Drinfeld-Jimbo **Quantum groups** $U_h(sl_2) \Leftrightarrow U_q(sl_2)$

$U_h(sl_2)$ = quantisation of Lie bialgebra sl_2

$:=$ a certain deformation of $U(sl_2)$

Def · Lie coalgebra is vector space over k

$+ \ [\,,\,] : \mathfrak{g} \wedge \mathfrak{g} \underset{}{\rightleftarrows} \mathfrak{g} : \delta$ cobracket

· (co)Jacobi identity

$$\begin{tikzcd}
\mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g} \arrow[r, shift left, "Alt"] \arrow[rrr, bend right, "=0"'] & \mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g} \arrow[l, shift left, "Alt"] \arrow[r, shift left, "id\otimes{[\,,\,]}"] & \mathfrak{g}\otimes\mathfrak{g} \arrow[l, shift left, "id\otimes\delta"] \arrow[r, shift left, "{[\,,\,]}"] & \mathfrak{g} \arrow[l, shift left, "\delta"] \arrow[lll, bend left, "=0"]
\end{tikzcd}$$

$$x\otimes y\otimes z \longmapsto x\otimes y\otimes z + y\otimes z\otimes x + z\otimes x\otimes y$$

$\mathfrak{g}^*$

Ex $sl_2 = k\langle e, f, h\rangle$ $\quad [h,e]=2e \quad [e,f]=h$

$[h,f]=-2f$

$\delta(e) = e\wedge h \quad \delta(f) = f\wedge h$

$\delta(h) = 0$

Def Lie bialgebra is $(\mathfrak{g}, [\,,\,], \delta)$

- $(\mathfrak{g}, [\,,\,])$ is Lie algebra
- $(\mathfrak{g}, \delta)$ is Lie coalgebra
- cocycle condition $\delta([a,b]) = (id \otimes [-,b] + [-,b] \otimes id)\delta(a)$

$$- \; a \leftrightarrow b$$

Ex: $\mathfrak{sl}_2$

Def A deformation of a Hopf algebra H_0 is a Hopf algebra H over $k[[h]]$ st $\cdot$ $H \cong H_0[[h]]$ as $k[[h]]$-moduls

$$H_0[[h]] = \left\{ \sum v_n h^n \mid v_n \in H_0 \right\}$$

- $H/hH \cong H_0$ as Hopf algebras

$U_h(\mathfrak{sl}_2) =$ deformation of $U(\mathfrak{sl}_2)$

$$+ \; \delta(x) = \frac{\Delta(\tilde{x}) - \tau\circ\Delta(\tilde{x})}{h} \mod h$$

$\forall x \in \mathfrak{sl}_2$, $\forall$ lifting $\tilde{x}$ in $U_h(\mathfrak{sl}_2)$

$$U_h(\mathfrak{sl}_2)/hU_h(\mathfrak{sl}_2) \cong U(\mathfrak{sl}_2)$$

$$\tilde{x} = x + \mathcal{O}(h)$$

$$\Delta(\tilde{x}) = \Delta(x) + \mathcal{O}(h)$$

$$\ldots = \mathcal{O}(h)$$

$$\Delta(\tilde{x}) - \tau \circ \Delta(\tilde{x}) = \underbrace{\Delta(x) - \tau \circ \Delta(x)} + O(h)$$

$U_h(sl_2(\mathbb{C}))$ is Hopf algebra over $\mathbb{C}[[h]]$ generated by E, F, H

and relations. $[H,E] = 2E$, $[H,F] = -2F$, $[E,F] = \dfrac{e^{hH} - e^{-hH}}{e^{h} - e^{-h}}$

- $\Delta(E) = E \otimes e^{hH} + 1 \otimes E$, $\Delta(F) = F \otimes 1 + e^{-hH} \otimes F$, $\Delta(H) = 1 \otimes H + H \otimes 1$
- $\chi(E) = -E e^{-hH}$, $\chi(F) = -e^{hH} F$, $\chi(H) = -H$
- $\varepsilon(E) = \varepsilon(F) = \varepsilon(H) = 0$
- $U_h(sl_2) \cong U(sl_2)[[h]]$ as $\mathbb{C}[[h]]$ - module
- $U_h(sl_2)/hU_h(sl_2) \cong U(sl_2)$

$$[E,F] = \frac{e^{hH} - e^{-hH}}{e^{h} - e^{-h}} = \frac{1 + hH - 1 + hH + O(h^2)}{1 + h - 1 + h + O(h^2)}$$

$$= \frac{2hH + O(h^2)}{2h + O(h^2)} = \frac{H + O(h)}{1 + O(h)} \xrightarrow{h \to 0} H$$

$$\Delta(E) - \tau \circ \underline{\Delta(E)} \mod h = E \otimes H - H \otimes E$$

$$\frac{\ldots}{h} \qquad = 2E\wedge H = 2\delta(E)$$

$U_h(sl_2)$ is quantisation of $(sl_2, [,], 2\delta)$

$U_h(\mathfrak{g})$

Dreinfeld Jimbo quantum group = quantisation $(\mathfrak{g}, [,], 2\delta)$

$U_q(sl_2(\mathbb{C}))$ for $q \in \mathbb{C} \setminus \{0, \pm 1\}$ Hopf-algebra generated by E, F, K, K^{-1} st

- $KK^{-1} = K^{-1}K = 1$, $KEK^{-1} = q^2 E$, $KFK^{-1} = q^{-2} F$, $[E,F] = \frac{K-K^{-1}}{q-q^{-1}}$
- $\Delta(E) = E\otimes K + 1\otimes E$, $\Delta(F) = F\otimes 1 + K^{-1}\otimes F$, $\Delta(K) = K\otimes K$
- $\chi(E) = -eK^{-1}$, $\chi(F) = -KF$, $\chi(K) = K^{-1}$
- $\varepsilon(E) = \varepsilon(F) = 0$, $\varepsilon(K) = 1$

$\mathbb{C}[q, q^{-1}, (q-q^{-1})^{-1}]$

$e^h = q \qquad e^{hH} = K$