

Affine group scheme over k field

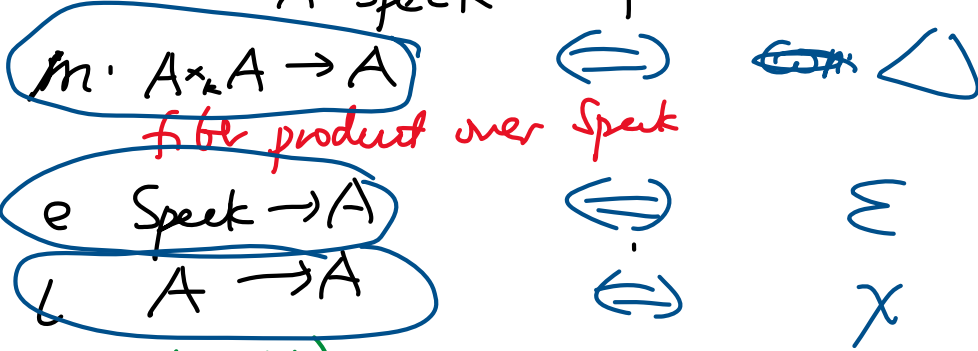
§1 Definition

3 ways!

(1) (Algebraic geometer)

group object in Aff/k

$$A = \text{Spec } R \rightarrow \text{Spec } k$$



(2) (category theorist)

affine scheme / k

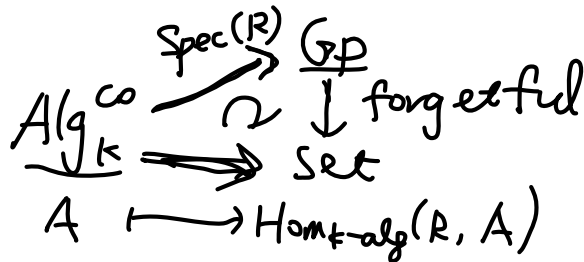
k -alg $\underline{R} \in \text{Ob}(\text{Alg}_k^{\text{co}})$

functor $\text{Alg}_k^{\text{co}} \rightarrow \text{Set}$

$$A \mapsto \text{Hom}_{k\text{-alg}}(R, A)$$

Gp category $\mathcal{C} \rightarrow \text{Set}$

affine group scheme over k



(3) (lover this Hopf algebra course)

a commutative Hopf algebra over k !

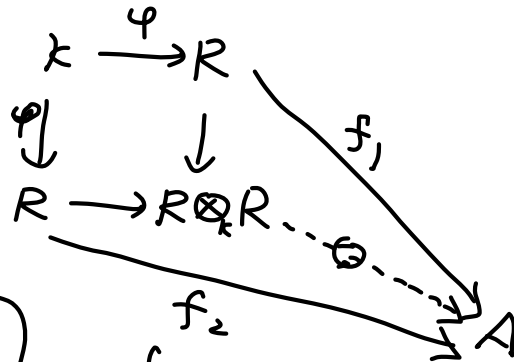
(2) \Rightarrow (3) See Notes 3.10.1

(3) \Rightarrow (2) ω Hopf alg R k

$$\text{Spec}(R): \underbrace{\text{Alg}_k^{\text{co}}}_{A} \rightarrow \text{Grp} \quad \varphi: k \rightarrow R \text{ unit}$$

$$A \mapsto \text{Hom}_{k\text{-alg}}(R, A) = \text{Spec}(R)(A)$$

$f_1: R \rightarrow A$
 $f_2: R \rightarrow A$
 $(f_1, f_2): R \rightarrow A?$



Comultiplication
 $R \xrightarrow{\Delta} R \otimes_k R \xrightarrow{f_1, f_2} A$
 $R \xrightarrow{f_1} A$
 $R \xrightarrow{f_2} A$

$f_1(r) \otimes f_2(r) \rightarrow A$
 $A \otimes A$ multiplication

Ex identity in $\text{Hom}_{k\text{-alg}}(R, A)$
inverse

Ex (for algebraic geometry) (1) \Leftrightarrow (3)

morphism (2) $G_1: \text{Alg}_k^{\text{co}} \rightarrow \text{Grp}$ natural transformation
 $G_2: \text{Alg}_k^{\text{co}} \rightarrow \text{Grp}$

(3) $G_1 = \text{Spec}(R_1)$ $G_1 \downarrow G_2$ Hopf algebra homomorphism
 $G_2 = \text{Spec}(R_2)$ $G_2 \downarrow$

(1) $\text{Spec}(R_1) \rightarrow \text{Spec}(R_2)$
induced group hom

§2 Examples

Ex1 (2) $\text{Alg}_k^\infty \rightarrow \text{Grp}$
 $A \mapsto (A, +)$
 additive group Ga

(3) $R = k[t]$
 $\Delta(t) = | \otimes t + t \otimes |$
 $\varepsilon(t) = 0$ $\varepsilon(a_0 t^0 + a_1 t^1 + \dots) = a_0$
 $\chi(t) = -t$

(1) $\text{Spec}(k[t])$ \mathbb{A}_k^1 affine line over k

Ex2 (2) $\text{Alg}_k^\infty \rightarrow \text{Grp}$
 $A \mapsto (A^\times)$ group of invertible elements
 multiplicative group Gm

(3) $k[x, y] / (x \cdot y - 1)$
 $\Delta(x) = x \otimes x$ $\Delta(y) = y \otimes y$
 $\varepsilon(x) = \varepsilon(y) = 1$ $\chi(x) = y$ $\chi(y) = x$

(1) $\text{Spec}(\dots)$ $\mathbb{A}_k^1 \setminus \{0\}$
 $\mathbb{P}_k^1 \setminus \{0, \infty\}$

Ex3 morphism: $\text{Gm} \rightarrow \text{Gm}$

(2) $A \in \text{Ob}(\text{Alg}_k^\infty)$, $\text{Gm}(A) \rightarrow \text{Gm}(A)$

(φ_A) $A^\times \rightarrow A^\times$
 $a \mapsto a^n$

$\text{Gm}[n](A) = \ker \varphi_A$
 $(\text{Gm}[n])$ $\text{Alg}_k^\infty \rightarrow \text{Grp}$
 (3) $k[x, x^{-1}] \rightarrow k[x, x^{-1}]$

$$\mathbb{G}_m[n] \hookrightarrow \mathbb{G}_m \quad x \mapsto x^n$$

Def $G_1 \subset \text{Spec}(R_1)$ is a closed embedding into $G_2 \subset \text{Spec}(R_2)$

$$\Leftrightarrow R_2 \twoheadrightarrow R_1$$

$$G_1 = \text{Spec}(R_1) = \mathbb{G}_m[n] \quad R_1 = k[x, x^{-1}] / (x^n - 1) = k[x] / (x^n - 1)$$

$$G_2 = \text{Spec}(R_2) = \mathbb{G}_m \quad R_2 = k[x, x^{-1}]$$

$$G_1 \rightarrow G_2$$

$$k[x, x^{-1}] \twoheadrightarrow k[x] / (x^n - 1)$$

quotient Hopf algebra

Ex 4 (2) $\text{Alg}_k^{\text{co}} \rightarrow \text{Grp}$

$$A \mapsto \text{Aut}_{A\text{-mod}}(k^n \otimes_k A)$$

GL_n general linear group

$GL_n(k)$
||
{invertible matrices over k}

(3) $R = \frac{k[t_{11}, \dots, t_{nn}, y]}{(\det \begin{pmatrix} t_{11} & \dots & t_{1n} \\ \vdots & \ddots & \vdots \\ t_{n1} & \dots & t_{nn} \end{pmatrix} \cdot y - 1)}$

Δ . See Notes (n=2)
 ε
 χ

(1) $\text{Spec}(\dots)$

Def an affine algebraic group over k is

Def an affine algebraic group over k is an affine group scheme / k $\text{Spec}(R)$ where R is a finitely generated k -alg

(Δ algebraic variety)

Thm 1 Every affine group scheme / k is a limit of affine algebraic groups / k .

Thm 2. Every affine algebraic gp / k is a closed embedding into GL_n for some n

affine group sch / k is a limit of matrix groups!

Ex 1 G_m is affine algebraic gp / k

$$G_m \cong GL_1$$

$$GL_1(A) = \text{Aut}_{A\text{-mod}}(A) \cong A^\times$$

$$\varphi \mapsto \varphi(1)$$

Ex 2 G_a is affine algebraic gp / k

$$\underline{G_a \hookrightarrow GL_2?}$$

$$G_a \cong \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \right\} \hookrightarrow GL_2$$

$$\frac{k[t_{11}, t_{12}, t_{21}, t_{22}, y]}{((t_{11}t_{22} - t_{12}t_{21})y - 1)}$$

$$t_{11} \mapsto 1$$

$$\begin{array}{l} t_{22} \mapsto 1 \\ t_{21} \mapsto 0 \\ t_{12} \mapsto t \\ y \mapsto 1 \end{array}$$