A ROUGH GUIDE TO SCHEMES

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1. Presheaves and sheaves

We will define presheaves and sheaves on a topological space X (of course we implicitly assume the topology as part of the data included in X). The open sets of X can be viewed as objects in a category $\mathbf{T}_X$ with a unique morphism $U \to V$ whenever $U \subseteq V$. For any two open sets U', U'' there are morphisms $U' \to U' \cup U'' \leftarrow U''$, so $\mathbf{T}_X$ is a *filtered category* (this is important for obtaining good properties of (co)limits).

It is also sometimes useful to consider $\mathbf{T}_X$ as ordered by *reverse inclusion* $\preccurlyeq$ where

$$V \preccurlyeq U \iff U \subseteq V.$$

For any two open sets U', U'' their intersection satisfies $U' \preccurlyeq U' \cap U''$ and $U'' \preccurlyeq U' \cap U''$, so $(\mathbf{T}_X, \preccurlyeq)$ is a *filtered ordered set* (this is important for obtaining good properties of colimits).

Now let $\mathbf{C}$ be a category which for simplicity we assume to be *concrete*, i.e., it admits a faithful functor into $\mathbf{Set}$ (so every object can be viewed as a set with additional structure and morphisms are actually functions). This is true for most of the categories we will encounter such as the categories of sets $\mathbf{Set}$, groups $\mathbf{Gp}$, abelian groups $\mathbf{AbGp}$, rings $\mathbf{Ring}$ and commutative rings $\mathbf{CoRing}$; notice in each of these categories, the one element objects are terminal. It is possible to define (pre)sheaves taking values in more general categories including non-concrete ones. Many of the categories that occur are *abelian* and lead to abelian categories of sheaves which tend not to be concrete.

Presheaves. A $\mathbf{C}$-*valued presheaf* (or $\mathbf{C}$-*presheaf*) $\mathscr{F}$ on X is a contravariant functor from $\mathbf{T}_X$ to $\mathbf{C}$, i.e., a functor $\mathscr{F} : \mathbf{T}_X^{\mathrm{o}} \to \mathbf{C}$ where $(-)^{\mathrm{o}}$ denotes the opposite category. More explicitly, for every open set U there is an object $\mathscr{F}(U)$ of $\mathbf{C}$ and whenever $V \subseteq U$ there is a morphism $\rho^U_V \colon \mathscr{F}(U) \to \mathscr{F}(V)$ so that the following rules are satisfied:

- If $W \subseteq V \subseteq U$ then $\rho^U_W = \rho^V_W \rho^U_V$.
- For any U, $\rho^U_U = 1_U$ (the identity morphism for U).

We will often specify the effect of a presheaf $\mathscr{F}$ on open sets by writing

$$U \mapsto \mathscr{F}(U).$$

If $\mathbf{C}$ is the category of sets or abelian groups or commutative rings or ... we say that a presheaf is a presheaf of sets or abelian groups or commutative rings or ....

A *morphism of* $\mathbf{C}$*-presheaves* $\mathscr{F} \to \mathscr{G}$ is a natural transformation $\mathscr{F} \xrightarrow{\cdot} \mathscr{G}$. The $\mathbf{C}$-presheaves on X clearly form a category $\mathbf{preSh}^X_{\mathbf{C}}$.

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Sheaves. Sheaves are presheaves with very strong conditions that allow for gluing of 'local' data on open covers to open sets.

A $\mathbf{C}$-presheaf $\mathscr{F}$ is a *sheaf* if for every collection of open sets $U_j \in \mathbf{T}_X$ $(j \in J)$ and $U = \bigcup_J U_j$, the following conditions hold:

- For $x, y \in \mathscr{F}(U)$,
$$\forall j \in J,\ \rho^U_{U_j}(x) = \rho^U_{U_j}(y) \quad \Longrightarrow \quad x = y.$$
- Let $z_j \in \mathscr{F}(U_j)$ $(j \in J)$ be a collection of elements such that for all $j', j'' \in J$, $\rho^{U_{j'}}_{U_{j'} \cap U_{j''}}(z_{j'}) = \rho^{U_{j''}}_{U_{j'} \cap U_{j''}}(z_{j''})$. Then there is an element $z \in \mathscr{F}(U)$ such that
$$\forall j \in J,\ \rho^U_{U_j}(z) = z_j.$$

These can be combined to give a single equivalent condition:

- Let $z_j \in \mathscr{F}(U_j)$ $(j \in J)$ be a collection of elements such that for all $j', j'' \in J$, $\rho^{U_{j'}}_{U_{j'} \cap U_{j''}}(z_{j'}) = \rho^{U_{j''}}_{U_{j'} \cap U_{j''}}(z_{j''})$. Then there is a unique element $z \in \mathscr{F}(U)$ such that
$$\forall j \in J,\ \rho^U_{U_j}(z) = z_j.$$

This can also be stated in terms of the following being an *equalizer diagram* (this would make sense in any category with products).

$$\begin{tikzcd}
\mathscr{F}(U) \arrow[r, "(\rho^U_{U_i})"] & \prod_i \mathscr{F}(U_i) \arrow[r, shift left, "(\rho^{U_i}_{U_i \cap U_j})"] \arrow[r, shift right, "(\rho^{U_j}_{U_i \cap U_j})"'] & \prod_{(i,j) \in J^2} \mathscr{F}(U_i \cap U_j)
\end{tikzcd}$$

Remark: If $\mathscr{F}$ is a sheaf then $\mathscr{F}(\varnothing)$ must be a terminal object in $\mathbf{C}$ since $\varnothing$ can be expressed as the empty union. For example: when $\mathbf{C} = \mathbf{Set}$, $\mathscr{F}(\varnothing)$ is a one element set; when $\mathbf{C} = \mathbf{Gp}$, $\mathscr{F}(\varnothing)$ is a trivial group; when $\mathbf{C} = \mathbf{CoRing}$, $\mathscr{F}(\varnothing)$ is a trivial ring.

Example 1.1 (Constant sheaves). Let A be a set. The *constant sheaf* $\underline{A}_X$ on X associated with A has for $\underline{A}_X(U)$ the set of continuous maps $U \to A$ where A is given the discrete topology. When U is connected, this is just a copy of A, but if U has path components U_j $(j \in J)$ then
$$\underline{A}_X(U) = \prod_J A.$$
If A is a group, abelian group, ring,..., then $\underline{A}_X$ is a sheaf of groups, abelian groups, rings,...

Definition and existence of sheaves. In practise it is possible to define a sheaf by specifying its values on a basis for the topology, similarly morphisms can be produced once their effect is specified on basic open sets.

Suppose that $\mathbf{B}$ is a basis for the topology on X and assume it is closed under finite intersections. Then $\mathscr{B}$ can be viewed as a defining a full subcategory of $\mathbf{T}_X$.

Proposition 1.2.
(a) *Let* $F\colon \mathbf{B}^{\circ} \to \mathbf{C}$ *be contravariant functor which satisfies*

- *For any collection of basic sets* $U_j \in \mathscr{B}$ *and elements* $z_j \in F(U_j)$ $(j \in J)$ *such that* $\bigcup_J U_j \in \mathscr{B}$ *and for all* $j', j'' \in J$, $\rho^{U_{j'}}_{U_{j'} \cap U_{j''}}(z_{j'}) = \rho^{U_{j''}}_{U_{j'} \cap U_{j''}}(z_{j''})$, *there is a unique element* $z \in F(U)$ *such that*
$$\forall j \in J,\ \rho^U_{U_j}(z) = z_j.$$

Then F extends uniquely to a sheaf $\mathscr{F}$ on X.
(b) *Suppose given two sheaves $\mathscr{F}$ and $\mathscr{G}$ on X. Then every natural transformation $\varphi\colon \mathscr{F}_{|\mathbf{B}} \dot{\to} \mathscr{G}_{|\mathbf{B}}$ between the restrictions of $\mathscr{F}$ and $\mathscr{G}$ to $\mathbf{B}$ has a unique extension to a morphism of sheaves $\tilde{\varphi}\colon \mathscr{F} \to \mathscr{G}$.*

The $\mathbf{C}$-sheaves form a full subcategory $\mathbf{Sh}_{\mathbf{C}}^{X}$ of $\mathbf{preSh}_{\mathbf{C}}^{X}$ (it is abelian if $\mathbf{C}$ is). The inclusion (forgetful) functor $(-)_\flat\colon \mathbf{Sh}_{\mathbf{C}}^{X} \to \mathbf{preSh}_{\mathbf{C}}^{X}$ has a left adjoint called the *sheafification functor* $(-)^\sharp\colon \mathbf{preSh}_{\mathbf{C}}^{X} \to \mathbf{Sh}_{\mathbf{C}}^{X}$. Hence for any presheaf $\mathscr{F}$ and sheaf $\mathscr{G}$, there is a natural bijection

$$\mathbf{preSh}_{\mathbf{C}}^{X}(\mathscr{F}, \mathscr{G}_\flat) \cong \mathbf{Sh}_{\mathbf{C}}^{X}(\mathscr{F}^\sharp, \mathscr{G}).$$

In particular there is a (natural) universal morphism of presheaves $\mathscr{F} \to (\mathscr{F}^\sharp)_\flat$ with the property that for any sheaf $\mathscr{G}$ and morphism of preheaves $\mathscr{F} \to \mathscr{G}_\flat$ there is a unique morphism of sheaves $\mathscr{F}^\sharp \to \mathscr{G}$ which makes the following diagram of morphisms of presheaves commute (where the morphisms are those just given).

$$\begin{tikzcd}
\mathscr{F} \arrow[rr] \arrow[dr] & & (\mathscr{F}^\sharp)_\flat \\
& \mathscr{G}_\flat \arrow[ur] &
\end{tikzcd}$$

Example 1.3 (Constant sheaves as sheafifications)**.** The constant sheaf $\underline{A}_X$ is the sheafification of the presheaf A_X which assigns A to each non-empty open set the set A. For any open set U, $\underline{A}_X(U)$ can also be interpreted as the set of continuous map $U \to A$ (where A has the discrete topology). This makes it clear that $\underline{A}_X(\varnothing)$ is a one element set.

Given a sheaf $\mathscr{F}$, each morphism $A \to \mathscr{F}(X)$ induces a unique morphism of presheaves $A_X \to \mathscr{F}$ which factors through a unique morphism of sheaves $\underline{A}_X \to \mathscr{F}$.

$$\begin{tikzcd}
A_X \arrow[rr] \arrow[dr] & & \mathscr{F} \\
& \underline{A}_X \arrow[ur] &
\end{tikzcd}$$

Example 1.4. If E is an initial object of $\mathbf{C}$, then the constant sheaf $\underline{E}_X$ is initial in $\mathbf{Sh}_{\mathbf{C}}^{X}$.

- $\underline{\varnothing}_X$ is initial in $\mathbf{Sh}_{\mathbf{Set}}^{X}$.
- $\underline{\{1\}}_X$ is initial in $\mathbf{Sh}_{\mathbf{Gp}}^{X}$ for any trivial group $\{1\}$. Similarly, $\underline{\{1\}}_X$ is initial in $\mathbf{Sh}_{\mathbf{AbGp}}^{X}$.
- $\underline{\mathbb{Z}}_X$ is initial in $\mathbf{Sh}_{\mathbf{Ring}}^{X}$ and $\mathbf{Sh}_{\mathbf{CoRing}}^{X}$.

Let $\mathscr{F}$ be a $\mathbf{C}$-presheaf on X. For a point $x \in X$ we can consider the open sets containing x which form a filtered directed system $(\mathbf{T}_X(x), \preccurlyeq)$ under *reverse* inclusion. The *stalk of $\mathscr{F}$ at x* is

$$\mathscr{F}_x = \operatorname*{colim}_{\mathbf{T}_X(x)} \mathscr{F} = \operatorname*{colim}_{U \in \mathbf{T}_X(x)} \mathscr{F}(U), \tag{1.1}$$

where the colimit is taken in $\mathbf{C}$, so we need to assume this makes sense, i.e., that $\mathbf{C}$ has colimits. For each U containing x there is a morphism $\rho_x^U\colon \mathscr{F}(U) \to \mathscr{F}_x$.

Here is a more explicit description of $\mathscr{F}_x$. Consider the set of all pairs (U, s) where $U \in \mathbf{T}_X(x)$ and $s \in \mathscr{F}(U)$. Define an equivalence relation $\sim$ by $(U, s) \sim (V, t)$ if and only if $\exists\, W \subseteq U \cap V$ such that $\rho_W^U(s) = \rho_W^V(t)$. Then $\mathscr{F}_x$ is the set of equivalence classes of $\sim$ and $\rho_x^U(s)$ is the equivalence class of (U, s). So the elements of $\mathscr{F}_x$ are 'germs defined on open neighbourhoods of x'.

Given a morphism of presheaves $\tau\colon \mathscr{F} \to \mathscr{G}$, for each open set U we can define the diagonal composition

$$\begin{tikzcd}
\mathscr{F}(U) \arrow[d, "\tau_U"'] \arrow[dr, dashed] & \\
\mathscr{G}(U) \arrow[r] & \mathscr{G}_x
\end{tikzcd}$$

and for $V \subseteq U$ there is a commutative diagram of solid arrows

$$\begin{tikzcd}
\mathscr{F}(U) \arrow[r] \arrow[d, "\tau_U"'] \arrow[drr] & \mathscr{F}(V) \arrow[r] \arrow[d, "\tau_V"'] \arrow[dr] & \mathscr{F}_x \arrow[d, dotted, "\tau_x"] \\
\mathscr{G}(U) \arrow[r] & \mathscr{G}(V) \arrow[r] & \mathscr{G}_x
\end{tikzcd}$$

and so there is a unique dotted arrow making the whole diagram commute. Therefore τ induces a morphism $\tau_x\colon \mathscr{F}_x \to \mathscr{G}_x$. It turns out that if $\mathscr{F}$ is a presheaf the universal morphism $\mathscr{F} \to \mathscr{F}^\sharp$ induces an isomorphism $\mathscr{F}_x \xrightarrow{\cong} \mathscr{F}_x^\sharp$ for every $x \in X$.

Example 1.5. The stalks of the constant presheaf A_X are clearly isomorphic to A, hence so is each $\underline{A}_{X,x}$.

Stalks are useful as they in effect allow us to represent values sheaves in terms of 'functions'.

Lemma 1.6. *Let $\mathscr{F}$ be a sheaf of sets on X. Then for each open set U, the function*

$$\varepsilon_U\colon \mathscr{F}(U) \to \prod_{x\in U} \mathscr{F}_x; \quad \varepsilon(t) = (\rho_x^U(t))$$

is injective.

Proof. Suppose that $\varepsilon_U(s) = \varepsilon_U(t)$ for some $s, t \in \mathscr{F}(U)$. Then for each $x \in U$, $\rho_x^U(s) = \rho_x^U(t)$ so there must be an open neighbourhood $U_x \subseteq U$ of x for which $\rho_{U_x}^U(s) = \rho_{U_x}^U(t) = r_x$. As the open sets U_x $(x \in U)$ cover U, the sheaf property of $\mathscr{F}$ implies that there is a *unique* element of $\mathscr{F}(U)$ restricting to r_x on each U_x. So $s = t$, hence ε_U is injective. $\square$

This allows us to interpret an element $t \in \mathscr{F}(U)$ as a function

$$U \to \coprod_{x\in U} \mathscr{F}_x; \quad x \mapsto \rho_x^U(t).$$

In particular, the set $\mathscr{F}(X)$ is the set of global functions of the sheaf. Similarly, elements of a stalk $\mathscr{F}_x$ can be viewed as *germs* of functions defined on small open neighbourhoods of x. This point of view is often adopted when discussing the *étale space* of the sheaf.

Naturality of presheaves with respect to continuous maps. Let $f\colon X \to Y$ be a continuous map.

Given a presheaf $\mathscr{F}$ on X we may define the *direct image* $f_*\mathscr{F}$ on Y by

$$f_*\mathscr{F}(W) = \mathscr{F}(f^{-1}W).$$

If $\mathscr{F}$ is a sheaf so is $f_*\mathscr{F}$. This defines functors $f_*\colon \mathbf{preSh}_{\mathbf{C}}^X \to \mathbf{preSh}_{\mathbf{C}}^Y$ and $f_*\colon \mathbf{Sh}_{\mathbf{C}}^X \to \mathbf{Sh}_{\mathbf{C}}^Y$.

If $\mathscr{G}$ is a presheaf on Y we define a presheaf on X by

$$U \mapsto \operatorname*{colim}_{\mathbf{T}_Y(fU)} \mathscr{G} = \operatorname*{colim}_{fU\subseteq W} \mathscr{G}(W)$$

where $\mathbf{T}_Y(fU)$ means the set of open subsets of Y containing fU under reverse inclusion $\preccurlyeq$. Even when $\mathscr{G}$ is a sheaf, this may not be, so to obtain the *preimage sheaf* $f^{-1}\mathscr{G}$ we must sheafify this presheaf.

The two functors f^{-1} and f_* are adjoint.

Proposition 1.7. *For sheaves $\mathscr{F}$ on X and $\mathscr{G}$ on Y there is a natural bijection*

$$\mathbf{Sh}_{\mathbf{C}}^{X}(f^{-1}\mathscr{G}, \mathscr{F}) \xrightarrow{\cong} \mathbf{Sh}_{\mathbf{C}}^{Y}(\mathscr{G}, f_*\mathscr{F}).$$

Algebraic properties of sheaves. Now suppose that $\mathbf{C}$ is an abelian category such as $\mathbf{AbGp}$ or the category of modules over a (commutative) ring. Let $\varphi\colon \mathscr{F} \to \mathscr{G}$ be a morphism of $\mathbf{C}$-sheaves. The presheaf

$$U \mapsto \ker \varphi_U \colon \mathscr{F}(U) \to \mathscr{G}(U)$$

is always a sheaf which we denote by $\ker\varphi$. On the other hand the presheaf

$$U \mapsto \operatorname{im} \varphi_U \colon \mathscr{F}(U) \to \mathscr{G}(U)$$

need not be a sheaf, however we can take its sheafification $\operatorname{im}\varphi$. Notice that if each φ_U is surjective then $\operatorname{im}\varphi = \mathscr{G}$.

Using these constructions, $\mathbf{Sh}_{\mathbf{C}}^{X}$ can be seen to be an abelian category.

A sequence of morphisms

$$\mathscr{F} \xrightarrow{\varphi} \mathscr{G} \xrightarrow{\theta} \mathscr{H}$$

is *exact* if $\ker\theta = \operatorname{im}\varphi$. We can extend this notion to longer sequences by requiring exactness at each object. It can be shown that this sequence is exact if an only if the induced sequences of stalks

$$\mathscr{F}_x \xrightarrow{\varphi_x} \mathscr{G}_x \xrightarrow{\theta_x} \mathscr{H}_x$$

are exact for every $x \in X$. The constant sheaf for the trivial object 0 is also denoted 0. An exact sequence

$$0 \to \mathscr{F} \xrightarrow{\varphi} \mathscr{G} \xrightarrow{\theta} \mathscr{H} \to 0$$

is called *short exact*; this is equivalent to each sequence of stalks

$$0 \to \mathscr{F}_x \xrightarrow{\varphi_x} \mathscr{G}_x \xrightarrow{\theta_x} \mathscr{H}_x \to 0$$

being short exact for every $x \in X$ (this uses the fact that filtered colimits preserve exactness).

Given a morphism $\varphi\colon \mathscr{F} \to \mathscr{G}$, the presheaf

$$U \mapsto \mathscr{G}(U)/\operatorname{im}\varphi_U = \operatorname{coker}\varphi_U$$

has sheafification $\operatorname{coker}\varphi$. Then there is an exact sequence

$$0 \to \ker\varphi \to \mathscr{F} \to \mathscr{G} \to \operatorname{coker}\varphi \to 0.$$

Given an open set U, there is a functor

$$\mathbf{Sh}_{\mathbf{C}}^{X} \to \mathbf{C}; \quad \mathscr{F} \mapsto \mathscr{F}(U).$$

Under this a short exact sequence

$$0 \to \mathscr{F} \xrightarrow{\varphi} \mathscr{G} \xrightarrow{\theta} \mathscr{H} \to 0$$

gets sent to a left exact sequence

$$0 \to \mathscr{F}(U) \xrightarrow{\varphi_U} \mathscr{G}(U) \xrightarrow{\theta_U} \mathscr{H}(U).$$

In particular we can take $U = X$ and this gives the *global sections* functor

$$\mathbf{Sh}_{\mathbf{C}}^{X} \to \mathbf{C}; \quad \mathscr{F} \mapsto \mathscr{F}(X).$$

Sheaf cohomology involves *right derived functors* of this functor which are computed using injective resolutions in $\mathbf{Sh}_{\mathbf{C}}^{X}$ (to do this requires that $\mathbf{C}$ has enough injectives).

2. LOCALLY RINGED SPACES

A *ringed space* $(X, \mathscr{O})$ is a topological space X equipped with a sheaf of commutative rings $\mathscr{O}$. A *locally ringed space* is a ringed space $(X, \mathscr{O})$ where each stalk $\mathscr{O}_x$ is a local ring.

A *morphism of ringed spaces* $(f, f^\natural)\colon (X, \mathscr{O}_X) \to (X, \mathscr{O}_Y)$ consists of continuous map $f\colon X \to Y$ and a morphism of sheaves of rings $f^\natural\colon \mathscr{O}_Y \to f_*\mathscr{O}_X$, or equivalently $f^\natural\colon f^{-1}\mathscr{O}_Y \to \mathscr{O}_X$ by Proposition 1.7. Notice that for every $x \in X$, on passing to stalks $f^\natural$ induces a morphism $f^\natural_x\colon \mathscr{O}_{Y,f(x)} \to \mathscr{O}_{X,x}$.

A *morphism of locally ringed spaces* $(f, f^\natural)\colon (X, \mathscr{O}_X) \to (X, \mathscr{O}_Y)$ is a morphism of ringed spaces such that for every $x \in X$, $f^\natural_x\colon \mathscr{O}_{Y,f(x)} \to \mathscr{O}_{X,x}$ is a local homomorphism, i.e., maps the maximal ideal of $\mathscr{O}_{Y,f(x)}$ into that of $\mathscr{O}_{X,x}$.

Given a ringed space $(X, \mathscr{O})$ an $\mathscr{O}$*-module* (pre)sheaf is a (pre)sheaf of abelian groups $\mathscr{M}$ so that for each open set U, $\mathscr{M}(U)$ is an $\mathscr{O}(U)$-module and whenever $V \subseteq U$ there is a commutative diagram of the following form.

$$\begin{tikzcd}
\mathscr{O}(U) \otimes \mathscr{M}(U) \arrow[r] \arrow[d, "\rho^U_V \otimes \rho^U_V"'] & \mathscr{M}(U) \arrow[d, "\rho^U_V"] \\
\mathscr{O}(V) \otimes \mathscr{M}(V) \arrow[r] & \mathscr{M}(V)
\end{tikzcd}$$

An alternative approach is to define the tensor product of two sheaves of abelian groups $\mathscr{A}, \mathscr{B}$ by considering the presheaf

$$U \mapsto \mathscr{A}(U) \otimes \mathscr{B}(U),$$

then defining $\mathscr{A} \otimes \mathscr{B}$ to be its sheafification. A sheaf of abelian groups $\mathscr{M}$ equipped with a morphism of sheaves of abelian groups $\mu\colon \mathscr{O} \otimes \mathscr{M} \to \mathscr{M}$ is an $\mathscr{O}$*-module* if μ satisfies appropriate associativity and unital conditions to analogous to those of a module over a ring. The unit condition depends on a morphism $\underline{\mathbb{Z}}_X \to \mathscr{O}$ and a natural isomorphism of sheaves of abelian groups

$$\underline{\mathbb{Z}}_X \otimes \mathscr{M} \xrightarrow{\cong} \mathscr{M}.$$

The category of $\mathscr{O}$-modules $\mathbf{Mod}_{(X,\mathscr{O})}$ is an abelian category, and furthermore we can define a tensor product $\mathscr{M} \otimes_{\mathscr{O}} \mathscr{N}$ of two $\mathscr{O}$-modules $\mathscr{M}$ and $\mathscr{N}$ by sheafifying the presheaf

$$U \mapsto \mathscr{M}(U) \otimes_{\mathscr{O}(U)} \mathscr{N}(U).$$

A morphism of locally ringed spaces $(f, f^\natural)\colon (X, \mathscr{O}_X) \to (X, \mathscr{O}_Y)$ induces adjoint functors $f_*\colon \mathbf{Mod}_{(X,\mathscr{O}_X)} \to \mathbf{Mod}_{(Y,\mathscr{O}_Y)}$ and $f^*\colon \mathbf{Mod}_{(Y,\mathscr{O}_Y)} \to \mathbf{Mod}_{(X,\mathscr{O}_X)}$. The first is familiar:

$$f_*\mathscr{M}(V) = \mathscr{M}(f^{-1}V)$$

and the $\mathcal{O}_Y$-module structure comes from the sequence of natural transformations

$$\mathcal{O}_Y \otimes f_*\mathcal{M} \xrightarrow{f^\natural \otimes \mathrm{id}} f_*\mathcal{O}_X \otimes f_*\mathcal{M} \xrightarrow{f_*\mathrm{mult}} f_*\mathcal{M}$$

Similarly,

$$f^*\mathcal{N} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{N}$$

where we use the morphism $f^\natural \colon f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ to define the presheaf

$$\mathcal{O}_X(U) \otimes_{f^{-1}\mathcal{O}_Y(U)} f^{-1}\mathcal{N}(U)$$

whose sheafification is $f^*\mathcal{N}$.

Proposition 2.1. *For a morphism of locally ringed spaces $(f, f^\natural) \colon (X, \mathcal{O}_X) \to (X, \mathcal{O}_Y)$, an $\mathcal{O}_X$-module $\mathcal{M}$ and an $\mathcal{O}_Y$-module $\mathcal{N}$ there are natural bijections*

$$\mathbf{Mod}_{(X,\mathcal{O}_X)}(f^*\mathcal{N}, \mathcal{M}) \cong \mathbf{Mod}_{(Y,\mathcal{O}_Y)}(\mathcal{N}, f_*\mathcal{M}).$$

3. SPECTRA OF COMMUTATIVE RINGS AND AFFINE SCHEMES

All rings are assumed commutative and unital, with unital homomorphisms; we allow trivial (1-element) rings which we usually just denote by 0. These form a category **CoRing** whose initial objects are isomorphic to $\mathbb{Z}$ and whose terminal objects are trivial. This category has coproducts, products, colimits and limits. It has a symmetric monoidal structure defined using its coproduct $\otimes = \otimes_{\mathbb{Z}}$.

Localisation. Let A be a commutative ring and let $S \subseteq A$ be a *multiplicative set*, i.e., $1 \in S$ and S is closed under multiplication. Then there is a ring homomorphism $i \colon A \to A[S^{-1}]$ which has the following universal property: Given any ring homomorphism $f \colon A \to B$ so that $fS \subseteq A^\times$, there is a unique homomorphism $\tilde{f} \colon A[S^{-1}] \to B$ making the following diagram commute.

$$\begin{tikzcd}
A \arrow[rr, "i"] \arrow[dr, "f"'] & & A[S^{-1}] \arrow[dl, "\tilde{f}"] \\
& B &
\end{tikzcd}$$

The ring $A[S^{-1}]$ can be constructed explicitly so that its elements are fractions a/s where $a \in A$ and $s \in S$, subject to the rule that

$$\frac{a_1}{s_1} = \frac{a_2}{s_2} \iff \exists t \in S \text{ such that } (a_1 s_2 - a_2 s_1)t = 0.$$

Notice that there is a one-to-one correspondence between prime ideals $\mathfrak{q} \lhd A[S^{-1}]$ and satisfying prime ideals $\mathfrak{p} \lhd A$ satisfying $\mathfrak{p} \cap S = \varnothing$, this is given by

$$\mathfrak{q} \longleftrightarrow i^{-1}\mathfrak{q}.$$

A very important example occurs when $\mathfrak{p} \lhd A$ is prime and $S = A \setminus \mathfrak{p}$; in this case we write $A_\mathfrak{p} = A[S^{-1}]$. When A is an integral domain, $(0) \lhd A$ is prime and $A_{(0)} = \mathrm{Fr}(A)$, the *field of fractions* (or *quotients*) of A.

At the other extreme, if T is the set of all elements of A which are not zero-divisors then $A[T^{-1}]$ is the *total ring of fractions* of A. Its prime ideals correspond to prime ideals of A which contain only zero-divisors.

3.1. **The spectrum as a topological space.** Let A be a commutative ring. The (*prime*) *spectrum* of A is the set of all prime ideals of A,

$$\operatorname{spec} A = \{\mathfrak{p} : \mathfrak{p} \lhd A \text{ prime}\},$$

which we interpret as $\varnothing$ if $A = 0$. The *maximal ideal spectrum* is the subset

$$\text{max-spec}\, A = \{\mathfrak{m} : \mathfrak{m} \lhd A \text{ maximal}\} \subseteq \operatorname{spec} A.$$

A ring homomorphism $f \colon A \to B$ induces a function

$$f^* \colon \operatorname{spec} B \to \operatorname{spec} A; \quad \mathfrak{q} \mapsto f^{-1}\mathfrak{q}.$$

Of course this does not usually restrict to a function $\text{max-spec}\, B \to \text{max-spec}\, A$. We can give $\operatorname{spec} A$ the *Zariski topology*: the closed sets have form

$$\mathrm{V}(\mathfrak{a}) = \mathrm{V}_A(\mathfrak{a}) = \mathrm{V}_{\operatorname{spec} A}(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{spec} A : \mathfrak{a} \subseteq \mathfrak{p}\}$$

where $\mathfrak{a} \lhd A$; we interpret this as meaning $\mathrm{V}(A) = \varnothing$. Notice that for a prime ideal $\mathfrak{p}$, the closure of $\{\mathfrak{p}\}$ is

$$\overline{\{\mathfrak{p}\}} = \{\mathfrak{q} \in \operatorname{spec} A : \mathfrak{p} \subseteq \mathfrak{q}\};$$

hence $\{\mathfrak{p}\}$ is closed if and only if $\mathfrak{p}$ is maximal. A prime $\mathfrak{p}$ is *generic* if $\overline{\{\mathfrak{p}\}} = \operatorname{spec} A$; for example, if A is an integral domain, (0) is generic.

There is a basis of open sets of the form

$$\mathrm{D}(u) = \mathrm{D}_A(u) = \mathrm{D}_{\operatorname{spec} A}(u) = \{\mathfrak{p} \in \operatorname{spec} A : u \notin \mathfrak{p}\}.$$

Notice that there is a bijection

$$\mathrm{D}_A(u) \leftrightarrow \operatorname{spec} A[u^{-1}],$$

which is actually a homeomorphism if we view the left hand side as a subspace of $\operatorname{spec} A$. Here $A[u^{-1}]$ denotes the localisation of A with respect to the multiplicative set of powers of u.

The Zariski topology is rarely Hausdorff, but it is always quasi-compact.

Proposition 3.1. *For any commutative ring A, $\operatorname{spec} A$ is quasi-compact.*

Proof. It is sufficient to show that any covering of $\operatorname{spec} A$ by a collection of basic open sets contains a finite subcover. So suppose that $\operatorname{spec} A = \bigcup_{j \in J} \mathrm{D}(u_j)$. Let $\mathfrak{u} = (u_j : j \in J)$ be the ideal generated by the u_j (this might be A itself).

For any prime ideal $\mathfrak{p} \in \operatorname{spec} A$ there is some j for which $\mathfrak{p} \in \mathrm{D}(u_j)$ and so $u_j \notin \mathfrak{p}$, hence $\mathfrak{u} \nsubseteq \mathfrak{p}$.

If $\mathfrak{u} \neq A$, then $\mathfrak{u}$ is contained in some maximal ideal $\mathfrak{m} \neq A$, but as $\mathfrak{m}$ is prime this gives a contradiction; therefore $\mathfrak{u} = A$. We can find $j_1, \dots, j_k \in J$ and $a_1, \dots, a_k \in A$ so that

$$a_1 u_{j_1} + \cdots + a_k u_{j_k} = 1.$$

Now for any prime ideal $\mathfrak{p} \in \operatorname{spec} A$, since $1 \notin \mathfrak{p}$, for at least one r, $u_{j_r} \notin \mathfrak{p}$, hence $\mathfrak{p} \in \mathrm{D}(u_{j_r})$. Therefore

$$\operatorname{spec} A = \bigcup_{1 \leqslant r \leqslant k} \mathrm{D}(u_{j_r}). \qquad \square$$

One technical point that is worth mentioning:

$$\mathrm{V}(\mathfrak{b}) \subseteq \mathrm{V}(\mathfrak{a}) \quad \iff \quad \sqrt{\mathfrak{a}} \subseteq \sqrt{\mathfrak{b}} \tag{3.1}$$

where

$$\sqrt{\mathfrak{c}} = \{x \in A : \exists m \geqslant 1 \text{ s.t. } x^m \in \mathfrak{c}\,\}$$

is the nilradical of $\mathfrak{c} \lhd A$; it is a standard result that

$$\sqrt{\mathfrak{c}} = \bigcap_{\mathfrak{c} \subseteq \mathfrak{p} \in \operatorname{spec} A} \mathfrak{p}.$$

It follows that

$$\mathrm{V}(\mathfrak{b}) = \mathrm{V}(\mathfrak{a}) \quad \iff \quad \sqrt{\mathfrak{a}} = \sqrt{\mathfrak{b}}.$$

Taking $\mathfrak{a} = (u)$ and $\mathfrak{b} = (v)$ to be principal ideals, these gives

$$\mathrm{D}(u) \subseteq \mathrm{D}(v) \quad \iff \quad \sqrt{(u)} \subseteq \sqrt{(v)}, \tag{3.2}$$

and this implies that for some $k \geqslant 1$ and $z \in A$, $u^k = zv$, hence there is a ring homomorphism $\rho^v_u\ A[v^{-1}] \to A[u^{-1}]$ making the following diagram commute.

$$\begin{tikzcd}
\mathrm{D}_A(v) \arrow[r, leftrightarrow] \arrow[d, "\rho^v_u"'] & \operatorname{spec} A[v^{-1}] \arrow[d, "(\rho^v_u)^*"] \\
\mathrm{D}_A(u) \arrow[r, leftrightarrow] & \operatorname{spec} A[u^{-1}]
\end{tikzcd}$$

Theorem 3.2. *The prime spectrum defines a contravariant functor*

$$\operatorname{spec}\colon \mathbf{CoRing}^{\mathrm{o}} \to \mathbf{Top}.$$

Proof. For $\mathfrak{a} \lhd A$,

$$(f^*)^{-1}\mathrm{V}_A(\mathfrak{a}) = (f^*)^{-1}\{\mathfrak{p} \in \operatorname{spec} A : \mathfrak{a} \subseteq \mathfrak{p}\} = \{f^{-1}\mathfrak{p} \in \operatorname{spec} B : \mathfrak{a} \subseteq \mathfrak{p}\} = \mathrm{V}_B(f^{-1}\mathfrak{a})$$

by the Correspondence Theorem. So f^* is continuous. $\square$

This functor spec is not very discerning. For example, for any field K,

$$\operatorname{spec} K = \text{max-spec}\, K = \{(0)\},$$

so for a homomorphism of fields $f\colon K \to K'$, $f^*\colon \operatorname{spec} K' \to \operatorname{spec} K$ is a homeomorphism. For a commutative ring let $\sqrt{0} \lhd A$ be its nilradical which is the set of nilpotent elements, or equivalently

$$\sqrt{0} = \bigcap_{\mathfrak{p} \in \operatorname{spec} A} \mathfrak{p}.$$

Then

$$\operatorname{spec} A = \operatorname{spec} A/\sqrt{0},$$

so spec doesn't see nilpotents. However, by adding more structure we can overcome such defects.

3.2. **The spectrum as a locally ringed space.** Let A be a commutative ring with $\operatorname{spec} A$ given its Zariski topology. We define a sheaf of commutative rings $\mathcal{O}_A$ on $\operatorname{spec} A$ as follows. For each basic open set $\mathrm{D}_A(u)$, let

$$\mathcal{O}_A(\mathrm{D}_A(u)) = A[u^{-1}].$$

Since

$$\mathrm{D}_A(u) \cap \mathrm{D}_A(v) = \mathrm{D}_A(uv),$$

and there are evident ring homomorphisms

$$A[u^{-1}] \to A[(uv)^{-1}] \leftarrow A[v^{-1}].$$

By Proposition 1.2 this extends to a sheaf.

Theorem 3.3. *There is a unique sheaf of rings $\mathcal{O}_A$ on $\operatorname{spec} A$ given on basic open sets by*

$$\mathcal{O}_A(\mathrm{D}_A(u)) = A[u^{-1}]$$

for all $u \in A$. For each $\mathfrak{p} \in \operatorname{spec} A$, the stalk at $\mathfrak{p}$ is the local ring

$$\mathcal{O}_{A,\mathfrak{p}} = A_{\mathfrak{p}},$$

hence $(\operatorname{spec} A, \mathcal{O}_A)$ is a locally ringed space.

Proof. To compute the stalk at a prime ideal $\mathfrak{p}$, notice that $\mathfrak{p} \in \mathrm{D}_A(u)$ precisely when $u \notin \mathfrak{p}$. Since these $\mathrm{D}_A(u)$ form a basis of open neighbourhoods of $\mathfrak{p}$, so

$$\mathcal{O}_{A,\mathfrak{p}} \cong \operatorname*{colim}_{\mathfrak{p} \in \mathrm{D}_A(u)} A[u^{-1}].$$

By definition of colimit, the evident homomorphism $A \to \operatorname{colim}_{\mathfrak{p} \in \mathrm{D}_A(u)} A[u^{-1}]$ satisfies the universal property for the localisation of A with respect to the multiplicative set $A \setminus \mathfrak{p}$, therefore

$$\operatorname*{colim}_{\mathfrak{p} \in \mathrm{D}_A(u)} A[u^{-1}] \cong A_{\mathfrak{p}}$$

and so $\mathcal{O}_{A,\mathfrak{p}} \cong A_{\mathfrak{p}}$. ☐

We will set $\operatorname{Spec} A = (\operatorname{spec} A, \mathcal{O}_A)$; this is the prototypical *affine scheme*. Notice that the set of global sections $\mathcal{O}_A(\operatorname{spec} A) = A$ recovers the ring A.

Definition 3.4. An *affine scheme* is a locally ringed space isomorphic to some $\operatorname{Spec} A$.

Associated to a point $\mathfrak{p} \in \operatorname{spec} A$ are two quotient rings, namely the integral domain $A/\mathfrak{p}$ and the residue field $\kappa(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}}$ where $\mathfrak{m}_{\mathfrak{p}} \lhd A_{\mathfrak{p}}$ is the unique maximal ideal generated by the image of $\mathfrak{p}$ under the localisation homomorphism $A \to A_{\mathfrak{p}}$. There is an evident commutative diagram of solid arrows

$$\begin{tikzcd}
A \arrow[rr] \arrow[d] & & A_{\mathfrak{p}} \arrow[d] \\
A/\mathfrak{p} \arrow[rr] \arrow[dr] & & \kappa(\mathfrak{p}) \\
& \operatorname{Fr}(A/\mathfrak{p}) \arrow[ur, dashed, "\exists!"'] &
\end{tikzcd}$$

and a unique extension to the larger diagram, where $\operatorname{Fr}(D)$ denotes the field of fractions (or quotients) of an integral domain D. It is straightforward to see that the dashed arrow is an isomorphism, i.e., $\operatorname{Fr}(A/\mathfrak{p}) \cong \kappa(\mathfrak{p})$. The one element space $\operatorname{spec} \kappa(\mathfrak{p})$ gives rise to the affine scheme

$\operatorname{Spec}\kappa(\mathfrak{p})$ and the above homomorphism $A \to \kappa(\mathfrak{p})$ induces a continuous map $\operatorname{spec}\kappa(\mathfrak{p}) \to \operatorname{spec} A$ whose image is $\{\mathfrak{p}\}$ which is closed if $\mathfrak{p}$ is maximal. When $\mathfrak{p}$ is maximal can think of the morphism of schemes $\operatorname{Spec}\kappa(\mathfrak{p}) \to \operatorname{Spec} A$ as picking out a 'geometric point' in $\operatorname{Spec} A$.

On the other hand, $A \to A/\mathfrak{p}$ induces a continuous injection $\operatorname{spec} A/\mathfrak{p} \to \operatorname{spec} A$ whose image is the closed set $\mathrm{V}(\mathfrak{p})$ (actually it is a homeomorphism onto this image). In each case we have a morphism of schemes,

$$\operatorname{Spec}\kappa(\mathfrak{p}) \to \operatorname{Spec} A, \quad \operatorname{Spec} A/\mathfrak{p} \to \operatorname{Spec} A.$$

A ring homomorphism $f\colon A \to B$ induces a continuous map $f^*\colon \operatorname{spec} B \to \operatorname{spec} A$. For a basic open set $\mathrm{D}(u) \subseteq \operatorname{spec} A$,

$$(f^*)^{-1}\mathrm{D}(u) = \{\mathfrak{q} \in \operatorname{spec} B : f(u) \notin \mathfrak{q}\} = \mathrm{D}(f(u)),$$

and it follows that there is an induced ring homomorphism $f^\natural$ making the following diagram commute

$$\begin{tikzcd}
\mathcal{O}_A(\mathrm{D}(u)) \arrow[r, "f^\natural"] \arrow[dd, equal] & (f^*)_*\mathcal{O}_B(\mathrm{D}(u)) \arrow[d, equal] \\
 & \mathcal{O}_B((f^*)^{-1}\mathrm{D}(u)) \arrow[d, equal] \\
A[u^{-1}] \arrow[r, "\widetilde{f}"] & B[f(u)^{-1}]
\end{tikzcd}$$

and where $\widetilde{f}$ is induced by f. This extends to give a morphism of sheaves of rings $f^\natural\colon \mathcal{O}_A \to (f^*)_*\mathcal{O}_B$, hence $(f^*, f^\natural)\colon \operatorname{Spec} B \to \operatorname{Spec} A$ is a morphism of ringed spaces. On stalks we have local ring homomorphisms.

$$\begin{tikzcd}
\mathcal{O}_{A,f^*\mathfrak{q}} \arrow[r, "f^\natural_{\mathfrak{q}}"] \arrow[d, equal] & \mathcal{O}_{B,\mathfrak{q}} \arrow[d, equal] \\
A_{f^{-1}\mathfrak{q}} \arrow[r, "f'"] & B_{\mathfrak{q}}
\end{tikzcd}$$

A quotient homomorphism $q\colon A \to A/\mathfrak{a}$ induces a morphism of affine schemes $\operatorname{Spec} A/\mathfrak{a} \to \operatorname{Spec} A$ where $q^*\colon \operatorname{spec} A/\mathfrak{a} \to \operatorname{spec} A$ is injective and has closed image

$$q^* \operatorname{spec} A/\mathfrak{a} = \mathrm{V}(\mathfrak{a}).$$

Example 3.5. A non-trivial idempotent $a \in A$ induces an epimorphism

$$A \to Ae; \quad x \mapsto ae,$$

inducing an embedding $\operatorname{Spec} Ae \to \operatorname{Spec} A$ which is both closed and open. Indeed, together with the complementary idempotent $1-e$, the Chinese Remainder Theorem gives an isomorphism

$$A \xrightarrow{\cong} Ae \times A(1-e)$$

and then

$$\operatorname{spec} A \doteq \operatorname{spec} Ae \amalg \operatorname{spec} A(1-e)$$

where $\operatorname{spec} Ae$ and $\operatorname{spec} A(1-e)$ are both clopen subsets. Notice that we also have

$$A[(1-e)^{-1}] \cong Ae, \quad A[e^{-1}] \cong A(1-e),$$

so these are also localisations of A.

Example 3.6. Recall the ring of Gaussian integers $\mathbb{Z}[i]$; this is a principal ideal domain. Here are its non-zero prime ideals (they are all maximal):

- $(1+i) \lhd \mathbb{Z}[i]$ and $(1+i) \cap \mathbb{Z} = (2) \lhd \mathbb{Z}$;
- if $p \equiv 3 \bmod 4$ is a prime then $(p) \lhd \mathbb{Z}[i]$ is prime with $(p) \cap \mathbb{Z} = (p) \lhd \mathbb{Z}$;
- if $p \equiv 1 \bmod 4$ is a prime then there are integers a, b satisfying $a^2 + b^2 = p$ and there are two distinct prime ideals $(a \pm bi) \lhd \mathbb{Z}[i]$ satisfying $(a \pm bi) \cap \mathbb{Z} = (p) \lhd \mathbb{Z}$.

The inclusion $\mathbb{Z} \to \mathbb{Z}[i]$ induces a finite covering map $\operatorname{spec} \mathbb{Z}[i] \to \operatorname{spec} \mathbb{Z}$ with

$$\begin{aligned} (1+i) &\mapsto (2), & & \\ (p) &\mapsto (p) & & \text{if } p \equiv 3 \bmod 4, \\ (a \pm bi) &\mapsto (p) & & \text{if } p \equiv 1 \bmod 4 \text{ and } p = a^2 + b^2. \end{aligned}$$

Notice that $(2) \lhd \mathbb{Z}[i]$ is not prime, and in fact $(1+i)^2 = (2)$ so 2 is *ramified* in $\mathbb{Z}[i]$; also, if $p \equiv 1 \bmod 4$ then $(a+bi)(a-bi) = (p)$ so p is *split* in $\mathbb{Z}[i]$.

Theorem 3.7. *The functor* $\operatorname{Spec}\colon \mathbf{CoRing}^{\circ} \to \mathbf{AffSch}$ *is an equivalence of categories.*

Furthermore, Spec *sends finite coproducts to products and finite products to coproducts.*

Proof. The main point to observe is that a morphism of affine schemes $\operatorname{Spec} B \to \operatorname{Spec} A$ determines a ring homomorphism

$$\begin{tikzcd} A \arrow[r, "="'] \arrow[rrr, bend left, "f"] & \mathcal{O}_A(\operatorname{spec} A) \arrow[r] & \mathcal{O}_B(\operatorname{spec} B) \arrow[r, "="'] & B \end{tikzcd}$$

which induces $(f^*, f^{\natural})$. This shows that there is a natural bijection

$$\mathbf{AffSch}(\operatorname{Spec} B, \operatorname{Spec} A) \xrightarrow{\cong} \mathbf{CoRing}(A, B),$$

therefore Spec provides an equivalence of categories. In particular this implies that any morphism of affine schemes $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is determined by the associated ring homomorphism $\mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$.

The statements about products and coproducts can be verified by construction. ☐

Corollary 3.8. *Let* $(X, \mathcal{O})$ *and* $\operatorname{Spec} A$ *be affine schemes. Then there is a natural bijection*

$$\mathbf{AffSch}((X, \mathcal{O}), \operatorname{Spec} A) \xrightarrow{\cong} \mathbf{CoRing}(A, \mathcal{O}(X)).$$

These results show that Spec captures more that just the topology of spec. For example, given two fields K, L, a morphism $\operatorname{Spec} L \to \operatorname{Spec} K$ corresponds to ring homomorphism $K \to L$, and an isomorphism corresponds to ring isomorphism. For any commutative ring A, there is always a closed embedding $\operatorname{spec} A/\sqrt{0} \to \operatorname{spec} A$ induced by the quotient homomorphism $A \to A/\sqrt{0}$, but the induced morphism of $\operatorname{Spec} A/\sqrt{0} \to \operatorname{Spec} A$ is only an isomorphism when $\sqrt{0} = (0)$, i.e., A is *reduced.*

The category of **CoRing** has finite coproducts. The coproduct of A and B is just $A \otimes B$: given ring homomorphisms $f\colon A \to C$ and $g\colon B \to C$ there is a unique ring homomorphism $f \otimes g\colon A \otimes B \to C$ making the following diagram commute.

$$\begin{tikzcd} A \arrow[r, "\cong"] \arrow[rrd, bend right, "f"'] & A \otimes \mathbb{Z} \arrow[r, "\text{unit}"] & A \otimes B \arrow[d, "f \otimes g"] & \mathbb{Z} \otimes B \arrow[l, "\text{unit}"'] & B \arrow[l, "\cong"'] \arrow[lld, bend left, "g"] \\ & & C & & \end{tikzcd}$$

The functor Spec sends coproducts to products so $\operatorname{Spec} A \times \operatorname{Spec} B = \operatorname{Spec} A \otimes B$. But it is not always true that the underlying space of $\operatorname{Spec} A \times \operatorname{Spec} B$, $\operatorname{spec} A \otimes B$, is homeomorphic to $\operatorname{spec} A \times \operatorname{spec} B$. For example, if $A = \mathbb{Z}/m$ and $B = \mathbb{Z}/n$,

$$\begin{aligned} \operatorname{spec} \mathbb{Z}/m \otimes \mathbb{Z}/n &= \operatorname{spec} \mathbb{Z}/\gcd(m,n) = \{p : p \text{ prime}, p \mid m \text{ and } p \mid n\}, \\ \operatorname{spec} \mathbb{Z}/m \times \operatorname{spec} \mathbb{Z}/n &= \{(p,q) : p, q \text{ prime}, p \mid m,\ q \mid n\}. \end{aligned}$$

When m, n are coprime, $\mathbb{Z}/m \otimes \mathbb{Z}/n = 0$ so $\operatorname{spec} \mathbb{Z}/m \otimes \mathbb{Z}/n = \varnothing$ but $\operatorname{spec} \mathbb{Z}/m \times \operatorname{spec} \mathbb{Z}/n \neq \varnothing$.

We can also define pushouts in **CoRing**. The pushout of the ring homomorphisms $p\colon R \to A$ and $q\colon R \to B$ is $A \otimes_R B$. As Spec sends pushouts to pullbacks,

$$\operatorname{Spec} A \times_{\operatorname{Spec} R} \operatorname{Spec} B = \operatorname{Spec} A \otimes_R B.$$

$$\begin{tikzcd} \operatorname{Spec} A \otimes_R B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \operatorname{Spec} B \arrow[d, "q^*"] \\ \operatorname{Spec} A \arrow[r, "p^*"'] & \operatorname{Spec} R \end{tikzcd}$$

Notice that a morphism of affine schemes $\operatorname{Spec} A \to \operatorname{Spec} R$ is equivalent to a ring homomorphism $R \to A$, so A is an R-algebra. This allows us to relativise affine schemes. For $S = \operatorname{Spec} R$, a morphism of affine schemes $p\colon T \to S$ is an *affine scheme over S*; given another such $p'\colon T' \to S$, a *morphism of affine schemes over S* is a morphism $f\colon T \to T'$ so that the following diagram commutes.

$$\begin{tikzcd} T \arrow[rr, "f"] \arrow[dr, "p"'] & & T' \arrow[dl, "p'"] \\ & S & \end{tikzcd}$$

If $T = \operatorname{Spec} A$ and $T' = \operatorname{Spec} A'$ then f is induced by a ring homomorphism $A' \to A$ which makes the following diagram commute,

$$\begin{tikzcd} & R \arrow[dl] \arrow[dr] & \\ A' \arrow[rr] & & A \end{tikzcd}$$

i.e., it is a homomorphism of R-algebras. We can define the category of affine schemes over S to be the subcategory $\mathbf{AffSch}_S$ of **AffSch** whose objects are schemes over S and whose morphisms are morphisms over S. There is a forgetful functor

$$\mathbf{AffSch}_S \to \mathbf{AffSch}; \quad (T \to S) \mapsto T,$$

and this has a left adjoint as we will see. Notice that if $S = \operatorname{Spec} \mathbb{Z}$ then $\mathbf{AffSch}_{\operatorname{Spec} \mathbb{Z}} = \mathbf{AffSch}$ since every commutative ring is a $\mathbb{Z}$-algebra because it has a unit homomorphism $\mathbb{Z} \to A$ which induces a morphism $\operatorname{Spec} A \to \operatorname{Spec} \mathbb{Z}$. It is normal to suppress mention of $\mathbb{Z}$ or $\operatorname{Spec} \mathbb{Z}$ in notation.

Another useful construction is to take an affine scheme over S say $p\colon T \to S$, and a morphism $g\colon S' \to S$. Then we can form the pullback $S' \times_S T$ which is an affine scheme over S'.

$$\begin{tikzcd}
S' \times_S T \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & T \arrow[d, "p"] \\
S' \arrow[r, "g"'] & S
\end{tikzcd}$$

If $T = \operatorname{Spec} A$, $S = \operatorname{Spec} R$ and $S' = \operatorname{Spec} R$ we have

$$S' \times_S T = \operatorname{Spec} R' \otimes_R A.$$

This process of turning an R-algebra into an R'-algebra is called *base change.*

Base change over $\mathbb{Z}$ provides a left adjoint for the forgetful functor $\mathbf{AffSch}_S \to \mathbf{AffSch}$ where $S = \operatorname{Spec} R$. Given $T \in \mathbf{AffSch}$ we can form $S \times_{\operatorname{Spec} \mathbb{Z}} T$. When $T = \operatorname{Spec} A$,

$$S \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} A = S \times \operatorname{Spec} A = \operatorname{Spec}(R \otimes A)$$

where $R \otimes A$ is clearly an R-algebra or equivalently there is a morphism $S \times \operatorname{Spec} A \to S$.

Theorem 3.9. *Base change over* $\mathbb{Z}$ *provides a right adjoint to the forgetful functor* $\mathbf{AffSch}_S \to \mathbf{AffSch}$, *so for an affine scheme* V *and a scheme* U *over* S *there is a natural bijection*

$$\mathbf{AffSch}_S(U \to S, S \times V \to S) \cong \mathbf{AffSch}(U, V).$$

Underlying this bijection is a natural bijection

$$\mathrm{CoAlg}_R(R \otimes A, B) \cong \mathbf{CoRing}(A, B)$$

for a commutative ring A and a commutative R-algebra $R \to B$.

Notice that if $\mathfrak{p} \in \operatorname{spec} R$ then there is a morphism of affine schemes $\operatorname{Spec} \kappa(\mathfrak{p}) \to \operatorname{Spec} R$ and we can do base change from R to the field $\kappa(\mathfrak{p})$ by forming $\operatorname{Spec} \kappa(\mathfrak{p}) \times_{\operatorname{Spec} R} T$. Thus if $T = \operatorname{Spec} A$ where A is an R-algebra,

$$\operatorname{Spec} \kappa(\mathfrak{p}) \times_{\operatorname{Spec} R} \operatorname{Spec} A = \operatorname{Spec} \kappa(\mathfrak{p}) \otimes_R A.$$

So we can pull back an affine scheme to the residue field at each prime ideal; this amounts to looking at points of T above the residue fields; when $\mathfrak{p} \lhd R$ is maximal, this means looking at points above geometric points of $\operatorname{Spec} R$.

When $R = \mathbb{Z}$, there are two types of prime ideals, namely (0) and (p) for a prime number p. These have residue fields $\kappa(0) = \mathbb{Q}$ and $\kappa(p) = \mathbb{F}_p$. For a commutative ring A,

$$\operatorname{Spec} \kappa(0) \times \operatorname{Spec} A = \operatorname{Spec}(\mathbb{Q} \otimes A), \quad \operatorname{Spec} \kappa(p) \times \operatorname{Spec} A = \operatorname{Spec}(\mathbb{F}_p \otimes A) = \operatorname{Spec}(A/(p)).$$

Affine n-space. Let R be a commutative ring and $S = \operatorname{Spec} R$. Then *affine n-space* over S (or over R) is the affine scheme

$$\mathrm{Aff}^n_S = \mathrm{Aff}^n_R = \operatorname{Spec} R[X_1, \ldots, X_n].$$

The inclusion of constants $R \to R[X_1, \ldots, X_n]$ makes the polynomial ring into an R-algebra so this is a scheme over S. For any affine scheme T over S,

$$\mathrm{Aff}_S(T) = \mathbf{AffSch}_S(T, \mathrm{Aff}^n_S)$$

is the set of T*-valued points* of Aff^n_R. If $T = \operatorname{Spec} A$ for an R-algebra A,

$$\mathrm{Aff}^n_R(\operatorname{Spec} A) = \mathrm{CoAlg}_R(R[X_1, \ldots, X_n], A) \cong A^n$$

since an R-algebra homomorphism $\varphi\colon R[X_1, \ldots, X_n] \to A$ is uniquely determined the vector $(\varphi(X_1), \ldots, \varphi(X_n)) \in A^n$. Classical algebraic geometry focuses on this situation for $R = K$ a field (often algebraically closed) where $\mathrm{Aff}^n_K(K) \cong K^n$.

This idea of taking an affine scheme U over S and then considering its set of T*-points* $\mathbf{AffSch}_S(T, U)$ for any scheme T over S leads to the idea that a scheme defines a functor $\mathbf{AffSch}_S(-, U)$. When $S = \operatorname{Spec} R$, $U = \operatorname{Spec} A$ and $T = \operatorname{Spec} B$ for R-algebras A, B,

$$\mathbf{AffSch}_S(T, U) \cong \mathrm{CoAlg}_R(A, B)$$

where $\mathrm{CoAlg}_R(A, -)$ is a covariant functor defined on CoAlg_R.

4. GLOBAL SCHEMES

A locally ringed space $(X, \mathcal{O})$ is a *(global) scheme* if every $x \in X$ has an open neighbourhood U and an isomorphism of affine schemes

$$(U, \mathcal{O}_{|U}) \xrightarrow[\cong]{(f, f^\natural)} \operatorname{Spec} A$$

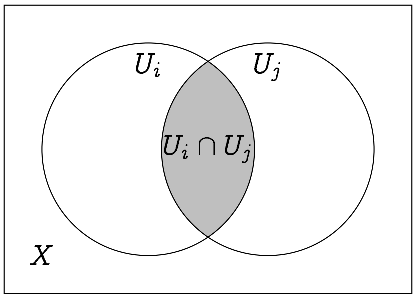
for some commutative ring A (which depends on x). In other words, each x has an affine neighbourhood so $(X, \mathcal{O})$ is *locally affine*.

So a (global) scheme is a locally ringed space together with an open cover U_j $(j \in J)$ where $(U_j, \mathcal{O}_{|U_j}) \cong \operatorname{Spec} A_j$ for suitable rings A_j.

When $U_i \cap U_j \neq \varnothing$, each element $x \in U_i \cap U_j$ has an open neighbourhood $V_{ij} \subseteq U_i \cap U_j$ for which

$$\operatorname{Spec} A_i[u_i^{-1}] \cong (V_{ij}, \mathcal{O}_{|V_{ij}}) \cong \operatorname{Spec} A_j[u_j^{-1}]$$

for suitable elements $u_i \in A_i$ and $u_j \in A_j$.



there is a commutative diagram of morphisms of affine schemes

$$\begin{tikzcd}
(U_i, \mathcal{O}_{|U_i}) \arrow[d, "\cong"'] & & (V_{ij}, \mathcal{O}_{|V_{ij}}) \arrow[ll] \arrow[rr] \arrow[dr, leftrightarrow, "\cong"] & & (U_j, \mathcal{O}_{|U_j}) \arrow[d, leftrightarrow, "\cong"] \\
\operatorname{Spec} A_i & \operatorname{Spec} A_i[u_i^{-1}] \arrow[l] \arrow[ur, leftrightarrow, "\cong"] \arrow[rr, dashed, shift left] & & \operatorname{Spec} A_j[u_j^{-1}] \arrow[ll, dashed, shift left] \arrow[r] & \operatorname{Spec} A_j
\end{tikzcd}$$

where

$$A_i \cong \mathcal{O}(U_i), \quad A_j \cong \mathcal{O}(U_j), \quad A_i[u_i^{-1}] \cong \mathcal{O}(V_{ij}) \cong A_j[u_j^{-1}].$$

It follows that the dashed arrows

$$\operatorname{Spec} A_i[u_i^{-1}] \rightleftarrows \operatorname{Spec} A_j[u_j^{-1}]$$

are inverse isomorphisms of schemes which identify the corresponding open subschemes of $\operatorname{Spec} A_i$ and $\operatorname{Spec} A_j$. Of course this is similar the way smooth manifolds are defined in terms of local patching data.

A morphism of schemes $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of locally ringed spaces between two schemes. These form a category **Sch** and there is a forgetful functor

$$\mathbf{AffSch} \to \mathbf{Sch}; \quad \operatorname{Spec} A \mapsto (\operatorname{spec} A, \mathcal{O}_A).$$

Here is an important result which shows how morphisms from a scheme into an affine scheme are controlled by ring theory.

Proposition 4.1. *Let $(X, \mathcal{O})$ be a scheme and A a commutative ring. Then there is a natural bijection*

$$\mathbf{Sch}((X, \mathcal{O}), \operatorname{Spec} A) \leftrightarrow \mathbf{CoRing}(A, \mathcal{O}(X)); \quad (f, f^\natural) \mapsto (f^\natural \colon A \to \mathcal{O}(X)).$$

Here we use the fact that $A = \mathcal{O}_A(\operatorname{spec} A)$.

Sketch of proof. By definition, X is covered by open sets U_i $(i \in J)$ which are affine, i.e., $(U_i, \mathcal{O}_{U_i}) \cong \operatorname{Spec} \mathcal{O}(U_i)$. Each restriction gives a morphism $\operatorname{Spec} \mathcal{O}(U_i) \to \operatorname{Spec} A$ determined by a ring homomorphism $\varphi_i \colon A \to \mathcal{O}(U_i)$. For each $a \in A$ the elements $\varphi_i(a) \in \mathcal{O}(U_i)$ are compatible on restrictions to intersections so give rise to a unique element of $\mathcal{O}(X)$. It is routine to verify that the resulting function $\varphi \colon A \to \mathcal{O}(X)$ is a ring homomorphism. □

If $(X, \mathcal{O})$ is a scheme and $W \subseteq X$ is an open subset, then the inclusion function $i \colon W \to X$ is continuous and we can define a sheaf of rings $\mathcal{O}_{|W}$ on W by setting

$$\mathcal{O}_{|W}(U) = \mathcal{O}(U).$$

Then for an open set $V \subseteq X$,

$$i_* \mathcal{O}_{|W}(V) = \mathcal{O}_{|W}(i^{-1}V) = \mathcal{O}_{|W}(V \cap W) \mathcal{O}(V \cap W)$$

and taking $i^\natural = \rho^V_{V \cap W}$ we obtain a morphism of locally ringed spaces

$$(i, i^\natural) \colon (W, \mathcal{O}_{|W}) \to (X, \mathcal{O}).$$

This defines an *open subscheme* of $(X, \mathcal{O})$ and we write $(W, \mathcal{O}_{|W}) \hookrightarrow (X, \mathcal{O})$ to indicate the inclusion morphism of such a subscheme. Two subschemes *intersect* when their underlying open sets do and we can then define another open subscheme on their intersection.

Gluing schemes. An important way to produce schemes is by gluing a collection of schemes using some sort of compatibility data. We require the *gluing conditions* listed below to be satisfied.

- A collection of schemes (X_i, O_i) $(i \in I)$.
- For all pairs $i, j \in I$, open subschemes $(X_{ij}, \mathcal{O}_{i,j} \hookrightarrow (X_i, O_i)$ together with gluing isomorphisms $(X_{ij}, \mathcal{O}_{i,j} \xrightarrow{\varphi_{ij}} (X_{ji}, \mathcal{O}_{j,i}$.
- The following conditions are satisfied for all triples $i, j, k \in I$:
  - $\varphi_{ii} = \mathrm{id}$.
  - $\varphi_{ji} \circ \varphi_{ij} = \mathrm{id}$.

- The following diagram commutes in the sense that $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ when restricted to the subset of X_i where both sides are defined.

$$\begin{tikzcd}
 & X_{ij} \arrow[rr, hook] \arrow[dl, "\varphi_{ij}"'] & & X_i & & X_{ik} \arrow[ll, hook'] \arrow[dr, "\varphi_{ik}"] & \\
X_{ji} \arrow[r, hook] & X_j & X_{jk} \arrow[l, hook'] \arrow[rr, "\varphi_{jk}"] & & X_{kj} \arrow[r, hook] & X_k & X_{ki} \arrow[l, hook']
\end{tikzcd}$$

Here is the precise statement about gluing.

Theorem 4.2. *Suppose that a collection of schemes* (X_i, O_i) $(i \in I)$ *satisfies the gluing conditions. Then there is a scheme* $(X, \mathcal{O})$ *and morphisms of schemes* $(\psi_i, \psi_i^\natural)\colon (X, \mathcal{O}) \to (X, \mathcal{O})$ $(i \in I)$ *satisfying*

- *for all* $i, j \in I$, $(\psi_i, \psi_i^\natural)$ *is an isomorphism onto an open subscheme* $(X'_i, \mathcal{O}_{|_{X'_i}})$ *and* $\psi_i X_{ij} = X'_i \cap X'_j$;
- *for all* $i, j \in I$, *the following diagram commutes.*

$$\begin{tikzcd}
X_{ij} \arrow[rr, "\varphi_{ij}"] \arrow[dr, "\psi_{i|_{X_{ij}}}"'] & & X_{ji} \arrow[dl, "\psi_{j|_{X_{ji}}}"] \\
 & X'_i \cap X'_j &
\end{tikzcd}$$

Example 4.3. Let R be a commutative ring. Take two copies of the affine line over R,

$$\mathbf{X} = \operatorname{Spec} R[X], \quad \mathbf{Y} = \operatorname{Spec} R[Y].$$

The open subsets

$$\mathrm{D}_{\mathbf{X}}(X) \subseteq \operatorname{spec} R[X], \quad \mathrm{D}_{\mathbf{Y}}(Y) \subseteq \operatorname{spec} R[Y]$$

define open subschemes of $\mathbf{X}$ and $\mathbf{Y}$.

Notice that $R[X][X^{-1}] = R[X, X^{-1}]$ and $R[Y][Y^{-1}] = R[Y, Y^{-1}]$ and there is a ring isomorphism

$$R[X, X^{-1}] \xrightarrow{\cong} R[Y, Y^{-1}]; \quad X \leftrightarrow Y^{-1}.$$

This induces a homeomorphism

$$\mathrm{D}_{\mathbf{Y}}(Y) \doteq \operatorname{spec} R[Y, Y^{-1}] \xrightarrow{\cong} \operatorname{spec} R[X, X^{-1}] \doteq \mathrm{D}_{\mathbf{X}}(X)$$

and an isomorphism between open subschemes of $\mathbf{Y}$ and $\mathbf{X}$.

If we glue these $\mathbf{X}$ and $\mathbf{Y}$ by identifying these open subschemes we obtain the *projective line over* R, $\mathrm{Proj}^1_R = \mathrm{Proj}^1_{\operatorname{Spec} R}$. The underlying space proj^1_R is covered by the open sets U_+ and U_- corresponding to $\mathbf{X}$ and $\mathbf{Y}$ whose overlap corresponds to $\operatorname{spec} R[X, X^{-1}] \cong \operatorname{spec} R[Y, Y^{-1}]$. The sheaf of rings $\mathcal{O}_{\mathrm{Proj}^1_R}$ has values

$$\mathcal{O}_{\mathrm{Proj}^1_R}(U_+) \cong R[X], \quad \mathcal{O}_{\mathrm{Proj}^1_R}(U_-) \cong R[Y]$$

while on the overlap

$$\mathcal{O}_{\mathrm{Proj}^1_R}(U_+ \cap U_-) \cong R[X, X^{-1}]$$

with restrictions given by

$$\rho^{U_+}_{U_+ \cap U_-}(X) = X, \quad \rho^{U_-}_{U_+ \cap U_-}(Y) = X^{-1}.$$

It follows that

$$\mathcal{O}_{\mathrm{Proj}^1_R}(\mathrm{proj}^1_R) = R.$$

This can be generalised to Proj^n_R for all $n \geqslant 1$.

Example 4.4. Another related but different example is obtained by again setting

$$\mathbf{X} = \operatorname{Spec} R[X], \quad \mathbf{Y} = \operatorname{Spec} R[Y].$$

and identifying the open subsets $\mathrm{D}_{\mathbf{X}}(X)$ and $\mathrm{D}_{\mathbf{Y}}(Y)$ using the ring isomorphism

$$R[Y, Y^{-1}] \xrightarrow{\cong} R[X, X^{-1}]; \quad Y \leftrightarrow X.$$

The resulting schemes is a copy of Aff^1_R with an additional point at the origin.

See [EH00, exercise I-44] for a picture. A similar construction can be made for manifolds thus showing that it is necessary to insist on a manifold being Hausdorff to avoid such pathological examples. For schemes we can eliminate such cases using a notion of *separability* which we will meet soon.

5. CATEGORICAL CONSTRUCTIONS FOR SCHEMES

Coproducts of schemes (or schemes over a given scheme) are easy to construct by taking coproducts of underlying spaces. On the other hand, products are more interesting.

From now on we denote a scheme by writing $X = (|X|, \mathcal{O}_X)$ so $|X|$ is the underlying space and $\mathcal{O}_X$ is the sheaf of rings. In particular,

$$\operatorname{Spec} A = (|\operatorname{Spec} A|, \mathcal{O}_{\operatorname{Spec} A}) = (\operatorname{spec} A, \mathcal{O}_A).$$

Recall that given three topological spaces and continuous maps

$$P \xrightarrow{p} R \xleftarrow{q} Q,$$

their *pullback* or *fibred product* over R is a space $P \times_R Q$ which fits into a commutative diagram

$$\begin{tikzcd}
P \times_R Q \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & Q \arrow[d, "q"] \\
P \arrow[r, "p"'] & R
\end{tikzcd}$$

called a *pullback diagram* and has the following universal property. Given a diagram of solid arrows

$$\begin{tikzcd}
T \arrow[dr, dashed, "\exists !"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& P \times_R Q \arrow[r] \arrow[d] & Q \arrow[d, "q"] \\
& P \arrow[r, "p"'] & R
\end{tikzcd} \tag{5.1}$$

there is a unique dashed arrow making the resulting diagram commute. It is easy to see that an explicit model for such a fibred product is given by

$$P \times_R Q = \{(x, y) \in P \times Q : p(x) = q(y)\}$$

given the subspace topology as a subset of the product space $P \times Q$; the maps to P and Q are the obvious projections. However, for any other example

$$P \leftarrow W \rightarrow Q$$

there is a unique homeomorphism (the dashed arrow) making the following diagram commute.

$$\begin{tikzcd}
 & P \times_R Q \arrow[dl] \arrow[dr] \arrow[dd, dashed, "\doteq"', "\exists!"] & \\
P & & Q \\
 & W \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

It is useful to note that in the diagram

$$\begin{tikzcd}
W' \arrow[r] \arrow[d] & P \times_R Q \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & Q \arrow[d, "q"] \\
P' \arrow[r] & P \arrow[r, "p"'] & R
\end{tikzcd}$$

the big rectangle is a pullback diagram if and only if the left hand square is a pullback diagram. This means that

$$P' \times_P (P \times_R Q) \doteq P' \times_R Q.$$

The notion of a fibred product makes sense **Sch** or more precisely in $\mathbf{Sch}_S$ for a given scheme S. A commutative diagram of schemes

$$\begin{tikzcd}
W \arrow[r] \arrow[d] & Y \arrow[d] \\
X \arrow[r] & S
\end{tikzcd}$$

is a *pullback diagram* if for any commutative diagram of solid arrows

$$\begin{tikzcd}
Z \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
 & W \arrow[r] \arrow[d] & Y \arrow[d] \\
 & X \arrow[r] & S
\end{tikzcd} \tag{5.2}$$

there is a unique dashed arrow making the resulting diagram commute. Such a W is unique up to scheme isomorphism compatible with the morphisms to X and Y and it is usual to denote this *fibred product over* S by $X \times_S Y$ although this is really only defined up to isomorphism. It is note immediately clear that such objects exist in full generality.

Theorem 5.1. *Let* $X \to S$ *and* $Y \to S$ *be schemes over scheme* S. *Then there is a fibred product*

$$X \leftarrow X \times_S Y \to Y.$$

For *affine* schemes we already know this is true and by Proposition 4.1, $\mathrm{Spec}\, A \times_{\mathrm{Spec}\, R} \mathrm{Spec}\, B$ is also the fibred product in $\mathbf{AffSch}_R$.

Given a fibred product of schemes $X \times_S Y$, there are maps $|X| \to |S|$ and $|Y| \to |S|$, hence we can form the space $|X| \times_{|S|} |Y|$ and a map $|X \times_S Y| \to |X| \times_{|S|} |Y|$.

Lemma 5.2. *The map canonical map* $|X \times_S Y| \to |X| \times_{|S|} |Y|$ *is always surjective but not injective in general.*

For a scheme over S, $X \to S$, the universal property of $X \times_S X$ applied to two copies of the identity morphism $X \to X$ leads to a diagonal morphism $\Delta\colon X \to X \times_S X$ making the following diagram commute.

$$\begin{tikzcd}
& X \arrow[d] \arrow[ddl, bend right, "\mathrm{id}"'] \arrow[ddr, bend left, "\mathrm{id}"] & \\
& X \times_S X \arrow[dl, "\mathrm{pr}_1"] \arrow[dr, "\mathrm{pr}_2"'] & \\
X & & X
\end{tikzcd}$$

Fibred products have various 'obvious' properties.

Proposition 5.3. *Let X, Y, Z be S-schemes; we also view S as an S-scheme using the identity morphism.*
(a) $X \times_S S \cong X \cong S \times_S X$.
(b) $X \times_S Y \cong Y \times_S X$.
(c) $(X \times_S Y) \times_S Z \cong X \times_S (Y \times_S Z)$.

Example 5.4. When $X = \operatorname{Spec} A$, $S = \operatorname{Spec} R$ and A is an R-algebra,

$$\operatorname{Spec} A \times_{\operatorname{Spec} R} \operatorname{Spec} A \cong \operatorname{Spec} A \otimes_R A$$

and $\Delta\colon \operatorname{Spec} A \to \operatorname{Spec} A \otimes_R A$ is induced by the multiplication map $\mu\colon A \otimes_R A \to A$ (this is an R-algebra homomorphism) which factors through the quotient ring $A \otimes_R A/\ker \mu \cong A$ (since μ is surjective). This means that

$$\Delta \operatorname{spec} A = \mathrm{V}(\ker \mu) \subseteq \operatorname{spec} A \otimes_R A$$

so $\Delta \operatorname{spec} A$ is a closed subset.

In general, the diagonal $\Delta\colon X \to X \times_S X$ does not have closed image $\Delta|X| \subseteq |X \times_S X|$. This leads to a very important notion.

The S-scheme $X \to S$ is *separated* if $\Delta|X| \subseteq |X \times_S X|$ is a closed subset. In particular, an affine schemes over another is always separated. When $S = \operatorname{Spec} \mathbb{Z}$, X is called (*absolutely*) *separated*.

The notion of separability of an S-scheme X can be reformulated in terms of morphisms into X.

Lemma 5.5. *The S-scheme $X \to S$ is separated if and only if for any S-scheme Y and S-morphisms $Y \underset{g}{\overset{f}{\rightrightarrows}} X$, the subset*

$$\{y \in |Y| : f(y) = g(y)\} \subseteq |Y|$$

is closed.

6. SUBSCHEMES AND IMMERSIONS

We have already seen a definition of an open subscheme of a scheme X corresponding to an open subset of $|X|$. Similarly, for an affine scheme $\operatorname{Spec} A$, a closed subscheme corresponds to the closed subset $\mathrm{V}(\mathfrak{a})$ associated with an ideal $\mathfrak{a} \lhd A$. To extend this to an arbitrary scheme X we need to introduce sheaves of ideals.

Let $X = (|X|, \mathscr{O})$ be a scheme. Then a sheaf $\mathscr{I}$ on $|X|$ is a *sheaf of ideals* if it is a sheaf of $\mathscr{O}$-submodules of $\mathscr{O}$. This means that where for every open subset $U \subseteq |X|$, $\mathscr{I}(U) \lhd \mathscr{O}(U)$ and when $V \subseteq U$ the ring homomorphism $\rho^U_V\colon \mathscr{O}(U) \to \mathscr{O}(U)$ restricts to $\rho^U_V\colon \mathscr{I}(U) \to \mathscr{I}(V)$ so that the multiplications restrict making the following diagram commute.

$$\begin{tikzcd}
\mathscr{O}(U) \otimes \mathscr{I}(U) \arrow[r] \arrow[d, "\rho^U_V \otimes \rho^U_V"'] & \mathscr{I}(U) \arrow[d, "\rho^U_V"] \\
\mathscr{O}(V) \otimes \mathscr{I}(V) \arrow[r] & \mathscr{I}(V)
\end{tikzcd}$$

For example, if $X = \operatorname{Spec} A$ and $\mathfrak{a} \lhd A$, then there is a sheaf of ideals $\tilde{\mathfrak{a}}$ satisfying

$$\tilde{\mathfrak{a}}(\mathrm{D}(u)) = A[u^{-1}]\mathfrak{a} \lhd A[u^{-1}]$$

for all $u \in A$. More generally, an A-module M has an associated $\mathscr{O}_A$-module sheaf $\widetilde{M}$ for which

$$\widetilde{M}(\mathrm{D}(u)) = A[u^{-1}] \otimes_A M.$$

For a scheme $X = (|X|, \mathscr{O})$, an $\mathscr{O}$-module sheaf $\mathscr{M}$ is *quasi-coherent* if for every affine open subset $U \subseteq |X|$, the restriction $\mathscr{M}_{|U}$ is isomorphic to $\widetilde{M}$ for some $\mathscr{O}(U)$-module M (depending on U). For a sheaf of ideals $\mathscr{I}$ we have an important characterisation of what it means to be quasi-coherent.

Lemma 6.1. *The $\mathscr{O}$-module $\mathscr{I}$ is quasi-coherent if and only if for every affine open subset $U \subseteq |X|$, $\mathfrak{I}_{|U}$ is isomorphic to $\tilde{\mathfrak{a}}$ for some ideal $\mathfrak{a} \lhd \mathscr{O}(U)$.*

Now for a quasi-coherent sheaf of ideals $\mathscr{I}$ we can define a 'zero set' $\mathrm{V}(\mathscr{I}) \subseteq |X|$. If $U \subseteq |X|$ is affine open subset (so $U \doteq \operatorname{spec} \mathscr{O}(U)$) let $\mathscr{I}_{|U} \cong \tilde{\mathfrak{a}}$ for some ideal $\mathfrak{a} \lhd \mathscr{O}(U)$. Then we can define the closed subset

$$\mathrm{V}_U(\mathscr{I}) = \mathrm{V}(\mathfrak{a}) \subseteq \operatorname{spec} \mathscr{O}(U) \doteq U.$$

Then we define

$$\mathrm{V}(\mathscr{I}) = \bigcup_U \mathrm{V}_U(\mathscr{I}) \subseteq |X|.$$

In fact $\mathrm{V}(\mathscr{I}) \subseteq |X|$ is closed (exercise!). This turns out to agree with the *support* of the quotient sheaf $\mathscr{O}/\mathfrak{I}$, where the *support* of any sheaf of abelian groups $\mathscr{F}$ on $|X|$ is

$$\operatorname{supp} \mathscr{F} = \{x \in |X| : \mathscr{F}_x \neq 0\} \subseteq |X|.$$

Then (exercise!)

$$(\operatorname{supp} \mathscr{O}/\mathscr{I}) \cap U = \mathrm{V}(\mathfrak{a}).$$

We can view the sheaf of rings $\mathscr{O}/\mathscr{I}$ as a sheaf of $\mathscr{O}$-algebras. It has the property that for each affine open subset $U \subseteq |X|$, $(\mathscr{O}/\mathscr{I})_{|U}$ is associated to the $\mathscr{O}(U)$-module $\mathscr{O}(U)/\mathscr{I}(U)$, so it is a *quasi-coherent* $\mathscr{O}$-algebra. Every quasi-coherent $\mathscr{O}$-algebra $\mathscr{A}$ has an associated scheme $\operatorname{Spec} \mathscr{A}$ which has a morphism $\operatorname{Spec} \mathscr{A} \to X$, in particular there is a morphism $\operatorname{Spec} \mathscr{O}/\mathscr{I} \to X$ and

$$|\operatorname{Spec} \mathscr{O}/\mathscr{I}| \doteq \operatorname{supp} \mathscr{O}/\mathscr{I} = \mathrm{V}(\mathscr{I}).$$

So we can view $\operatorname{supp} \mathscr{O}/\mathscr{I} = \mathrm{V}(\mathscr{I})$ as defining a closed subscheme of X.

In fact every closed subset arises in this way.

Proposition 6.2. *Let $Y \subseteq |X|$ be a closed subset. Then there is a quasi-coherent ideal sheaf $\mathscr{I}_Y$ of $\mathcal{O}$ given by*

$$U \mapsto \{t \in \mathcal{O}(U) : \forall x \in Y \cap U,\ \rho_x^U(t) = 0\}$$

and satisfying

$$\mathrm{V}(\mathscr{I}_Y) = Y.$$

Then there is a subscheme $(Y, \mathcal{O}_Y)$ of X where $(Y, \mathcal{O}_Y) \cong \operatorname{Spec} \mathcal{O}/\mathscr{I}_Y$.

This scheme is sometimes referred to as the *canonical reduced subscheme* for Y because $\mathcal{O}/\mathscr{I}_Y$ has no nilpotent elements.

Now we can give the definition of an *immersion*. A morphism of schemes $f\colon (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ is an *open/closed immersion* if there is an open/closed subscheme $Y' = (Y', \mathcal{O}_{Y'})$ of $X = (X, \mathcal{O}_X)$ and a factorisation of f through an isomorphism $Y \xrightarrow{\cong} Y'$.

$$\begin{tikzcd} Y \arrow[r, "\cong", "f'"'] \arrow[rr, bend left, "f"] & Y' \arrow[r, "\mathrm{inc}"'] & X \end{tikzcd}$$

Here are some results on immersions.

Proposition 6.3. *Let $f\colon Y \to \operatorname{Spec} A$ be a morphism of schemes. Then the following are equivalent:*

- *f is a closed immersion;*
- *Y is affine, so $Y \cong \operatorname{Spec} B$ and f is induced by a surjective ring homomorphism $A \to B$.*

Proposition 6.4. *Let $f = (f, f^\natural)\colon Y \to X$ be a morphism of schemes. Then the following are equivalent:*

- *f is a closed immersion;*
- *for every affine open subset $U \subseteq X$, the induced morphism $f^{-1}U \to U$ is a closed immersion, so by* Proposition 6.3 *$f^{-1}U$ is affine and $\mathcal{O}_X(U) \to \mathcal{O}_Y(f^{-1}U)$ is a surjective ring homomorphism;*
- *there is an affine open covering U_i $(i \in I)$ of X such that each $f^{-1}U_i \to U_i$ a closed immersion, so $f^{-1}U_i$ is affine and induced from a surjective ring homomorphism $\mathcal{O}_X(U_i) \to \mathcal{O}_Y(f^{-1}U_i)$.*

There is another important notion of a *locally closed subscheme*. First we recall a topological notion: Let X be a topological space and $Y \subseteq X$ a subset. Then Y is *locally closed* if for every $y \in Y$, there is an open neighbourhood $U_y \subseteq X$ such that $U_y \cap Y \subseteq U_y$ is a closed subset; in particular this means that $Y \subseteq \bigcup_{y \in Y} U_y$ is a closed subset.

Now let X be a scheme. Then a scheme Y is a *locally closed subscheme* of X is there is an open subscheme $U \subseteq X$ so that Y is a closed subscheme of U. A morphism of schemes $f\colon Y \to X$ is a *locally closed immersion* if it factorises as

$$\begin{tikzcd} Y \arrow[r, "\cong", "f'"'] \arrow[rr, bend left, "f"] & Y' \arrow[r, "\mathrm{inc}"'] & X \end{tikzcd}$$

where Y' is a locally closed subscheme of X.

By Proposition 6.2, every locally closed subset $Y \subseteq |X|$ can be viewed as the underlying space of a locally closed subscheme of X.

Proposition 6.5. *Let $f\colon Y \to X$ be a locally closed immersion and $U \subseteq X$ and open subscheme such that $f|Y| \subseteq |U|$ is a closed subset. Then the restriction to $Y \to U$ is a closed immersion.*

Proposition 6.6. *Open, closed and locally closed immersions are all preserved under compositions and base change.*

Now we return to separated schemes.

Proposition 6.7. *Let $X \to S$ be a scheme over a scheme S. Then $X \to S$ is separated if and only if the diagonal morphism $\Delta\colon X \to X \times_S X$ is a closed immersion.*

Proposition 6.8. *Let $X \to S$ be a scheme over an affine scheme $S = \operatorname{Spec} R$. Let U_i $(i \in I)$ be an affine open covering of $|X|$. Then the following are equivalent:*

- *$X \to S$ is separated;*
- *for all $i, j \in I$, the diagonal morphism Δ induces a closed immersion $U_i \cap U_j \to U_i \times_S U_j$;*
- *for all $i, j \in I$, $U_i \cap U_j$ is affine and Δ induces a surjective ring homomorphism*

$$\mathcal{O}_X(U_i) \otimes_R \mathcal{O}_X(U_j) \to \mathcal{O}_X(U_i \cap U_j).$$

Corollary 6.9. *Let $X \to S$ be a separated scheme over an affine scheme S. If U and V are affine open subschemes of X then $U \cap V$ is also an affine subscheme.*

6.1. **Base change and separation.** Let $X \to S \leftarrow Y$ be morphisms of schemes and let $S \to T$ be another morphism of schemes. Then the diagonal $\Delta\colon S \to S \times_T S$ induces a pullback square.

$$\begin{tikzcd}
X \times_S Y \arrow[r, "\tau"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \times_T Y \arrow[d] \\
S \arrow[r, "\Delta"'] & S \times_T S
\end{tikzcd}$$

It follows that $\tau\colon X \times_S Y \to X \times_T Y$ is a locally closed immersion. If $S \to T$ is separated then τ is closed immersion.

Proposition 6.10. *Let $f\colon X \to Y$ be a morphism of schemes over a scheme S. Then the graph morphism $\Gamma_f\colon X \to X \times_S Y$ is an immersion, and if $\to S$ is separated Γ_f is a closed immersion.*

If $f\colon X \to S$ is a morphism of schemes, a *section* is a morphism $s\colon S \to X$ such that $f \circ s = \mathrm{id}_S$. Then the graph $\Gamma_s\colon S \to S \times_S X \cong X$ coincides with s. So s is an immersion and even a closed immersion if f is separated.

From this various results follow.

Proposition 6.11.
(a) *The composition of separated morphisms is separated.*
(b) *If $f\colon X \to Y$ and $f'\colon X' \to Y'$ are separated morphisms of schemes over S then $f \times f'\colon X \times_S X' \to Y \times_S Y'$ is separated.*

(c) *Separated morphisms of schemes over S are stable under base change.*
(d) *Let $f\colon X \to Y$ and $g\colon Y \to Z$ be morphisms of schemes. If $g \circ f$ is separated then so is f.*

7. Finiteness conditions

Let $\varphi\colon A \to B$ be a ring homomorphism.

φ is *finite* if B is a finitely generated A-module where for $a \in A$ and $b \in B$,

$$a \cdot b = \varphi(a)b.$$

It is standard that a finite homomorphism makes B *integral* over A, i.e., every $x \in B$ satisfies a polynomial identity of form

$$x^n + a_{n-1} \cdot x^{n-1} + \cdots a_1 \cdot x + a_0.1 = 0$$

for some $a_i \in A$ and $n \geqslant 1$.

φ is of *finite type* if it factors as

$$\begin{tikzcd}
A \arrow[rr, "\varphi"] \arrow[dr] & & B \\
& A[T_1, \dots, T_n] \arrow[ur, "\varphi'"'] &
\end{tikzcd}$$

where φ' is surjective; if also $\ker \varphi' \lhd A[T_1, \dots, T_n]$ is finitely generated then φ is of *finite presentation*. In practise we think of B as an A-algebra and then refer to B as a finite or finitely presented A-algebra. Of course if A is Noetherian, then every finite type algebra is finitely presented.

If $s \in A$ then the localisation $A \to A[s^{-1}]$ is of finite type since it factors as

$$\begin{tikzcd}
A \arrow[rr] \arrow[dr] & & A[s^{-1}] & & s^{-1} \\
& A[T] \arrow[ur] & & T \arrow[ur, mapsto] &
\end{tikzcd}$$

where the right hand homomorphism is surjective. More generally, localisations with respect to finitely many elements have finite type.

A morphism of schemes $f = (f, f^{\natural})\colon X \to Y$ is

- *affine* if there is an affine open cover $\{V_i\}_{i \in I}$ of Y for which each $f^{-1}V_i$ is affine;
- *finite* if there is an affine open cover $\{V_i\}_{i \in I}$ of Y for which each $f^{-1}V_i$ is affine and each ring homomorphism $f^{\natural}\colon \mathcal{O}_Y(V_i) \to \mathcal{O}_X(f^{-1}V_i)$ is finite.

Lemma 7.1. *Let $f = (f, f^{\natural})\colon X \to Y$ be a morphism of schemes where Y is affine. Then*

- *f is affine if and only if X is affine;*
- *f is finite if and only if X is affine and the induced ring homomorphism $\mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$ is finite.*

A morphism of schemes $f\colon X \to Y$ is *locally of finite type/presentation at $x \in |X|$* if there are affine open subschemes $U \subseteq X$ and $V \subseteq Y$ such that $x \in U \subseteq f^{-1}V$ and the induced ring homomorphism $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ is of finite type/presentation. f is *locally of finite type/presentation* if locally of finite type/presentation at every point of X.

A morphism of schemes $f\colon X \to Y$ is *closed* if the image $fZ \subseteq |Y|$ of every closed subset $Z \subseteq |X|$ is closed; f is *universally closed* if for every base change $Y' \to Y$,

$$f \times_Y \mathrm{id}_{Y'}\colon X \times_Y Y' \to Y \times_Y Y' \cong Y'$$

is closed. f is *proper* if it is separated, of finite type and universally closed, and similarly an S-scheme $X \to S$ is proper if the morphism is proper.

Lemma 7.2.
(a) *Every finite morphism of schemes is proper.*
(b) *Let* $X \xrightarrow{f} Y \xrightarrow{g} Z$ *be morphisms of schemes so that* $g \circ f$ *is proper. Then*

- *if* g *is separated,* f *is proper;*
- *if* g *is separated and of finite type and* f *is surjective,* g *is proper.*

Recall that the Zariski topology is not usually Hausdorff but is compact in the sense that every open cover has a finite subcover. This combination of properties is often called *quasi-compactness* rather than compactness although outwith algebraic geometry the latter is common.

Let X be a scheme; then an open subset of X is quasi-compact if and only if it is a finite union of affine open subsets of X.

A morphism of schemes $f\colon X \to Y$ is *quasi-compact* if for every quasi-compact subset $V \subseteq |Y|$, $f^{-1}V \subseteq |X|$ is quasi-compact. f is *quasi-separated* if the diagonal morphism $\Delta\colon X \to X \times_Y X$ is quasi-compact.

Lemma 7.3. *Let* $f\colon X \to Y$ *be a morphism of schemes.*
(a) f *is quasi-compact if and only if* Y *has an affine open covering* $\{V_j\}_{j\in J}$ *such that every preimage* $f^{-1}V_j \subseteq |X|$ *is quasi-compact.*
(b) f *is quasi-separated if and only if* Y *has an affine open covering* $\{W_k\}_{k\in K}$ *such that for every pair of affine open subsets* $U, V \subseteq |X|$ *where* $fU \cup fV \subseteq W_k$ *for some* k, $U \cap V$ *is quasi-compact.*

Here is an important result.

Proposition 7.4. *Suppose that* $f\colon X \to Y$ *is a morphism of schemes which is quasi-compact and quasi-separated. If* $\mathcal{M}$ *is a quasi-coherent* $\mathcal{O}_X$*-module then* $f_*\mathcal{M}$ *is a quasi-coherent* $\mathcal{O}_Y$*-module.*

8. Grothendieck topologies

A good source for this material is the collection [FGI$^+$05].

Let $\mathbf{C}$ be a category with products and pullbacks. A *Grothendieck topology* on $\mathbf{C}$ assigns to each object U of $\mathbf{C}$ a collection of sets of morphisms $\{U_i \to U\}_{i\in I}$ called *coverings of* U satisfying the following conditions.

- If $V \to U$ is an isomorphism then $\{V \to U\}$ is a covering.
- If $\{U_i \to U\}_{i\in I}$ is a covering and $V \to U$ is a morphism, then the set of projections $\{U_i \times_U V \to V\}_{i\in I}$ is a covering.
- If $\{U_i \to U\}_{i\in I}$ is a covering and for each $i \in I$ there is a covering $\{V_{ij} \to U_i\}_{j\in J_i}$, then the set of compositions $\{V_{ij} \to U_i \to U\}_{i\in I,\, j\in J_i}$ is a covering of U.

A category with a Grothendieck topology is called a *site*.

Notice that in a Grothendieck topology, if $\{U_i \to U\}_{i\in I}$ and $\{V_j \to U\}_{j\in J}$ are two coverings of U then it follows that the set of projections $\{U_i \times_U V_j \to U\}_{i\in I, j\in J}$ is also a covering of U.

Here are some important examples.

A set of morphisms $\{U_i \to U\}_{i\in I}$ of spaces or schemes is called *jointly surjective* if the set-theoretic union of the images is equal to U.

Example 8.1 (The topological site of a space). Let X be a space and let $\mathbf{T}_X$ be its category of open sets. Then we associate to each open set U its open coverings $\{U_i \to U\}_{i\in I}$ where each $U_i \to U$ is the inclusion of an open subset and fibred products are given by intersections, i.e.,

$$U_i \times_U U_j = U_i \cap U_j.$$

Example 8.2 (The global classical topology). Take $\mathbf{C} = \mathbf{Top}$ the category of spaces and continuous maps. Then a covering $\{U_i \to U\}_{i\in I}$ of a space U is a jointly surjective collection of open embeddings (i.e., open continuous injections).

Example 8.3 (The global étale topology for spaces). Take $\mathbf{C} = \mathbf{Top}$ the category of spaces and continuous maps. Then a covering $\{U_i \to U\}_{i\in I}$ of a space U is a jointly surjective collection of local homeomorphisms.

Example 8.4 (The global Zariski topology for schemes over a base). Take $\mathbf{C} = \mathbf{Sch}_S$, the category of schemes over a scheme S. Then a covering $\{U_i \to U\}_{i\in I}$ of $U \to S$ is a collection of open embeddings covering U, where an open embedding means a morphism $V \to U$ giving an isomorphism of V to an open subscheme of U.

A morphism of schemes $f\colon X \to Y$ is *flat* if for each $x \in |X|$, the induced stalk homomorphism $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ makes $\mathcal{O}_{X,x}$ a flat $\mathcal{O}_{Y,f(x)}$-module; it is *faithfully flat* if it is flat and surjective. A flat morphism which is locally of finite presentation is open.

Example 8.5 (The fppf = fidelement plat et de presentation finie topology). Take $\mathbf{C} = \mathbf{Sch}_S$. Then a covering $\{U_i \to U\}_{i\in I}$ of $U \to S$ is a jointly surjective collection of flat maps locally of finite presentation.

Finally we mention an important topology which we won't completely define.

Example 8.6 (The global étale topology). Take $\mathbf{C} = \mathbf{Sch}_S$. Then a covering $\{U_i \to U\}_{i\in I}$ of $U \to S$ is a jointly surjective collection of étale maps locally of finite presentation.

Sheaves on a site. Suppose we have a site on category $\mathbf{C}$. Let $\mathcal{F} : \mathbf{C}^{\mathrm{o}} \to \mathbf{Set}$ be a functor. Then $\mathcal{F}$ is a *sheaf* on the site if the following conditions is satisfied:

- For every covering $\{U_i \to U\}_{i\in I}$ and collection of elements $a_i \in \mathcal{F}(U_i)$ $(i \in I)$ for which the projections $\mathrm{pr}_1\colon U_i \times_U U_j \to U_i$ and $\mathrm{pr}_2\colon U_i \times_U U_j \to U_j$ satisfy

  $$\mathrm{pr}_1^* a_i = \mathrm{pr}_2^* a_j,$$

  there is a unique element $a \in \mathcal{F}(U)$ such that $\rho^U_{U_i}(a) = a_i$ for every $i \in I$.

A *morphism of sheaves* is just a natural transformation.

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