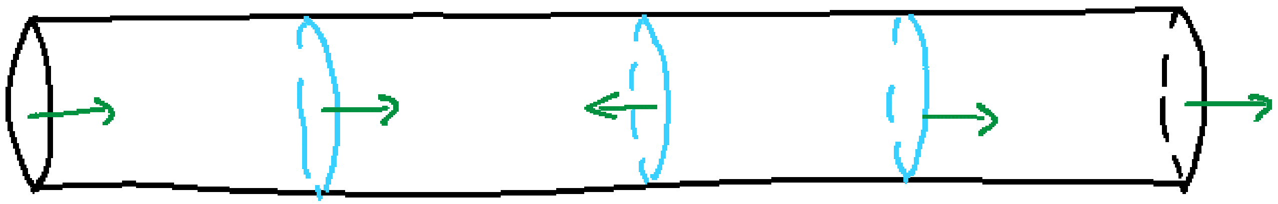


Get two cobordisms from one.

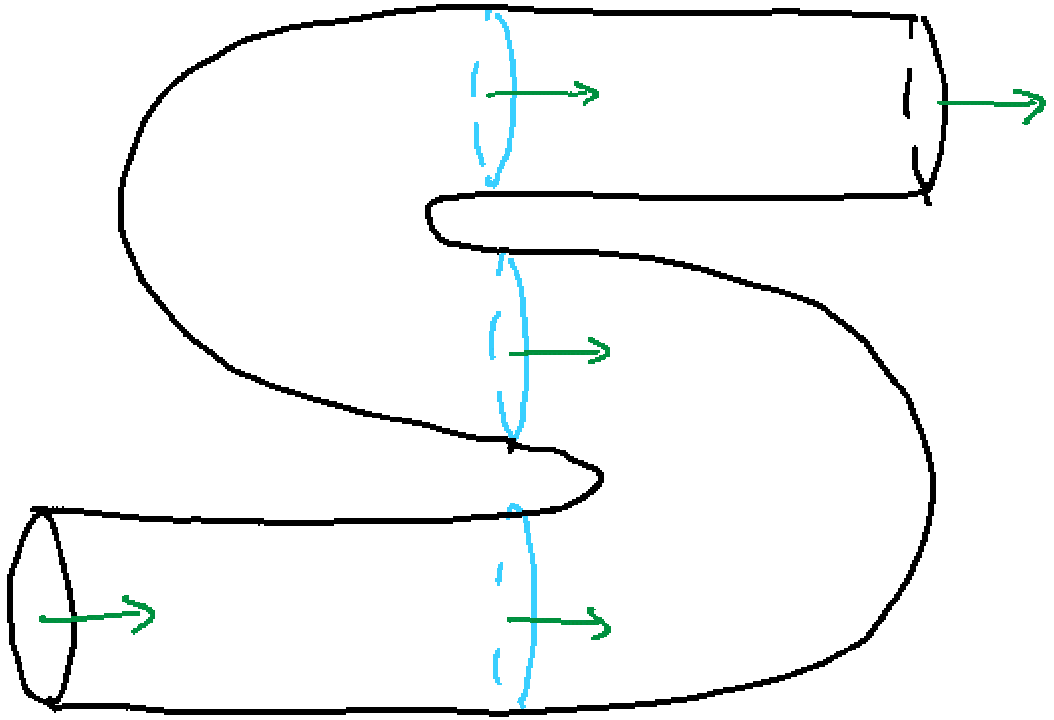
This submfld Σ must be oriented so that a +ve normal for it points towards the out-part of M.

Eg. Given the cobordism $\Sigma \times I$ from Σ to itself, can decompose it as follows:





more clearly as two cobordisms:



Surely we should be able to combine two cobordisms in the obvious way? $\Sigma_0 \overset{M_1}{\Longrightarrow} \Sigma_1$ & $\Sigma_1 \overset{M_2}{\Longrightarrow} \Sigma_2 \rightsquigarrow \Sigma_0 \overset{M_1 M_2}{\Longrightarrow} \Sigma_2$?

<u>Problem 1</u>: no canonical choice of smooth structure on $M_1 M_2$ near Σ_1

<u>Problem 2</u>: what is the "identity"? Should be the cylinder of length 0... but that's not an n-mfd!

Both problems are solved by passing to cobordism classes.

<u>Th=</u> (Milnor, ≤ '65): $\exists$ smooth structure on $M_1 M_2 := M_1 \amalg_{\Sigma_1} M_2$ s.t. the embeddings $M_1 \hookrightarrow M_1 M_2$ & $M_2 \hookrightarrow M_1 M_2$ are diffeo$^{\underline{m}}$s onto their images. Moreover, this smooth structure is unique up to diffeo$^{\underline{m}}$ fixing $\Sigma_0, \Sigma_1,$ & Σ_2.

Yay! Can now define:

Def^n: The category $\underline{nCob}$ is given as follows:

- objects are closed, oriented $(n-1)$-mflds
- arrows are cobordism classes.

Easy to show this $\underline{is}$ a category.

Now:

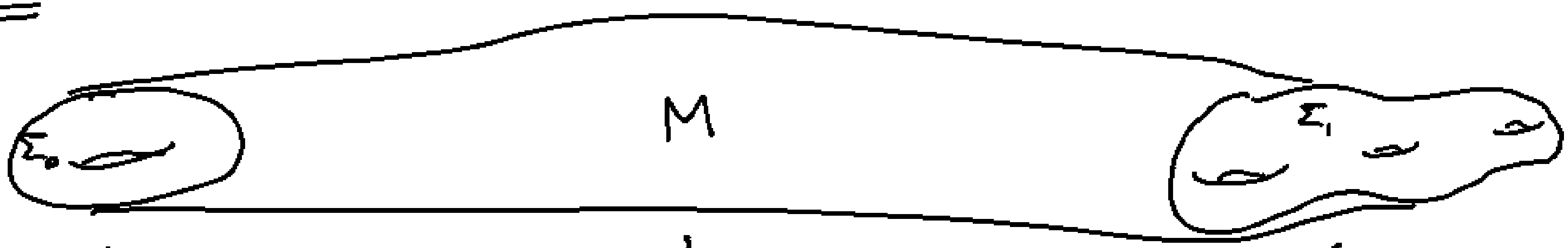
- The empty n-mfld $\emptyset_n$ is a cobordism $\emptyset_{n-1} \overset{\emptyset_n}{\Longrightarrow} \emptyset_{n-1}$
- Disjoint union of cobordisms is a cobordism between disjoint unions:

$$\left.\begin{array}{l} M : \Sigma_0 \Longrightarrow \Sigma_1 \\ M' : \Sigma_0' \Longrightarrow \Sigma_1' \end{array}\right\} \quad M \amalg M' : \Sigma_0 \amalg \Sigma_0' \Longrightarrow \Sigma_1 \amalg \Sigma_1'.$$

- Cobordism induced by twist diffeo^m $\Sigma \amalg \Sigma' \xrightarrow{\cong} \Sigma' \amalg \Sigma$ is called the **twist cobordism** $T_{\Sigma,\Sigma'} : \Sigma \amalg \Sigma' \Longrightarrow \Sigma' \amalg \Sigma$.

Can check that $(\underline{nCob}, \amalg, \emptyset, T)$ is a symmetric monoidal category.

TQFTs:



k-vector space $\longrightarrow$ $\Sigma_0\mathcal{A} \xrightarrow[M\mathcal{A}]{} \Sigma_1\mathcal{A}$ $\longleftarrow$ k-vector space

k-linear map

Defn: An n-dim^l topological quantum field theory (n-TQFT) $\mathcal{A}$ over a field k is a rule which associates:

- to each closed oriented $(n-1)$-mfd Σ, a k-vector space $\Sigma\mathcal{A}$
- to each cobordism $\Sigma_0 \overset{M}{\Longrightarrow} \Sigma_1$ a linear map $M\mathcal{A} : \Sigma_0\mathcal{A} \longrightarrow \Sigma_1\mathcal{A}$

subject to conditions (on next slide).

If you want to do physics: think of the Σ as spaces & cobordisms as spacetimes; get output "state spaces" & "time evolution operators".

We require the following:

- $M \cong M' \Rightarrow M\mathcal{A} = M'\mathcal{A}$
- Cylindrical cobordism $\Sigma \times I$ gets sent to $\mathrm{id}_{\Sigma\mathcal{A}}$

"topological"

- If M decomposes as $M'M''$, then $M\mathcal{A} = M''\mathcal{A} \circ M'\mathcal{A}$
- Disjoint union goes to tensor product:

$$\Sigma = \Sigma' \sqcup \Sigma'' \Rightarrow \Sigma\mathcal{A} = \Sigma'\mathcal{A} \otimes \Sigma''\mathcal{A}$$

$$\& \quad M = M' \sqcup M'' \Rightarrow M\mathcal{A} = M'\mathcal{A} \otimes M''\mathcal{A}$$

- $\Sigma = \emptyset$ gets sent to k

(⇒ empty cobordism sent to id_k)

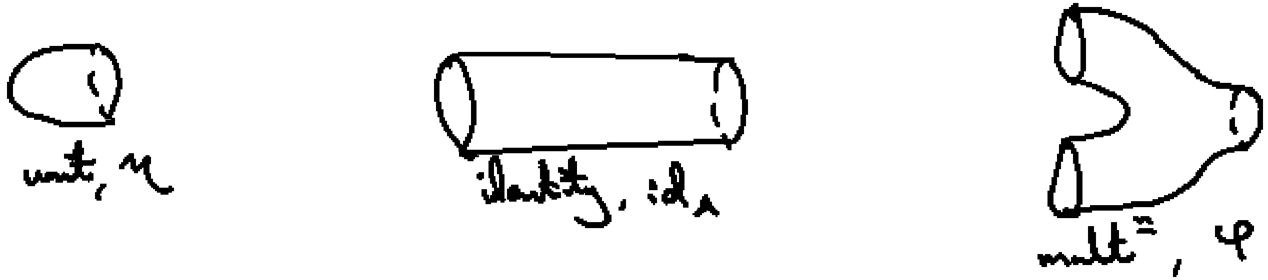
"quantum"

So have a monoidal functor $(\underline{n\mathrm{Cob}}, \sqcup, \emptyset) \longrightarrow (\underline{\mathrm{Vect}}_k, \otimes_k, k)$.

In fact we ask it's symmetric $(\underline{n\mathrm{Cob}}, \sqcup, \emptyset, T) \longrightarrow (\underline{\mathrm{Vect}}_k, \otimes_k, k, \sigma)$. $v \otimes w \longmapsto w \otimes v$

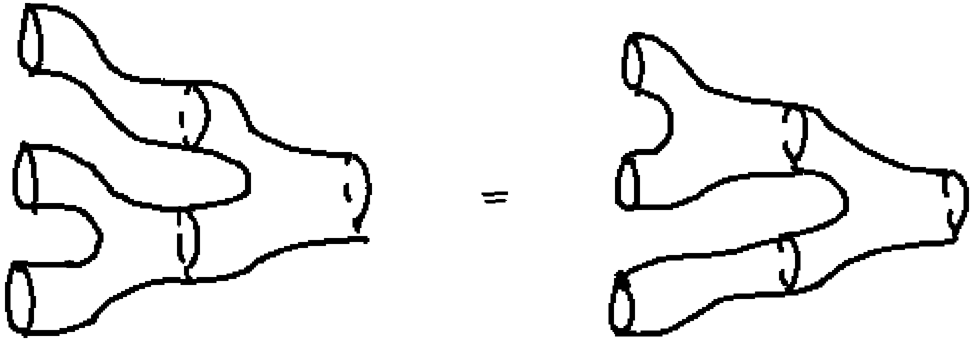
Towards the main thm

For a k-algebra (A, φ, η), we introduce the following notation:

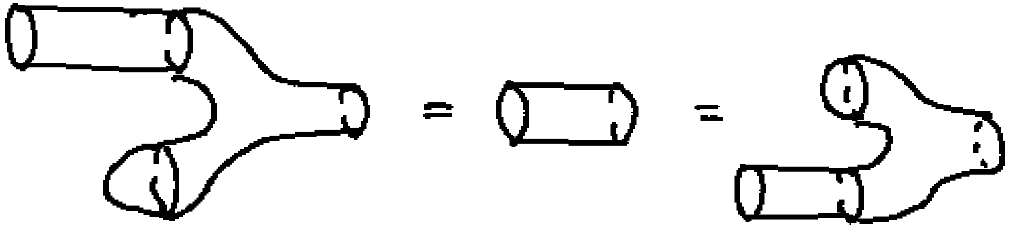


A k-linear map $A^{\otimes n} \longrightarrow A^{\otimes m}$ will have n circles on the left, ordered so that the bottom one corresponds to the first copy of A, & m circles on the right with the analogous ordering. The tensor product of two maps is drawn as the disjoint union of the two symbols, arranged in a column, with the same ordering convention. Composition of maps is concatenation of symbols.

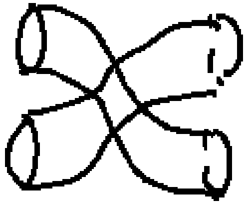
So axioms for algebra:



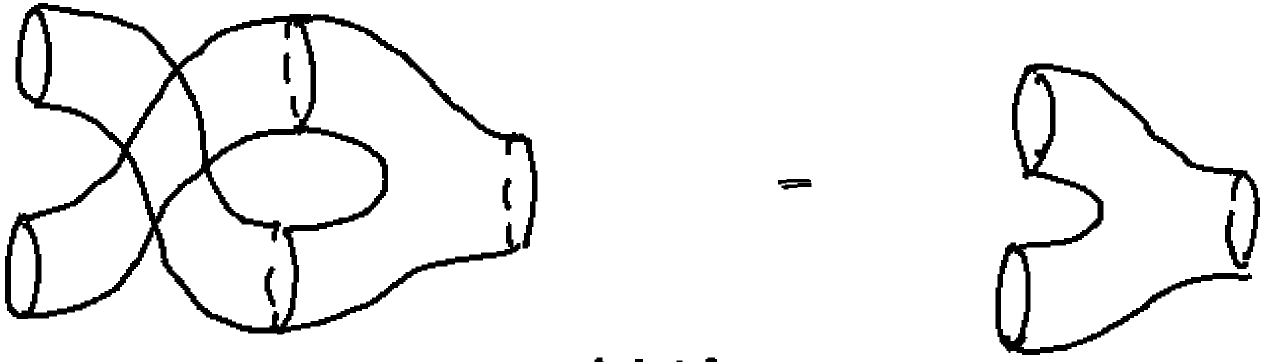
associativity



unit axiom

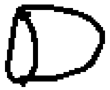
& denote the twist map $A \otimes A \longrightarrow A \otimes A$, $v \otimes w \longmapsto w \otimes v$ by 

so that:

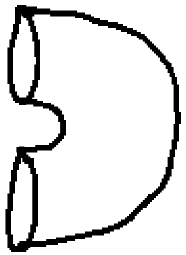


commutativity

Now draw:

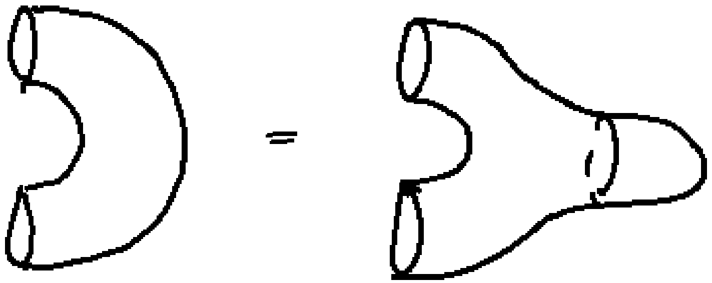


Frobenius form, ε

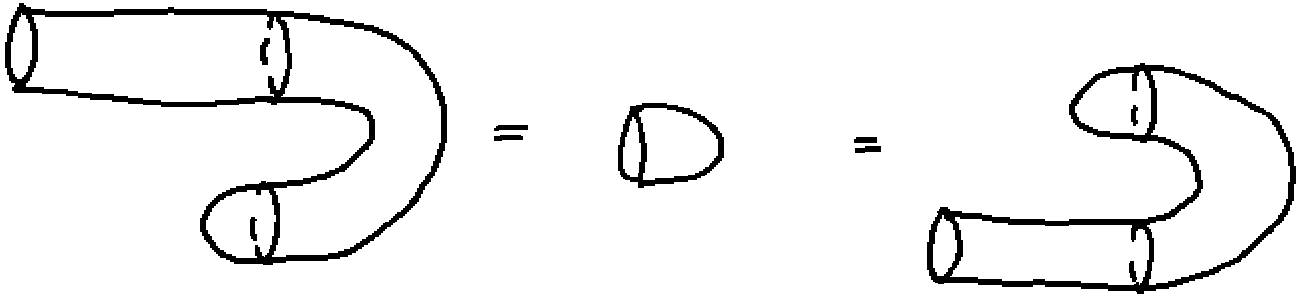


Frobenius pairing, β

& so

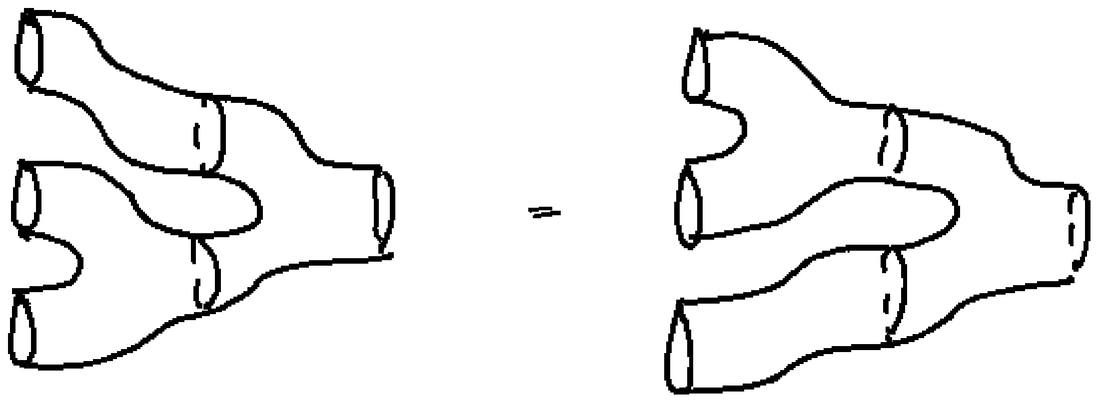


$$\beta(x, y) = \varepsilon(xy)$$



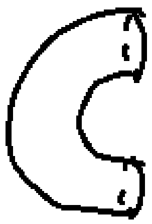
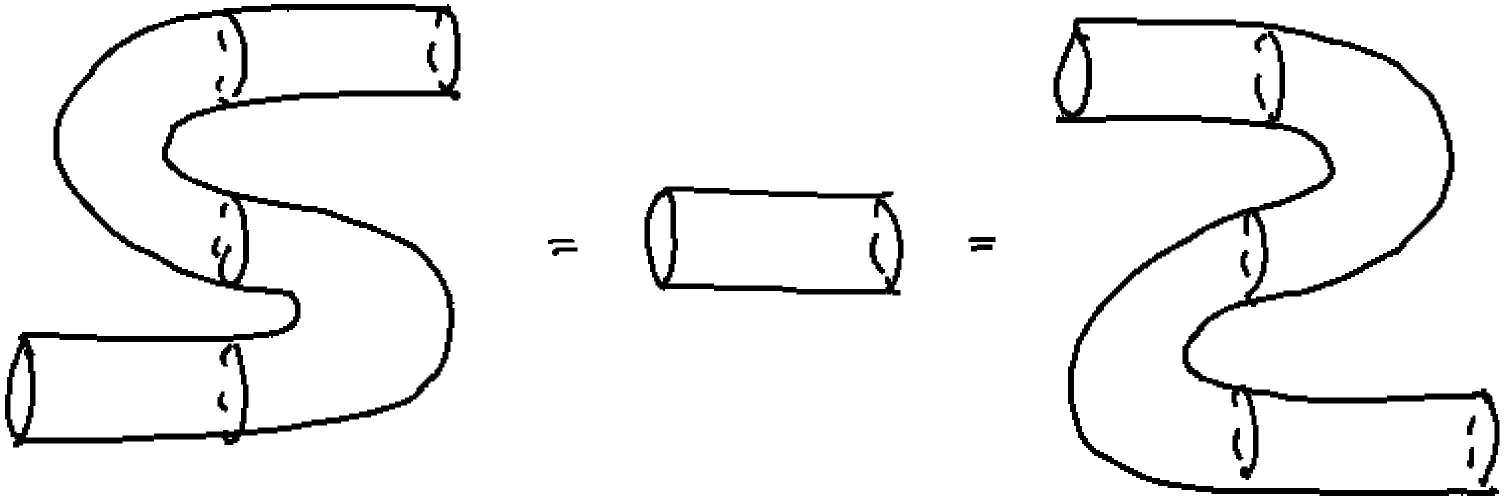
$$\beta(1_A, x) = \varepsilon(x) = \beta(x, 1_A).$$

Can easily write down the two conditions we need on β in order that (A, β) is a Frobenius algebra:

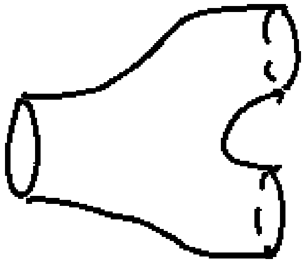
-  ← associativity of β

- Non-degeneracy:

"the snake relation"

$\exists$  s.t. 

A Frobenius algebra has a natural coalgebra structure for which ε is the counit. You can probably guess:



$\longleftarrow$ comultiplication, ψ

(! by !ness of copairing)

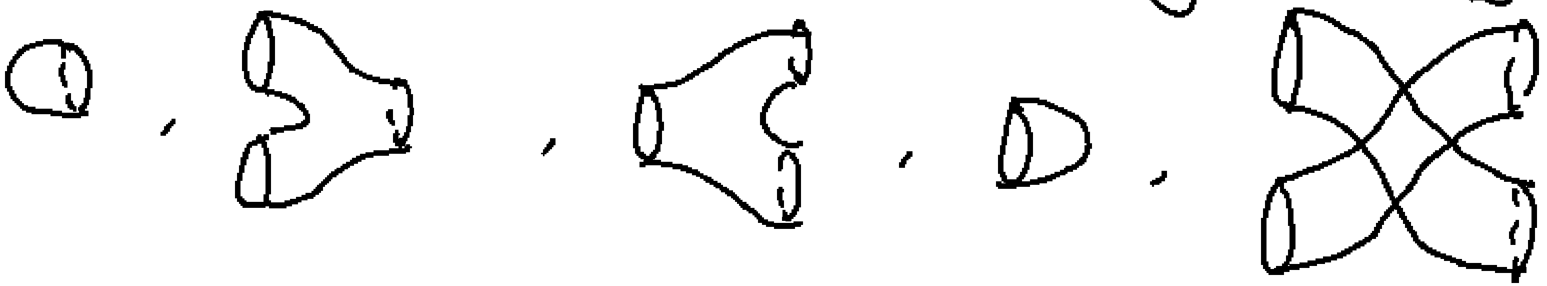
Frobenius algebras are <u>not</u> in general bialgebras.

Note that for f.d. Hopf algebras, the Frobenius comult$^{\underline{n}}$ is distinct from the bialgebra comult$^{\underline{n}}$.

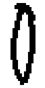
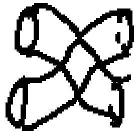
Make the category $\underline{\mathrm{FA}}_k$: objects are Frobenius algebras $/k$ & arrows are $\underbrace{\text{Frobenius algebra homo}^{\underline{m}}\text{s}}_{\text{algebra homo}^{\underline{m}}\text{s that are also coalgebra homo}^{\underline{m}}\text{s}}$

Recall an n-dim$^{\underline{l}}$ TQFT was a symmetric monoidal functor $(\underline{n\mathrm{Cob}}, \amalg, \emptyset, T) \longrightarrow (\underline{\mathrm{Vect}}_k, \otimes_k, k, \sigma)$. Such form the objects of a category $\underline{n\mathrm{TQFT}}_k = \underline{\mathrm{Rep}}_k(\underline{n\mathrm{Cob}})$, & the arrows are the monoidal natural transformations.

Turns out that $\underline{2\mathrm{Cob}}$ is generated by



, subject to some relations.

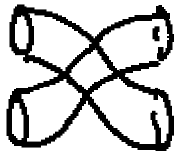
A monoidal functor is determined entirely by its values on generators. So to specify a 2TQFT $\mathcal{A}$, need to specify a vector space A as the image of  (1 circle) & linear maps for each generator. Since $\mathcal{A}$ is <u>symmetric</u> monoidal,  had better be sent to the usual twist map σ. So:

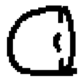
$\underline{2\mathrm{Cob}}_k \longrightarrow \underline{\mathrm{Vect}}_k$

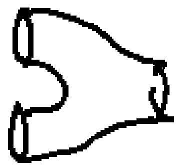
1 circle  $\longmapsto A$

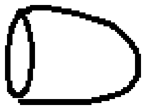
n circles  $\longmapsto A^{\otimes n}$

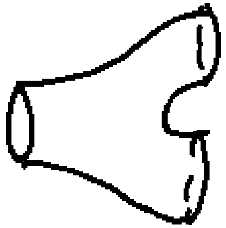
 $\longmapsto id_A : A \longrightarrow A$

 $\longmapsto \sigma : A \otimes A \longrightarrow A \otimes A$

 $\longmapsto \eta : k \longrightarrow A$ unit

 $\longmapsto \varphi : A \otimes A \longrightarrow A$ mult^{n}

 $\longmapsto \varepsilon : A \longrightarrow k$ counit

 $\longmapsto \psi : A \longrightarrow A \otimes A$ comult^{n}

subject to the relations in $\underline{2\mathrm{Cob}}_k$ ... which turn out to be precisely our axioms for a commutative Frobenius algebra.

That is, given $\mathcal{A}$ a 2TQFT, the image A (i.e. the image of 1 circle) in $\underline{\mathrm{Vect}}_k$ is actually a commutative Frobenius k-algebra.

Correspondence actually also works on arrows, so we get the equivalence of categories

$$\underline{2\mathrm{TQFT}}_k \simeq {}_c\mathrm{FA}_k .$$