Affine group scheme over k ← field

§1 Definition

3 ways:

(1) (Algebraic geometer)

group object in Aff/k

$A = \operatorname{Spec} R \longrightarrow \operatorname{Spec} k$

$m: A \times_k A \to A$ $\Longleftrightarrow$ ~~com~~ Δ

fiber product over $\operatorname{Spec} k$

$e \quad \operatorname{Spec} k \to A$ $\Longleftrightarrow$ ε

$\iota \quad A \to A$ $\Longleftrightarrow$ χ

(2) (category theorist)

affine scheme$/k$ — k-alg $R \in \mathrm{Ob}(\mathrm{Alg}_k^{co})$

functor $\mathrm{Alg}_k^{co} \longrightarrow \mathrm{Set}$

$A \longmapsto \mathrm{Hom}_{k\text{-alg}}(R, A)$

Gp category $C \to \mathrm{Set}$

affine group scheme over k

$$\begin{tikzcd} & \mathrm{Gp} \arrow[d, "\text{forgetful}"] \\ \mathrm{Alg}_k^{co} \arrow[ur, "\operatorname{Spec}(R)"] \arrow[r] \arrow[ur, phantom, shift right=4, "\circlearrowleft"] & \mathrm{Set} \end{tikzcd}$$

$A \longmapsto \mathrm{Hom}_{k\text{-alg}}(R, A)$

(3) (lover this Hopf algebra course)

a commutative Hopf algebra over k!

(2) $\Rightarrow$ (3) See Notes 3.10

(3) $\Rightarrow$ (2) co Hopf alg R / k

$\mathrm{Spec}(R): Alg_k^{co} \to Gp$ $\qquad \varphi: k \to R$ unit

$A \mapsto \mathrm{Hom}_{k\text{-alg}}(R, A) = \mathrm{Spec}(R)(A)$

$f_1: R \to A$

$f_2: R \to A$

$f_1 \cdot f_2: R \to A$?

$$\begin{tikzcd} k \arrow[r, "\varphi"] \arrow[d, "\varphi"'] & R \arrow[d] \arrow[ddr, bend left, "f_1"] & \\ R \arrow[r] \arrow[drr, bend right, "f_2"'] & R \otimes_k R \arrow[dr, dashed, "\otimes"] & \\ & & A \end{tikzcd}$$

comultiplication

$$R \xrightarrow{\Delta} R \otimes_k R \xrightarrow{f_1 \ f_2} A$$

$R \xrightarrow{f_1} A$

$R \to$

$f_1(r) \otimes f_2(r) \to A$

$\underline{A \otimes A \xrightarrow{\text{multiplication}} A}$

Ex identity in $\mathrm{Hom}_{k\text{-}Alg}(R, A)$

inverse

Ex (for algebraic geometers) (1) $\Leftrightarrow$ (3)

morphism

(2) $G_1 : Alg_k^{co} \to Gp$ natural

$\downarrow$

$G_2 : Alg_k^{co} \to Gp$ transformation

(3) $G_1 = \mathrm{Spec}(R_1)$ $\qquad G_1 \to G_2$ $\qquad R_2 \to R_1$ Hopf algebra homomorphism

$G_2 = \mathrm{Spec}(R_2)$

(1) $\mathrm{Spec}(R_1) \to \mathrm{Spec}(R_2)$

induced group hom

§2 Examples

Ex1 (2) $\mathrm{Alg}_k^{co} \to \mathrm{Gp}$

$A \mapsto (A, +)$

additive group $\mathbb{G}_a$

(3) $R = k[t]$

$\Delta(t) = 1 \otimes t + t \otimes 1$

$\boxed{\varepsilon(t) = 0}$ $\quad \varepsilon(a_0 + a_1 t + a_2 t^2 + \cdots) = a_0$

$\chi(t) = -t$

(1) $\mathrm{Spec}(k[t])$ $\quad \mathbb{A}_k^1$ affine line over k

Ex2 (2) $\mathrm{Alg}_k^{co} \to \mathrm{Gp}$

$A \mapsto A^\times$ group of invertible elements

multiplicative group $\mathbb{G}_m$

(3) $k[x, y]/(x \cdot y - 1)$

$\Delta(x) = x \otimes x \quad \Delta(y) = y \otimes y$

$\varepsilon(x) = \varepsilon(y) = 1 \quad \chi(x) = y \quad \chi(y) = x$

(1) $\mathrm{Spec}(\cdots\cdots)$ $\quad \mathbb{A}_k^1 \setminus \{0\}$

$\mathbb{P}_k^1 \setminus \{0, \infty\}$

Ex3 morphism: $\underline{\mathbb{G}_m} \longrightarrow \mathbb{G}_m$

(2) $A \in \mathrm{Ob}(\mathrm{Alg}_k^{co})$,

$$\begin{array}{ccc} \mathbb{G}_m(A) & \longrightarrow & \mathbb{G}_m(A) \\ \wr\! \| & & \wr\! \| \\ \varphi_A : A^\times & \longrightarrow & A^\times \\ a & \longmapsto & a^n \end{array}$$

$\underline{\mathbb{G}_m[n](A) = \ker \varphi_A}$

$\mathbb{G}_m[n] : \mathrm{Alg}_k^{co} \to \mathrm{Gp}$

(3) $k[x, x^{-1}] \to k[x, x^{-1}]$

$\mathbb{G}_m[n] \hookrightarrow \mathbb{G}_m \qquad x \mapsto x^n$

Def $\underset{\text{Spec}(R_1)}{G_1}$ is a <u>closed embedding</u> into $\underset{\text{Spec}(R_2)}{G_2}$

$\Leftrightarrow \quad R_2 \twoheadrightarrow R_1$

$G_1 = \text{Spec}(R_1) = \mathbb{G}_m[n] \qquad R_1 = k[x,x^{-1}]/(x^n-1) = k[x]/(x^n-1)$

$G_2 = \text{Spec}(R_2) = \mathbb{G}_m \qquad R_2 = k[x,x^{-1}]$

$G_1 \to G_2 \qquad k[x,x^{-1}] \twoheadrightarrow k[x]/(x^n-1)$

quotient Hopf algebra

Ex 4 (2) $\text{Alg}_k^{co} \to \text{Gp}$

$A \mapsto \text{Aut}_{A\text{-mod}}(\underbrace{k^n \otimes_k A}_{k\text{-mod}})$

$GL_n(k) = \{\text{invertible matrices over } k\}$

GL_n general linear group

(3) ~~coefficients~~

$$R = \frac{k[t_{11}, \cdots, t_{nn}, y]}{\left(\det\begin{pmatrix} t_{11} & \cdots & t_{1n} \\ \vdots & & \vdots \\ t_{n1} & \cdots & t_{nn}\end{pmatrix} \cdot y - 1\right)}$$

Δ, See Notes $(n=2)$

ε

χ

(1) Spec(..)

Def an affine algebraic group over k is

Def an affine algebraic group over k is
an affine group scheme $/ k$
$\mathrm{Spec}(R)$
where R is a finitely generated k-alg

(⚠ algebraic variety)

Thm 1 Every affine group scheme $/k$ is a limit of affine algebraic groups $/k$.

Thm 2. Every affine algebraic gp $/k$ is a closed embedding into GL_n for some n

affine group sch $/k$ is a limit of matrix groups!

Ex1 $\mathbb{G}_m$ is affine algebraic grp $/k$

$$\mathbb{G}_m \xrightarrow{\cong} GL_1$$

$$GL_1(A) = \mathrm{Aut}_{A\text{-mod}}(A) \cong A^\times$$
$$\varphi \longmapsto \varphi(1)$$

Ex2 $\mathbb{G}_a$ is affine algebraic grp $/k$

$\mathbb{G}_a \hookrightarrow GL_2$?

$$\mathbb{G}_a \cong \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \right\} \hookrightarrow GL_2$$

$$\frac{k[t_{11}, t_{12}, t_{21}, t_{22}, y]}{((t_{11}t_{22} - t_{12}t_{21})y - 1)} \twoheadrightarrow k[t]$$
$$\rho$$
$$t_{11} \longmapsto 1$$

$t_{22} \longmapsto 1$

$t_{21} \longmapsto 0$

$t_{12} \longmapsto t$

$y \longmapsto 1$