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C^*- Hopf Algebra

0- Motivations

Applications of C^*-algebra in gp representation, Quantum field theories, Quantum statistical mechanics, ...

Structure of Hopf algebra

S. Vaes, A.Van Daele : C^*-algebra $\rightsquigarrow$ locally compact quantum gps

I- Some functional analysis

Def: Let X be a vector space over a field $\mathbb{K} \subset \mathbb{C}$.
A **norm** on X is a map $\|\cdot\| : X \to \mathbb{R}$ s.t. $\forall x, y \in X$ and $\lambda \in \mathbb{K}$,

1. non negativity : $\|x\| \geq 0$
2. positive definiteness : $\|x\| = 0 \iff x = 0$
3. absolute homogeneity : $\|\lambda x\| = |\lambda| \, \|x\|$
   absolute value in $\mathbb{C}$
4. triangle inequality : $\|x+y\| \leq \|x\| + \|y\|$

Def: A normed space $(X, \|\cdot\|)$ is **complete** if every Cauchy sequence in X converges in X.

Def: A **Banach space** is a complete normed space $(X, \|\cdot\|)$.

Def: Let X be a vector space over a field $\mathbb{K} \subset \mathbb{C}$.
An inner product on X is a map $X \times X \to \mathbb{K}$, $x, y \mapsto \langle x, y \rangle$

such that $\forall\, x, y, z \in X$ and $a, b \in \mathbb{K}$, ← conjugate in $\mathbb{C}$

1. Conjugate symmetry: $\langle x, y \rangle = \overline{\langle y, x \rangle}$
2. linearity in 1^{st} elt: $\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$
3. positive definiteness: $\|x\| = 0 \Leftrightarrow x = 0$.

Def: A Hilbert space is a real/complex inner product space $(H, \langle \cdot, \cdot \rangle)$ that is also a complete metric space wrt the distance function induced by the inner product.

Def: An operator between two vector spaces X, Y is a linear transformation $T: X \to Y$.
If $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ are normed v. s., an operator $T: X \to Y$ is bounded if there is $k \geq 0$ s.t. $\|Tx\|_Y \leq k \|x\|_X$ $\forall\, x \in X$. If T is bounded, then the operator norm of T is

$$\|T\| = \sup_{\|x\|_X = 1} \|Tx\|_Y.$$

Not°
$\mathcal{L}(X, Y)$ is the set of bounded operators $X \to Y$
$\mathcal{L}(X)$ " " " " " " $X \to X$

Def: Let $T: \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded operator on Hilbert spaces. Then its adjoint T^* is defined by $\langle T^* y, x \rangle_1 = \langle y, Tx \rangle_2 \quad \forall x \in (\mathcal{H}_1, \|\cdot\|_1)$, $y \in (\mathcal{H}_2, \|\cdot\|_2)$.

II - C^*- Algebra

Def: A normed Banach algebra is a complex algebra (which is a Banach space) under a norm which is multiplicative, i.e.,
$\|xy\| \leq \|x\| \|y\| \quad \forall x, y \in A$.

- An involution on a normed Banach algebra A is a morphism $A \longrightarrow A$, $x \longmapsto x^*$ s.t. $\forall x, y \in A$ and $\lambda \in \mathbb{C}$,

- $(x+y)^* = x^* + y^*$
- $(xy)^* = y^* x^*$
- $(\lambda x)^* = \bar{\lambda} x^*$
- $(x^*)^* = x$
- $\|x^*\| = \|x\|$

- A (Banach) $*$-algebra is a normed Banach algebra with an involution

- A C^*- algebra is a Banach $*$algebra A satisfying the C^*- axiom:
$\|x^* x\|$ for all $x \in A$.

$\|x^*x\|$ for all $x \in A$.

Examples: i) $M_{n,\mathbb{C}}$: set of $n\times n$ matrices over $\mathbb{C}$ with involution beeing the conjugate transpose is a C^*-algebra.

ii) X - locally compact Hausdorff space

every pt has a nbd itself contained in a compact set

$\forall x_1 \neq x_2 \in X, \exists U_1 \ni x_1$, $U_2 \ni x_2$ nbd s.t. $U_1 \cap U_2 \neq \emptyset$.

$C_0(X)$: complex valued functions on X vanishing at ∞.

$C_0(X)$ with pointwise operations, supremum norm and involution : $f^*(x) = \overline{f(x)}$ is a commutative C^*-algebra.

iii) Let X a Banach space, then $\mathcal{L}(X)$ with involution beeing the adjoint operator is a Banach algebra

iv) Let $\mathcal{H}$ be a Hilbert space, then $\mathcal{L}(\mathcal{H})$ with involution beeing the adjoint operator is a C^*-algebra.

III - C^*-Hopf-algebra

Def: Let A be a C^*-subalgebra of a C^*-algebra B. We say that $b \in B$ is a left (respectively, right) multiplier of A if $bA \subset A$ (respectively $Ab \subset A$)

We say that $b \in B$ is a two-sided multiplier (or

We say that $b \in B$ is a two-sided multiplier (or just multiplier), if it is a left and right multiplier.

$M(A)$: C^*-algebra of multipliers of A.

[S. Vaes and A. Van Daele, Hopf C^*-Algebra 1999]

Def: A comultiplication on a C^*-algebra is a non-degenerate $*$-homomorphism

$\exists$ an approached unit $(e_i) \subset A$ s.t. $\varphi(e_i) \xrightarrow[\text{strongly}]{} 1$ — respects involution.

$$\varphi : A \to M(A \times A) \text{ s.t. } (\varphi \otimes \underset{\text{identity}}{\iota}) \otimes \varphi = (\underset{\text{identity}}{\iota} \otimes \varphi) \otimes \varphi$$

Define the maps $T_1, T_2 : \underset{\text{algebraic tensor product}}{A \odot A} \to A \otimes A$

$$T_1(a \odot b) = \varphi(a)(1 \otimes b)$$
$$T_2(a \odot b) = (a \otimes 1)\varphi(b)$$

Def: Let (A, φ) be a pair of C^*-algebra A with a comultiplication φ s.t. $\varphi(A)(1 \odot A)$ and $\varphi(A)(A \otimes 1)$ are subspaces of $A \odot A$. We call this pair a Hopf C^*-algebra if the maps T_1 and T_2, as defined above, are injective on $A \otimes_h A$

($A \odot A$ with Haagerup norm)

(HOA with Haagerup norm)

In their paper, Vaes and van Daele construct a counit and an antipode on those Hopf C^*-algebra.