Recall: Quantum groups are non-commutative, non-cocommutative quasitriangular Hopf algebras.

$\exists$ R-matrix

$U_q \mathfrak{sl}_2$ is the "quantum deformation" of the universal enveloping algebra of the semi-simple Lie algebra $\mathfrak{sl}_2$.

It is a unital algebra with generators E, F, K, K^{-1} satisfying

1) $KEK^{-1} = q^2 E$
2) $KFK^{-1} = q^{-2} F$
3) $EF - FE = \dfrac{K - K^{-1}}{q - q^{-1}}$

Where $q \in \mathbb{C}^\times$.

When q is not a root of unity, the representation theory of $U_q \mathfrak{sl}_2$ is similar to the non-deformed case.

In particular, finite-dimensional modules have a weight space decomposition

$$M = \bigoplus_{\lambda \in \Lambda} M_\lambda$$

weight modules

Thus, every such module is completely reducible, i.e. decomposes into a sum of irreducible modules with highest weights $\underline{\lambda = \pm q^n}$.

We will write $M_\lambda = V(n)$. $\longrightarrow$ $\dim = n+1$

Crystals

$U_q \mathfrak{sl}_2$ Cartan A_1

$\alpha_i = \alpha_1$ $\{1\}$ $\mathbb{Z}$

Let Φ be a root system with index set I and weight lattice Λ.

A Kashiwara crystal is a set $\mathcal{B}$ with maps

$\neq \emptyset$

$\mathrm{wt} : \mathcal{B} \longrightarrow \Lambda$ weight

$e_i, f_i : \mathcal{B} \longrightarrow \mathcal{B} \sqcup \{0\}$ Kashiwara operators

$\varphi_i, \varepsilon_i : \mathcal{B} \longrightarrow \mathbb{Z} \cup \{-\infty\}$ string lengths

$f_i(x) = y \Leftrightarrow e_i(y) = x$

satisfying

$x \xrightarrow{i} y$

1) If $x, y \in \mathcal{B}$, $e_i(x) = y \Longleftrightarrow f_i(y) = x$

and $\mathrm{wt}(y) = \mathrm{wt}(x) + \alpha_i$,

$\varepsilon_i(y) = \varepsilon_i(x) - 1$

$\varphi_i(y) = \varphi_i(x) + 1$

$\underbrace{x \xrightarrow{i} \cdot}_{\varepsilon_i} \underbrace{\xrightarrow{i} \cdots\cdots y}_{\varphi_i}$

2) $\varphi_i(x) = \langle \mathrm{wt}(x), \alpha_i^\vee \rangle + \varepsilon_i(x) \quad \forall x \in \mathcal{B}, i \in I.$

If $\varphi_i(x) = -\infty$, then $\varepsilon_i(x) = -\infty$ and $e_i(x) = f_i(x) = 0$.

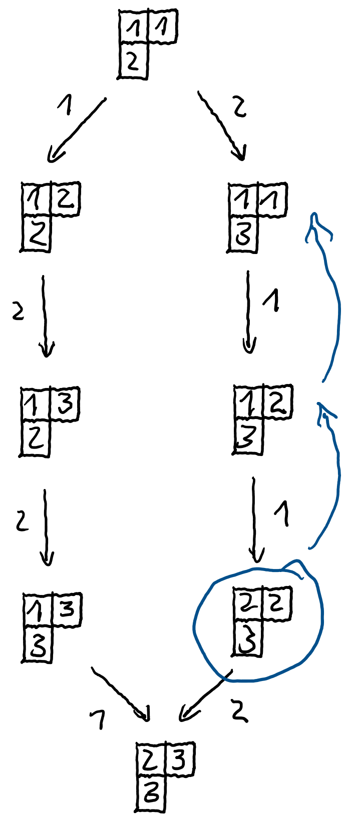
Examples

- Type A_{n-1} standard crystal: $\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \cdots \xrightarrow{n-1} \boxed{n}$

- Crystal graph for some irrep $V(m)$ of $U_q \mathfrak{sl}_2$: $\underline{u_0} \longrightarrow u_1 \longrightarrow u_2 \longrightarrow \cdots \longrightarrow \underline{u_m}$

highest weight vector lowest weight

- $U_q\mathfrak{sl}_3$, adjoint rep:



$wt(T) = (2,1)$

#1s   #2s

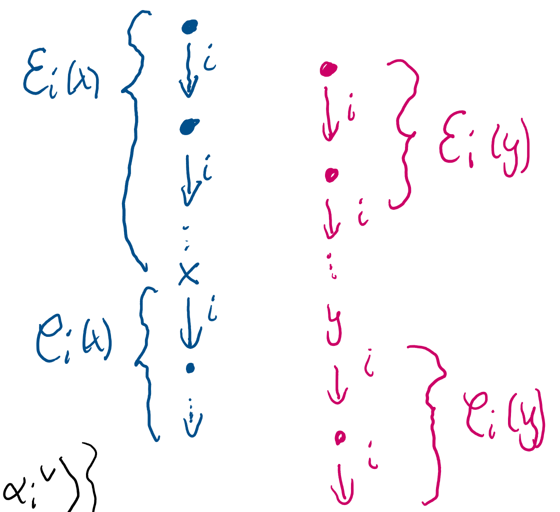
$i=1$

$\varepsilon_2\left(\begin{smallmatrix}2&2\\3\end{smallmatrix}\right) = 2$

Tensor Product

Let $\mathcal{B}, \mathcal{C}$ be two crystals with the same root system Φ.
Then $\mathcal{B} \otimes \mathcal{C}$ is a crystal with vertex set $\mathcal{B} \times \mathcal{C}$.
For $(x,y) =: x \otimes y \in \mathcal{B} \otimes \mathcal{C}$, we have

- $wt(x \otimes y) = wt(x) + wt(y)$
- $f_i(x \otimes y) = \begin{cases} f_i(x) \otimes y & \text{if } \varphi_i(x) > \varepsilon_i(y) \\ x \otimes f_i(y) & \text{if } \varphi_i(x) \leq \varepsilon_i(y) \end{cases}$
- $e_i(x \otimes y) = \begin{cases} e_i(x) \otimes y & \text{if } \varphi_i(x) \geq \varepsilon_i(y) \\ x \otimes e_i(y) & \text{if } \varphi_i(x) < \varepsilon_i(y) \end{cases}$
- $\varphi_i(x \otimes y) = \max\{\varphi_i(y), \varphi_i(x) + \langle wt(y), \alpha_i^\vee \rangle\}$



- $\varepsilon_i(x \otimes y) = \max\{\varepsilon_i(x), \varepsilon_i(y) - \langle wt(x), \alpha_i^\vee \rangle\}$

Example:

Consider two fin. dim. irreducible modules of $U_q\mathfrak{sl}_2$, $V(2)$ and $V(3)$, with bases $\{u_0, u_1, u_2\}$ and $\{v_0, v_1, v_2, v_3\}$, respectively.

Recall that $\qquad \varepsilon(u_0) = 0 = \varepsilon(v_0) \qquad \varphi(u_0) = 2$

$\varphi(v_0) = 3$

$\mathcal{B}(2): \quad u_0 \longrightarrow u_1 \longrightarrow u_2$

$\mathcal{B}(3): \quad v_0 \longrightarrow v_1 \longrightarrow v_2 \longrightarrow v_3$.

$f(v_3) = 0$

We want to find $\mathcal{B}(2) \otimes \mathcal{B}(3)$:

$\mathcal{B}(5)$

$$\begin{tikzcd}
u_0 \otimes v_0 \arrow[r] & u_1 \otimes v_0 \arrow[r] & u_2 \otimes v_0 \arrow[d] \\
u_0 \otimes v_1 \arrow[r] & u_1 \otimes v_1 \arrow[d] & u_2 \otimes v_1 \arrow[d] \\
u_0 \otimes v_2 \arrow[d] & u_1 \otimes v_2 \arrow[d] & u_2 \otimes v_2 \arrow[d] \\
u_0 \otimes v_3 & u_1 \otimes v_3 & u_2 \otimes v_3
\end{tikzcd}$$

$\mathcal{B}(1) \qquad \mathcal{B}(3)$

↪ No longer connected!?

- 3 connected components, isomorphic to $\mathcal{B}(5), \mathcal{B}(3), \mathcal{B}(1)$.

$\Rightarrow V(2) \otimes V(3) \cong V(5) \oplus V(3) \oplus V(1)$.