Hopf Algebras

String Diagrams

Willoughby Seago

6th February 2025

SMSTC
The University of Glasgow

String Diagrams

Set Up

We work in an arbitrary

- monoidal category $(\mathsf{C}, \otimes, I, \alpha, \lambda, \rho)$; or
- braided/symmetric monoidal category $(\mathsf{C}, \otimes, I, \alpha, \lambda, \rho, \sigma)$

as needed.

Set Up

We work in an arbitrary

- monoidal category $(\mathsf{C}, \otimes, I, \alpha, \lambda, \rho)$; or
- braided/symmetric monoidal category $(\mathsf{C}, \otimes, I, \alpha, \lambda, \rho, \sigma)$

as needed.

Goal: an alternative notation to commutative diagrams

A Horrible Commutative Diagram

$$
\begin{tikzcd}
 & & C \arrow[dll, bend right, "\Delta"'] \arrow[drr, bend left, "\Delta"] \arrow[ddd, dashed, "(f * g) * h = f * (g * h)" description] & & \\
C \otimes C \arrow[r, "\Delta \otimes \mathrm{id}_C"'] \arrow[d, "(f * g) \otimes h"'] & (C \otimes C) \otimes C \arrow[rr, "\alpha"] \arrow[d, "(f \otimes g) \otimes h"'] & & C \otimes (C \otimes C) \arrow[d, "f \otimes (g \otimes h)"] & C \otimes C \arrow[l, "\mathrm{id} \otimes \Delta"] \arrow[d, "f \otimes (g * h)"] \\
A \otimes A \arrow[drr, bend right, "\mu"'] & (A \otimes A) \otimes A \arrow[l, "\mu \otimes \mathrm{id}"'] \arrow[rr, "\alpha"'] & & A \otimes (A \otimes A) \arrow[r, "\mathrm{id} \otimes \mu"] & A \otimes A \arrow[dll, bend left, "\mu"] \\
 & & A & &
\end{tikzcd}
$$

Morphisms

$f\colon A \to B \qquad \rightsquigarrow$

Identity/Object

$$\mathrm{id}_A \colon A \to A \qquad \rightsquigarrow$$

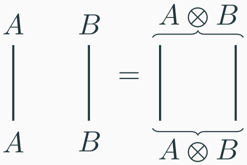
Composition

$$\left.\begin{array}{l} f\colon A \to B \\ g\colon B \to C \end{array}\right\} \quad g \circ f\colon A \to C \qquad \rightsquigarrow$$

Monoidal Product

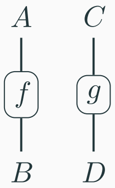
$$\mathrm{id}_A \otimes \mathrm{id}_B = \mathrm{id}_{A \otimes B} \colon A \otimes B \to A \otimes B$$

$$\rightsquigarrow$$



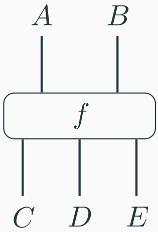
Monoidal Product

$$\left.\begin{aligned} f&\colon A \to B \\ g&\colon C \to D \end{aligned}\right\} \quad f \otimes g\colon A \otimes C \to B \otimes D \qquad \rightsquigarrow$$



More Complicated Morphism

$$f\colon A \otimes B \to C \otimes D \otimes E \qquad \rightsquigarrow$$



Rules

- Wires are “pinned” at the top and bottom

Rules

- Wires are “pinned” at the top and bottom
- Move things around, but don’t cross wires

Rules

- Wires are “pinned” at the top and bottom
- Move things around, but don’t cross wires
- Can’t leave the bounding box of the diagram (so no going over the top of a wire)

Rules

- Wires are “pinned” at the top and bottom
- Move things around, but don’t cross wires
- Can’t leave the bounding box of the diagram (so no going over the top of a wire)
- Don’t write anything for the coherence morphisms (α, λ, ρ) or for I

Notation Encapsulates Monoidal Structure

Theorem (Coherence Theorem (roughly))
Any reasonable diagram made only from α, λ, ρ, their inverses, id, $\otimes$, and $\circ$ commutes.

Notation Encapsulates Monoidal Structure

Theorem (Coherence Theorem (roughly))
Any reasonable diagram made only from α, λ, ρ, their inverses, id, $\otimes$, and $\circ$ commutes.

Theorem (Coherence Theorem (strictification version))
Every monoidal category is monoidally equivalent to a strict monoidal category.

Notation Encapsulates Monoidal Structure

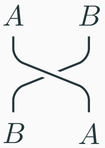
Theorem (Coherence Theorem (roughly))
Any reasonable diagram made only from α, λ, ρ, their inverses, id, $\otimes$, and $\circ$ commutes.

Theorem (Coherence Theorem (strictification version))
Every monoidal category is monoidally equivalent to a strict monoidal category.

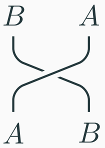
Theorem (Correctness of the Graphical Calculus)
A well-typed equation between morphisms in a monoidal category follows from the axioms if and only if it holds in the the graphical language up to planar isotopy.

Braiding

$\sigma_{A,B} \colon A \otimes B \to B \otimes A \qquad \rightsquigarrow$

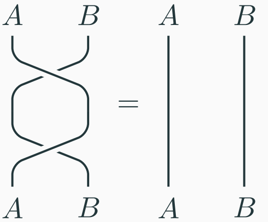


$\sigma^{-1}_{A,B} \colon B \otimes A \to A \otimes B \qquad \rightsquigarrow$



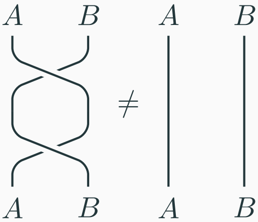
Inverse Braiding

$$\sigma^{-1}_{A,B} \circ \sigma_{A,B} = \mathrm{id}_{A\otimes B} \qquad \rightsquigarrow$$



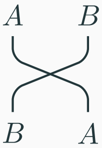
Braided Monoidal Category

$$\sigma_{B,A} \circ \sigma_{A,B} \neq \mathrm{id}_{A\otimes B} \qquad \rightsquigarrow$$



Symmetry

$$\sigma_{A,B} = \sigma_{B,A}^{-1} \colon A \otimes B \to B \otimes A \qquad \rightsquigarrow$$



Rules

- Braided: Wires can pass under/over each other

Rules

- Braided: Wires can pass under/over each other
- Symmetric: Wires can pass *through* each other

Rules

- Braided: Wires can pass under/over each other
- Symmetric: Wires can pass *through* each other

Theorem (Correctness of the Graphical Calculus)
A well-typed equation between morphisms in a braided monoidal category follows from the axioms if and only if it holds in the the graphical language up to spatial isotopy.

Theorem (Correctness of the Graphical Calculus)
A well-typed equation between morphisms in a symmetric monoidal category follows from the axioms if and only if it holds in the the graphical language up to graphical equivalence[1].

[1] probably 4-dimensional isotopy, certainly 3-dimensional isotopy plus the ability to pass wires through each other

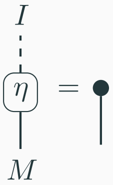
Monoids

Multiplication

$$\mu\colon M \otimes M \to M \qquad \rightsquigarrow$$

Unit

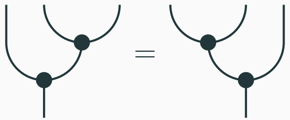
$\eta\colon I \to M \qquad \rightsquigarrow$



Monoid Laws

Associativity:

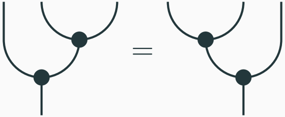
$$a(bc) = (ab)c \qquad \rightsquigarrow$$



Monoid Laws

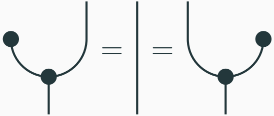
Associativity:

$$a(bc) = (ab)c \quad \rightsquigarrow$$



Unit law:

$$1a = a = a1 \quad \rightsquigarrow$$



Comonoids

Comultiplication

Just flip everything upside down!

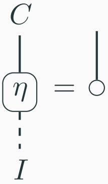
Comultiplication

Just flip everything upside down!

$$\Delta\colon C \to C \otimes C \qquad \rightsquigarrow$$

Counit

$$\varepsilon\colon C \to I \qquad \rightsquigarrow$$



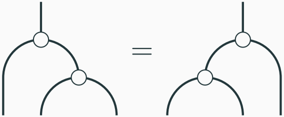
Comonoid Laws

Turn them upside down!

Comonoid Laws

Turn them upside down!

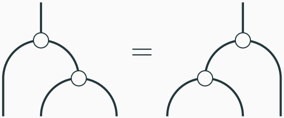
Coassociativity:



$$\rightsquigarrow \qquad (\mathrm{id} \otimes \Delta) \circ \Delta = \alpha \circ (\Delta \otimes \mathrm{id}) \circ \Delta$$

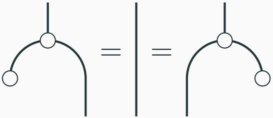
Turn them upside down!

Coassociativity:



$$\rightsquigarrow \qquad (\mathrm{id} \otimes \Delta) \circ \Delta = \alpha \circ (\Delta \otimes \mathrm{id}) \circ \Delta$$

Unit law:



$$\rightsquigarrow \qquad \begin{gathered} \lambda_C \circ (\varepsilon \otimes \mathrm{id}) \circ \Delta \\ = \mathrm{id} = \\ \rho_C \circ (\mathrm{id} \otimes \varepsilon) \circ \Delta \end{gathered}$$

Monoidal Product of Comonoids

Theorem
Work in a braided monoidal category. Let (C, Δ, ε) *and* $(C', \Delta', \varepsilon')$ *be comonoids. Then* $(C \otimes C', \tilde{\Delta}, \tilde{\varepsilon})$ *is a comonoid where*

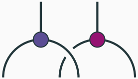
$$\tilde{\Delta} = \alpha^{-1} \circ (\mathrm{id} \otimes \alpha) \circ (\mathrm{id} \otimes \sigma \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \alpha^{-1}) \circ \alpha \circ (\Delta \otimes \Delta')$$

and

$$\tilde{\varepsilon} = \lambda_I \circ (\varepsilon \otimes \varepsilon')$$

(Without coherence morphisms (i.e., in a strict braided monoidal category) we have $\tilde{\Delta} = (\mathrm{id} \otimes \sigma \otimes \mathrm{id}) \circ (\Delta \otimes \Delta')$*, and* $\tilde{\varepsilon} = \varepsilon \otimes \varepsilon'$*)*

Proof.
Let $\Delta =$ , $\varepsilon =$ , $\Delta' =$ , and $\varepsilon' =$ . Then

$$\tilde{\Delta} =$$ 

and $\tilde{\varepsilon} =$ . These two comonoid structure sit on top of each other and don't interact, therefore the comonoid laws of each comonoid hold separately, and thus $C \otimes C'$ inherits these laws from C and C'. $\square$

Monoid Homomorphism

$f\colon M \to N$ is a monoid homomorphism if

- $f(ab) = f(a)f(b)$; and
- $f(1_M) = 1_N$;

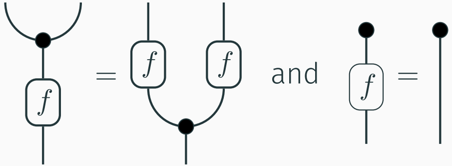
in diagrams:

Monoid Homomorphism

$f\colon M \to N$ is a monoid homomorphism if

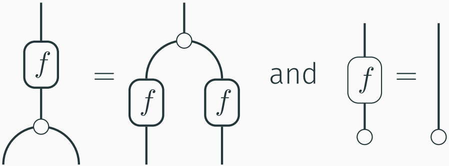
- $f(ab) = f(a)f(b)$; and
- $f(1_M) = 1_N$;

in diagrams:



Comonoid Homomorphism

$f\colon C \to D$ is a comonoid homomorphism if



Bialgebras

Bialgebra

Recall that a bialgebra, $(B, \mu, \eta, \Delta, \varepsilon)$,is

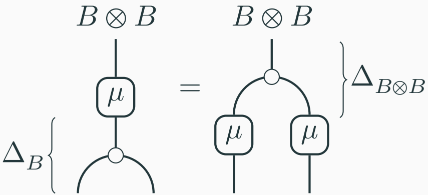
- an algebra, (B, μ, η);
- a coalgebra, (B, Δ, ε);

in such a way that $\mu\colon B \otimes B \to B$ and $\eta\colon I \to B$ are coalgebra homomorphisms.

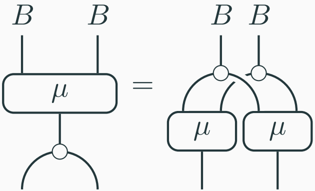
Let $\mu =$ , $\eta =$ , $\Delta =$ , and $\varepsilon =$ .

Then $(B \otimes B,$ , $)$ is a coalgebra and $(I, \lambda_I^{-1}, \mathrm{id}_I)$ is the trivial coalgebra.

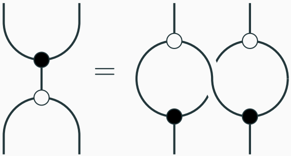
μ is a Coalgebra Homomorphism



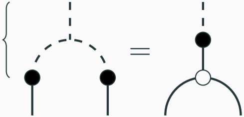
μ is a Coalgebra Homomorphism



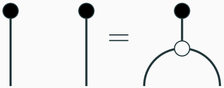
μ is a Coalgebra Homomorphism



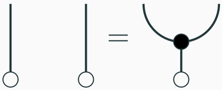
η is a Coalgebra Homomorhpism

$\Delta_I = \lambda_I^{-1}$ 

η is a Coalgebra Homomorhpism



ε is an Algebra Homomorphism



Hopf Algebras

$$\begin{tikzcd}
 & H \otimes H \arrow[rr, "\chi\otimes\mathrm{id}"] & & H \otimes H \arrow[dr, "\mu"] & \\
H \arrow[ur, "\Delta"] \arrow[dr, "\Delta"'] \arrow[rr, "\varepsilon"] & & I \arrow[rr, "\eta"] & & H \\
 & H \otimes H \arrow[rr, "\mathrm{id}\otimes\chi"'] & & H \otimes H \arrow[ur, "\mu"'] &
\end{tikzcd}$$

Hopf Algebras

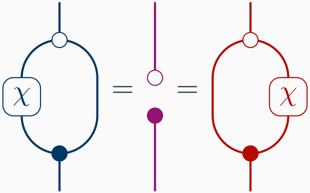
$$\begin{tikzcd}
& H \otimes H \arrow[rr, "\chi\otimes\mathrm{id}"] & & H \otimes H \arrow[dr, "\mu"] & \\
H \arrow[ur, "\Delta"] \arrow[rr, "\varepsilon"] \arrow[dr, "\Delta"'] & & I \arrow[rr, "\eta"] & & H \\
& H \otimes H \arrow[rr, "\mathrm{id}\otimes\chi"'] & & H \otimes H \arrow[ur, "\mu"'] &
\end{tikzcd}$$

$$\mu \circ (\chi \otimes \mathrm{id}) \circ \Delta$$

$$\eta \circ \varepsilon$$

$$\mu \circ (\mathrm{id} \otimes \chi) \circ \Delta$$

Hopf Algebras



Convolution Monoid

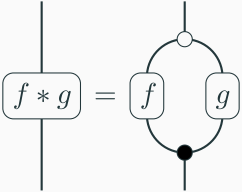
Definition

- Algebra (A, μ, η);
- Coalgebra (C, Δ, ε);

Equip the vector space $\hom_{\Bbbk}(C, A)$ with multiplication, $*$, given by

$$f * g = \mu \circ (f \otimes g) \circ \Delta$$

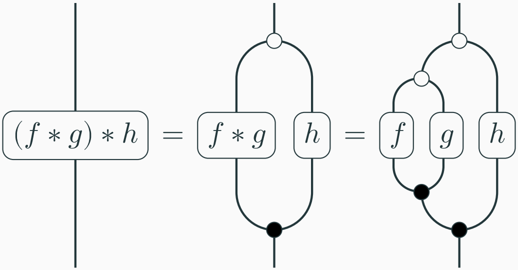
or in diagrams,



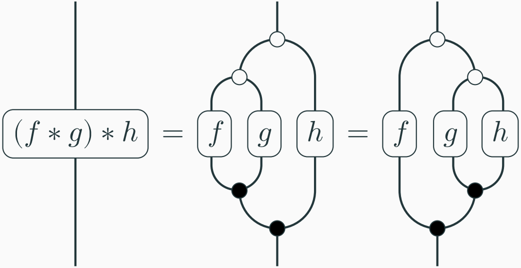
The Convolution Monoid is a Monoid

Lemma
$(\hom_{\Bbbk}(C, A), *, \eta \circ \varepsilon)$ *is a monoid.*

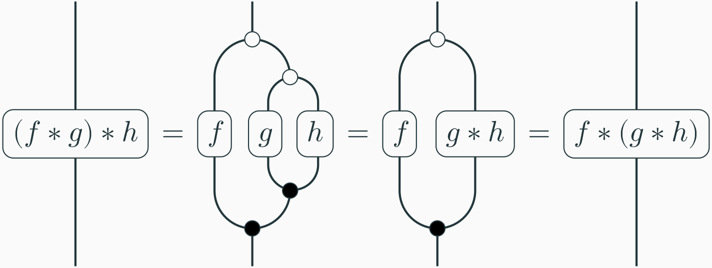
Associativity



Associativity



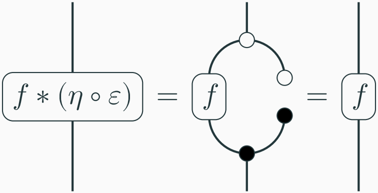
Associativity



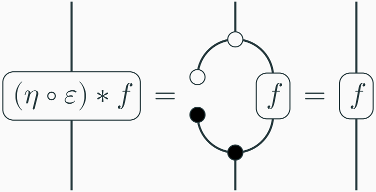
No More Horrible Commutative Diagram

$$
\begin{tikzcd}
& & C \arrow[dll, bend right, "\Delta"'] \arrow[drr, bend left, "\Delta"] \arrow[ddd, dashed, "(f * g) * h = f * (g * h)" description] & & \\
C \otimes C \arrow[r, "\Delta \otimes \mathrm{id}_C"'] \arrow[d, "(f * g) \otimes h"'] & (C \otimes C) \otimes C \arrow[rr, "\alpha"] \arrow[d, "(f \otimes g) \otimes h"'] & & C \otimes (C \otimes C) \arrow[d, "f \otimes (g \otimes h)"] & C \otimes C \arrow[l, "\mathrm{id} \otimes \Delta"] \arrow[d, "f \otimes (g * h)"] \\
A \otimes A \arrow[drr, bend right, "\mu"'] & (A \otimes A) \otimes A \arrow[l, "\mu \otimes \mathrm{id}"'] \arrow[rr, "\alpha"'] & & A \otimes (A \otimes A) \arrow[r, "\mathrm{id} \otimes \mu"] & A \otimes A \arrow[dll, bend left, "\mu"] \\
& & A & &
\end{tikzcd}
$$

Right Unit



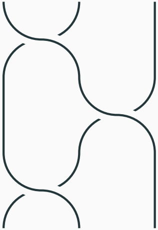
Left Unit



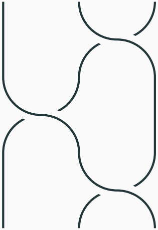
No More Horrible Commutative Diagram

$$\begin{tikzcd}
 & C \arrow[dl, "\Delta"'] \arrow[dr, "\Delta"] \arrow[dd, "\mathrm{id}"] \arrow[ddddd, bend right=90, looseness=1.5, "(\eta\circ\varepsilon)*f"'] \arrow[ddddd, bend left=90, looseness=1.5, "f*(\eta\circ\varepsilon)"] & \\
C\otimes C \arrow[d, "\varepsilon\otimes\mathrm{id}"'] & & C\otimes C \arrow[d, "\mathrm{id}\otimes\varepsilon"] \\
I\otimes C \arrow[r, "\lambda_C"] \arrow[d, "\mathrm{id}\otimes f"'] & C \arrow[d, "f"'] & C\otimes I \arrow[l, "\rho_C"'] \arrow[d, "f\otimes\mathrm{id}"] \\
I\otimes A \arrow[r, "\lambda_A"] \arrow[d, "\eta\otimes\mathrm{id}"'] & A \arrow[dd, "\mathrm{id}"] & A\otimes I \arrow[l, "\rho_A"'] \arrow[d, "\mathrm{id}\otimes\eta"] \\
A\otimes A \arrow[dr, "\mu"'] & & A\otimes A \arrow[dl, "\mu"] \\
 & A &
\end{tikzcd}$$

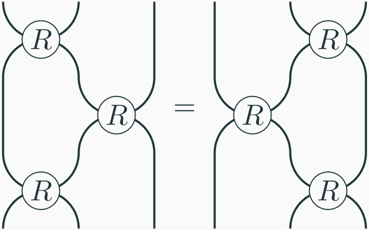
Yang Baxter Equation



$=$



R-Matrices



Resources

Resources

- Categories for Quantum Theory by Chris Heunen and Jamie Vicary (and the course Chris taught at Edinburgh) (diagrams read bottom to top)
- Categories for the Working Mathematician by Saunders Mac Lane for the coherence theorem (Part VII Chapter 2)
- Physics, Topology, Logic and Computation: A Rosetta Stone by John Baez and Mike Stay `https://arxiv.org/abs/0903.0340`