1. Some category theory odds and ends

We assume basic familiarity with categories, functors etc.

Given a category $\mathbf{C}$, we identify objects with their identity morphisms and write $\mathbf{C}(c,d)$ for the set of morphisms $c \to d$.

Definition 1.1. Let $\mathbf{C}$ be a category.

An object $t \in \mathbf{C}$ is *terminal* if for each $c \in \mathbf{C}$, $\mathbf{C}(c,t)$ has exactly one element.

An object $i \in \mathbf{C}$ is *initial* if for each $c \in \mathbf{C}$, $\mathbf{C}(i,c)$ has exactly one element.

If $\mathbf{C}$ has a terminal object and an initial object which are isomorphic then any such object is a *null object*.

It is easy to see that any two terminal objects are isomorphic, and similarly for initial objects. When there is a null object, it is isomorphic to every terminal and initial object. Initial and terminal objects are typically denoted $\mathbf{0}$ and $\mathbf{1}$.

The category of sets **Set** has $\varnothing$ as its unique initial object, and any set with one element as a terminal object.

The category of based (pointed) sets $\mathbf{Set}_*$ has any with one element as a null object.

The categories of groups **Gp** and abelian groups **AbGp** have the trivial groups as null objects.

The category of (unital) rings **Ring** has $\mathbb{Z}$ as an initial object. If we allow a trivial ring $\{0\}$ then it is a terminal object.

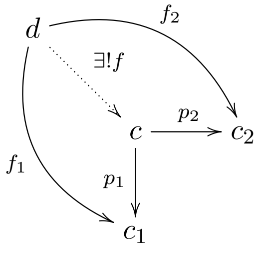
In any abelian category there is a null object, also called a zero object. For example, this applies the category of abelian groups or modules over a ring or sheaves of modules over a ring on a space.

Definition 1.2. Let $\mathbf{C}$ be a category.

Suppose that we have a set of morphisms $\{c \xrightarrow{p_i} c_i : i \in I\}$ in $\mathbf{C}$ where I is some indexing set. Then c is a *product* of the c_i (or for the p_i) if given any set of morphisms $\{d \xrightarrow{f_i} c_i : i \in I\}$ there is a unique morphism $f\colon d \to c$ such that for all $i \in I$, $f_i = p_i f$. If $I = \varnothing$ the product is a terminal object.

Suppose that we have a set of morphisms $\{c_i \xrightarrow{j_i} c : i \in I\}$ in $\mathbf{C}$ where I is some indexing set. Then c is a *coproduct* of the c_i (or for the j_i) if given any set of morphisms $\{c_i \xrightarrow{g_i} d : i \in I\}$ there is a unique morphism $g\colon c \to d$ such that for all $i \in I$, $g_i = g j_i$. If $I = \varnothing$ the coproduct is an initial object.

When the indexing set is $I = \{1,2\}$ we can express this diagrammatically: Given the diagram of solid arrows



there is a unique dotted arrow f making the whole diagram commute. A similar diagram with all arrows reversed applies for the coproduct.

Given two products $\{c \xrightarrow{p_i} c_i : i \in I\}$ and $\{c' \xrightarrow{p'_i} c_i : i \in I\}$ it turns out that there is an isomorphism $c \cong c'$ so it is usual to refer to *the* product and denote it $\prod_I c_i$; when $I = \{1,2\}$ this is written $c_1 \Pi c_2$. Similarly the coproduct is unique up to isomorphism and denoted $\coprod_I c_i$ and $c_1 \amalg c_2$.

It is also important that products and coproducts are functorial in their variables: Given products $\{c \xrightarrow{p_i} c_i : i \in I\}$ and $\{d \xrightarrow{q_i} d_i : i \in I\}$ and morphisms $f_i \colon c_i \to d_i$, there is a unique morphism $\prod_I f_i \colon \prod_I c_i \to \prod_I d_i$ such that for every $i_0 \in I$,

$$q_{i_0}(\prod_I f_i) = f_{i_0} p_{i_0}.$$

A similar result holds for coproducts.

In the category **Set**, the categorical product is the Cartesian product, the coproduct is disjoint union.

In **Gp** the categorical product is the Cartesian product, the coproduct is free product.

In an abelian category, finite products and coproducts agree and are denoted with $\bigoplus$ and $\oplus$.

Proposition 1.3. *Suppose that in the category* **C** *all products of two objects exist and there is a terminal object* **1**. *Then all finite products exist, and for each object c,*

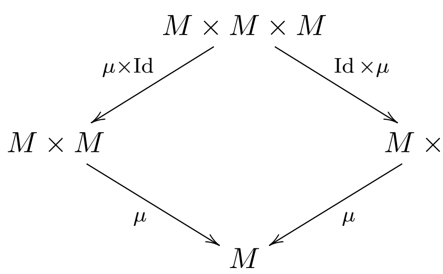
$$\mathbf{1} \Pi c \cong c \cong c \Pi \mathbf{1}.$$

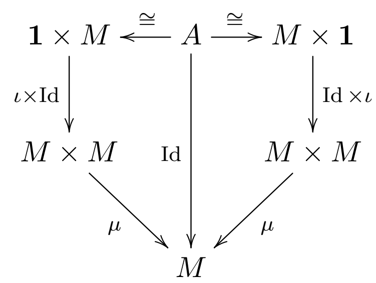
Similarly if all coproducts of two objects exist and there is an initial object **1**, *then all finite coproducts exist, and for each object c,*

$$\mathbf{0} \amalg c \cong c \cong c \amalg \mathbf{0}.$$

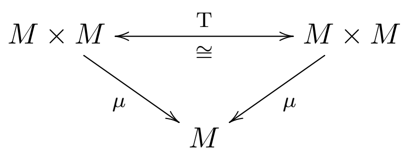
Remark 1.4. In an abelian category, finite products and coproducts exist and are essentially equivalent notions. That is why when discussing vector spaces or modules, $\times$ and $\oplus$ are often used interchangeably since $M \times N \cong M \oplus N$.

Monoids and comonoids. Let's recall the notion of a *monoid* in algebra. A set M together with a product/multiplcation $\mu \colon M \times M \to M$ and a map $\iota \colon \mathbf{1} \to M$ (where **1** is a one element set) defines a *monoid* (M, μ, ι) if the following diagrams commute (in **Set**).





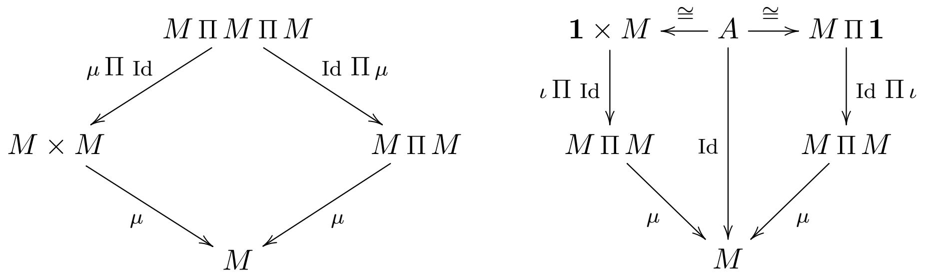
If the following diagram commutes then M is *commutative*.



Of course a *group* is a monoid together with a self-map $\chi \colon M \to M$ that satisfies some additional commutative diagrams defining left and right inverses.

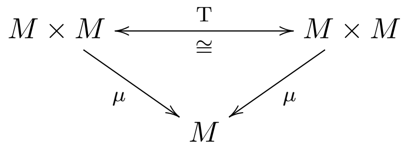
Now let's generalise to a category $\mathbf{C}$ with finite products and terminal objects. A *monoid* in $\mathbf{C}$ is a triple $(M, \mu\colon M \amalg M \to M, \iota\colon \mathbf{1} \to M)$ where the following diagrams in $\mathbf{C}$ commute.

(1.1)



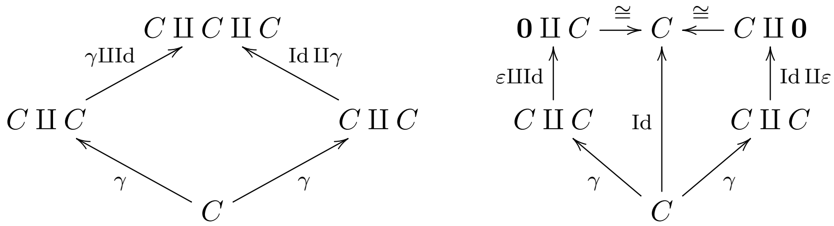
If the following diagram commutes then M is *commutative*.

(1.2)



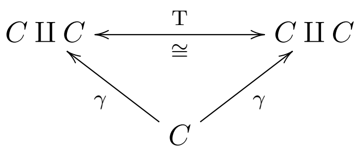
Now one of the magical tricks of Category Theory is that any definition involving commutative diagrams can be dualised by reversing arrows, replacing products by coproducts and terminal objects by initial objects. So if $\mathbf{C}$ has finite coproducts and initial objects then a *comonoid* in $\mathbf{C}$ is a triple $(C, \gamma\colon C \to C \amalg C, \varepsilon\colon C \to \mathbf{0})$ making the following diagrams commute.

(1.3)



If the following diagram commutes then C is *cocommutative*.

(1.4)



Note that comonoids don't exist in **Set**, but do exist in other settings such as the homotopy category of based spaces (where they are called co-H-spaces). A monoid in a category $\mathbf{C}$ gives rise to a comonoid in the opposite category $\mathbf{C}^{\mathrm{op}}$ and vice versa.

We can also introduce notions of monoids and comonoids in a monoidal category $(\mathbf{C}, \otimes, \mathbf{1})$. A monoid M then has morphisms $M \otimes M \to M$ and $\mathbf{1} \to M$ fitting into commutative diagrams like (**??**) while a comonoid C has morphisms $C \to C \otimes C$ and $C \to \mathbf{1}$ with diagrams like (**??**).

Here is a really important and illuminating example.

Example 1.5. Let **AbGp** be the abelian category of abelian groups made symmetric monoidal using the tensor product $\otimes$. The unit object here is $\mathbb{Z}$ since for any abelian group M,

$$\mathbb{Z} \otimes M \cong M \cong M \otimes \mathbb{Z}.$$

A monoid here is an abelian group R equipped with a homomorphism $\varphi\colon R \otimes R \to R$ which gives a map

$$R \times R \to R \otimes R \to R; \qquad (x, y) \mapsto xy = \varphi(x \otimes y)$$

and this is associative. The unit homomorphism $\eta\colon \mathbb{Z} \to R$ satisfies

$$\eta(1)x = x = x\eta(1),$$

so $1_R = \eta(1)$ behaves like a unity in a ring should.

The distributive laws are hidden in the fact that φ is a homomorphism and the tensor product is constructed to be bilinear so that

$$(x_1 + x_2) \otimes y = x_1 \otimes y + x_2 \otimes y, \quad x \otimes (y_1 + y_2) = x \otimes y_1 + x \otimes y_2.$$

So a monoid in $(\mathbf{AbGp}, \otimes, \mathbb{Z})$ is a (unital) ring and a commutative monoid is just a commutative (unital) ring.