

SMSTC Supplementary module
An introduction to Hopf algebras over fields

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Prerequisites: As background, basic knowledge of rings, modules and representation theory of groups would be useful, as well some familiarity with category theory, commutative diagrams and homological algebra.

Learning outcomes: As well as exposure to important topics in modern abstract algebra, this is a good place to gain familiarity with the use of category theory in mathematics.

Who might be interested: Likely to be of interest to mathematicians working on topics such as representation theory of finite dimensional algebras and finite groups, knot theory, Lie theory, algebraic topology, algebraic geometry, non-commutative geometry.

Practicalities: There may be an opportunity for participants to give short talks on topics that particularly interest them; this can be discussed during the course.

Quick introduction

A \mathbb{k} -algebra (A, φ, η) over a field \mathbb{k} is a *monoid* in the monoidal category $(\mathbf{Vect}_{\mathbb{k}}, \otimes)$, i.e., a \mathbb{k} -vector space A with a *product* $\varphi: A \otimes A \rightarrow A$ and *unit* $\eta: \mathbb{k} \rightarrow A$, which make the following diagrams in $\mathbf{Vect}_{\mathbb{k}}$ commute.

$$\begin{array}{ccc} (A \otimes A) \otimes A & \xleftarrow{\cong} & A \otimes (A \otimes A) \\ \downarrow \varphi \otimes \text{Id} & & \downarrow \text{Id} \otimes \varphi \\ A \otimes A & & A \otimes A \\ \searrow \varphi & & \swarrow \varphi \\ & A & \end{array}$$

$$\begin{array}{ccccc} \mathbb{k} \otimes A & \xleftarrow{\cong} & A & \xrightarrow{\cong} & A \otimes \mathbb{k} \\ \downarrow \eta \otimes \text{Id} & & \downarrow \text{Id} & & \downarrow \text{Id} \otimes \eta \\ A \otimes A & & A & & A \otimes A \\ \searrow \varphi & & & & \swarrow \varphi \\ & A & & & \end{array}$$

A is *commutative* if in addition the following diagram commutes.

$$\begin{array}{ccc} A \otimes A & \xrightleftharpoons[\cong]{\text{switch}} & A \otimes A \\ \searrow \varphi & & \swarrow \varphi \\ & A & \end{array}$$

The dual notion is that of a \mathbb{k} -coalgebra, which is a triple (C, ψ, ε) , with C a \mathbb{k} -vector space, a *coproduct* $\psi: C \rightarrow C \otimes C$, and a *counit* $\varepsilon: C \rightarrow \mathbb{k}$ fitting into the commutative diagrams shown.

$$\begin{array}{ccc}
 (C \otimes C) \otimes C & \xleftarrow{\cong} & C \otimes (C \otimes C) \\
 \psi \otimes \text{Id} \uparrow & & \uparrow \text{Id} \otimes \psi \\
 C \otimes C & & C \otimes C \\
 \psi \swarrow & & \searrow \psi \\
 & C &
 \end{array}
 \qquad
 \begin{array}{ccccc}
 \mathbb{k} \otimes C & \xrightarrow{\cong} & C & \xleftarrow{\cong} & C \otimes \mathbb{k} \\
 \varepsilon \otimes \text{Id} \uparrow & & \uparrow \text{Id} & & \uparrow \text{Id} \otimes \varepsilon \\
 C \otimes C & & C & & C \otimes C \\
 \psi \swarrow & & \searrow \psi & & \\
 & C & & &
 \end{array}$$

This says that (C, ψ, ε) is a *comonoid* in $\mathbf{Vect}_{\mathbb{k}}$.

If the following diagram commutes then C is *cocommutative*.

$$\begin{array}{ccc}
 C \otimes C & \xleftarrow{\text{switch}} & C \otimes C \\
 & \cong & \\
 \psi \swarrow & & \searrow \psi \\
 & C &
 \end{array}$$

A \mathbb{k} -Hopf algebra H is a \mathbb{k} -vector space which is both an algebra and a coalgebra together with an antipode $\chi: H \rightarrow H$, so that all of this structure interacts in a certain way. A Hopf algebra is a *group object* in the category of coalgebras or a *cogroup object* in the category of algebras, so Hopf algebras generalise groups!

Some examples:

- ▶ The *group algebra* $\mathbb{k}[G]$ of a group G is a cocommutative Hopf algebra with the elements of G as a basis, product $\varphi(g_1 \otimes g_2) = g_1 g_2$, coproduct $\psi(g) = g \otimes g$ and antipode $\chi(g) = g^{-1}$.
- ▶ If G is a compact Lie group or more generally an H -space, $H_*(G; \mathbb{k})$ and $H^*(G; \mathbb{k})$ are Hopf algebras.
- ▶ If \mathfrak{g} is a Lie algebra, its universal enveloping algebra $U(\mathfrak{g})$ is a cocommutative Hopf algebra.
- ▶ Quantum groups are Hopf algebras which are neither commutative nor cocommutative.
- ▶ Affine group schemes have associated commutative Hopf algebras.
- ▶ Examples from combinatorics.

Outline of course content

Depending on time and the audience's interests, I expect to discuss most of the topics below, depending on the audience's interests.

- ▶ Some category theory: monoidal categories (vector spaces over a field as an important example), adjoint functors.
- ▶ Algebras and coalgebras over a field; bialgebras and Hopf algebras.
- ▶ Lots of examples.
- ▶ SubHopf algebras, adjoint actions and normal subHopf algebras.
- ▶ Modules and comodules, representation theory of a Hopf algebra.
- ▶ Hopf modules and finite dimensional Hopf algebras; every finite dimensional Hopf algebra is Frobenius.
- ▶ Quantum Groups.
- ▶ Homological algebra for modules over Hopf algebras.