BRAVE NEW HOPF ALGEBROIDS AND THE ADAMS SPECTRAL SEQUENCE FOR R-MODULES

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Introduction

This note is intended to draw attention to some phenomena arising naturally within the framework of brave new ring spectra which has recently been constructed by May et al [5]. Actually, our first contact with the algebraic aspects was in the framework of Robinson's theory of A_∞ ring spectra [2, 3], however their most natural interpretation seems to be within the commutative theory.

Since writing this document, we became aware of recent work of A. Lazarev [6] in which similar algebra plays a part, see also [4].

Let R be a commutative S-algebra in the sense of [5] and E a commutative R-ring spectrum. When M is a left R-module, we will write $E^R_*M = \pi_* E \underset{R}{\wedge} M$.

We will describe the E-theory Adams spectral sequence in the homotopy category of R-module spectra. It turns out that the E_2-term is built up from Ext-groups over the brave new Hopf algebroid E^R_*E. Dually, it can be described in terms of the function spectrum $\mathrm{REnd}_R(E)$.

1. Brave new Hopf algebroids

Throughout we will work in a good category of spectra $\mathcal{S}$ such as that of [5]. Associated to this is the category of S-modules $\mathcal{M}_S$ and its derived category $\mathcal{D}_S$.

Let R be a commutative S-algebra in the sense of [5]. There is an associated category of R-modules $\mathcal{M}_R$ and its derived category $\mathcal{D}_R$.

For a commutative S-algebra R, an *R-ring spectrum* is an R-module A which has a unit $\eta\colon R \longrightarrow A$, product $\varphi\colon A \underset{R}{\wedge} A \longrightarrow A$ and the following diagrams commute in $\mathcal{D}_R$, but not necessarily in $\mathcal{M}_R$:

$$\begin{array}{ccc} A \underset{R}{\wedge} A \underset{R}{\wedge} A & \xrightarrow{\varphi\wedge\mathrm{id}} & A \underset{R}{\wedge} A \\ {\scriptstyle \mathrm{id}\wedge\varphi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ A \underset{R}{\wedge} A & \xrightarrow[\varphi]{} & A \end{array} \qquad \begin{array}{ccccc} R \underset{R}{\wedge} A & \xrightarrow{\eta\wedge\mathrm{id}} & A \underset{R}{\wedge} A & \xleftarrow{\mathrm{id}\wedge\eta} & A \underset{R}{\wedge} R \\ & {\scriptstyle\mu}\searrow & \downarrow{\scriptstyle\varphi} & \swarrow{\scriptstyle\mu} & \\ & & A & & \end{array}$$

A is *commutative* if the following diagram commutes in $\mathcal{D}_R$:

$$\begin{array}{ccc} A \underset{R}{\wedge} A & \xrightarrow{\tau} & A \underset{R}{\wedge} A \\ & {\scriptstyle\varphi}\searrow \quad \swarrow{\scriptstyle\varphi} & \\ & A & \end{array}$$

Let E be such a commutative R-ring spectrum. Then the smash product $E \underset{R}{\wedge} E$ is also a commutative R-ring spectrum. It is also naturally an E-algebra spectrum in two different ways

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induced from the left and right units

$$E \xrightarrow{\cong} E \underset{R}{\wedge} R \longrightarrow E \underset{R}{\wedge} E \longleftarrow E \underset{R}{\wedge} R \xleftarrow{\cong} E.$$

Theorem 1.1. *Let E^R_*E be flat as a left or equivalently right E_*-module. Then*
i) *(E_*, E^R_*E) is a Hopf algebroid over R_*;*
ii) *for any R-module M, E^R_*M is a left E^R_*E-comodule.*

Proof. This is proved using essentially the same argument as in [1, 8]. The natural map

$$E \underset{R}{\wedge} M \xrightarrow{\cong} E \underset{R}{\wedge} R \underset{R}{\wedge} M \longrightarrow E \underset{R}{\wedge} E \underset{R}{\wedge} M$$

induces the coaction

$$\psi\colon E^R_*M \longrightarrow \pi_* E \underset{R}{\wedge} E \underset{R}{\wedge} M \xrightarrow{\cong} E^R_*E \underset{E_*}{\otimes} E^R_*M,$$

the flatness condition being used to show that

$$\pi_* E \underset{R}{\wedge} E \underset{R}{\wedge} M \cong E^R_*E \underset{E_*}{\otimes} E^R_*M.$$

□

2. Some examples

The examples in this section were first noted in the late 1980's and mentioned in the concluding remarks of [2]; the work of that paper and its companion [3] was carried out in the framework of Robinson's theory of A_∞ spectra. It is only with the benefit of the theory of commutative ring spectra that the significance of such constructions has become clear to us.

2.1. $\boldsymbol{BP \longrightarrow H\mathbb{F}_p}$**.** Let BP be a commutative ring spectrum model for the Brown-Peterson spectrum at a prime p which is claimed to exist by work of I. Kriz. By considering the Eilenberg-MacLane spectrum $H\mathbb{F}_p$ as a commutative BP-algebra [5], we can form $H\mathbb{F}_p \underset{BP}{\wedge} H\mathbb{F}_p$. By [5], there is a Künneth spectral sequence,

$$\mathrm{E}^2_{s,t} = \mathrm{Tor}^{BP_*}_{s,t}(\mathbb{F}_p, \mathbb{F}_p) \Longrightarrow H\mathbb{F}_p{}^{BP}_{s+t} H\mathbb{F}_p.$$

Using a Koszul complex over BP_*, it is straightforward to see that

$$\mathrm{E}^2_{*,*} = \Lambda_{\mathbb{F}_p}(\tau_j : j \geqslant 0),$$

where Λ denotes an exterior algebra and $\tau_j \in \mathrm{E}^2_{1,2(p^j-1)}$. Of course, this is naturally a quotient Hopf algebra over $\mathbb{F}_p$ of the dual Steenrod algebra $H\mathbb{F}_{p*}H\mathbb{F}_p$.

2.2. $\boldsymbol{BP \longrightarrow E(n)}$**.** By [5, 12], the Johnson-Wilson spectrum $E(n)$ is a commutative BP-ring spectrum and we can form $E(n) \underset{BP}{\wedge} E(n)$. There is a Künneth spectral sequence,

$$\mathrm{E}^2_{s,t} = \mathrm{Tor}^{BP_*}_{s,t}(E(n)_*, E(n)_*) \Longrightarrow E(n)^{BP}_{s+t} E(n).$$

By using a Koszul complex over BP_* for $BP\langle n\rangle_*$ and localizing at v_n, we find that

$$\mathrm{E}^2_{*,*} = \Lambda_{E(n)_*}(\tau_j : j \geqslant n+1),$$

where Λ denotes an exterior algebra and $\tau_j \in \mathrm{E}^2_{1,2(p^j-1)}$. So as an $E(n)_*$-algebra,

$$E(n)^{BP}_* E(n) = \Lambda_{E(n)_*}(\tau_j : j \geqslant n+1).$$

2.3. $\widehat{E(n)} \longrightarrow K(n)$. Let $\widehat{E(n)}$ be the I_n-adic completion of the Johnson-Wilson spectrum $E(n)$, known to be a commutative S-algebra by work of P. Goerss and M. Hopkins. Morava K-theory $K(n)$ is a commutative $\widehat{E(n)}$-ring spectrum [12]. There is a Künneth spectral sequence

$$\mathrm{E}^2_{s,t} = \mathrm{Tor}^{\widehat{E(n)}_*}_{s,t}(K(n)_*, K(n)_*) \Longrightarrow K(n)^{\widehat{E(n)}}_{s+t} K(n).$$

We find that

$$\mathrm{E}^2_{*,*} = \Lambda_{K(n)_*}(\tau_j : n-1 \geqslant j \geqslant 0),$$

which is naturally a quotient of $K(n)_*K(n) = K(n)_*(K(n))$ as a Hopf algebra over $K(n)_*$.

3. The Adams spectral sequence for R-modules

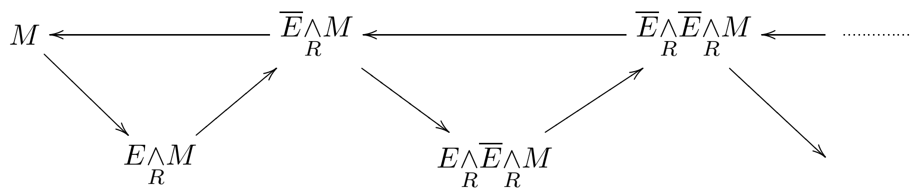
Let L, M be R-modules and E a commutative R-ring spectrum with E^R_*E flat as a left or right E_*-module.

Theorem 3.1. *If E^R_*L is projective as an E_*-module, there is an Adams spectral sequence with*

$$\mathrm{E}_2^{s,t}(L, M) = \mathrm{Ext}^{s,t}_{E^R_*E}(E^R_*L, E^R_*M).$$

If π_*M is connective, this converges to $\mathcal{D}_{\mathrm{L}^R_E R}(\Sigma^{s+t}\mathrm{L}^R_E L, \mathrm{L}^R_E M)$, where L^R_E is the E^R_*-localization functor on R-modules. In particular, if $L = R$ then the spectral sequence converges to $\pi_{s+t}\mathrm{L}^R_E M$.

Proof. The proof follows that of Adams [1], replacing the sphere spectrum S with R and working in the derived category $\mathcal{D}_R$ throughout. The Adams resolution of M is built up in the usual way by splicing together cofibre triangles:



Identification of the E_2-term and convergence are demonstrated as in Adams. □

4. Some examples of brave new Adams spectral sequences

We give some sample calculations based on the examples of §2.

4.1. $BP \longrightarrow H\mathbb{F}_p$. Taking $R = BP$ and $E = H\mathbb{F}_p$, we obtain a spectral sequence

$$\mathrm{E}_2^{s,t}(BP, M) = \mathrm{Ext}^{s,t}_{\Lambda_{\mathbb{F}_p}(\tau_j : j \geqslant 0)}(\mathbb{F}_p, H\mathbb{F}_p{}^{BP}_* M) \Longrightarrow \pi_{s+t}\mathrm{L}^{BP}_{H\mathbb{F}_p} M.$$

Here $\mathrm{L}^{BP}_{H\mathbb{F}_p} M$ is related to the p-adic completion of M. For a connective BP-module spectrum M of finite type with no BP_*-torsion in M_*,

$$\pi_n \mathrm{L}^{BP}_{H\mathbb{F}_p} M = (\pi_n M)\widehat{_p}.$$

When $M = BP$,

$$\mathrm{E}_2^{s,t}(BP, BP) = \mathrm{Ext}^{s,t}_{\Lambda_{\mathbb{F}_p}(\tau_j : j \geqslant 0)}(\mathbb{F}_p, \mathbb{F}_p) \Longrightarrow (BP_{s+t})\widehat{_p}.$$

4.2. $\boldsymbol{BP \longrightarrow E(n)}$. Taking $R = BP$ and $E = E(n)$, we obtain a spectral sequence with

$$\mathrm{E}_2^{s,t}(BP, M) = \mathrm{Ext}^{s,t}_{\Lambda_{E(n)_*}(\tau_j : j \geqslant n+1)}(E(n)_*, E(n)^{BP}_* M),$$

however convergence here is problematic. The target of this spectral sequence does not always appear to be $\pi_* \mathrm{L}^{BP}_{E(n)} M$ even when M_* is a finitely generated BP_*-module, as the example of $M = BP$ shows. We then have

$$\mathrm{E}_2^{s,t}(BP, BP) = \mathrm{Ext}^{s,t}_{\Lambda_{E(n)_*}(\tau_j : j \geqslant n+1)}(E(n)_*, E(n)_*) \Longrightarrow (v_n^{-1} BP)_{s+t},$$

since here $\mathrm{E}_2^{*,*}(BP, BP)$ is a polynomial algebra over $E(n)_*$ on generators

$$V_k \in \mathrm{E}_2^{1, 2p^k - 1}(BP, BP) \quad (k \geqslant n+1),$$

where V_k detects the elements $v_k \in BP_*$. But from [7] it is known that $\pi_* \mathrm{L}^{BP}_{E(n)} BP \neq v_n^{-1} BP_*$.

4.3. $\boldsymbol{\widehat{E(n)} \longrightarrow K(n)}$. Taking $R = \widehat{E(n)}$ and $E = K(n)$, we obtain a spectral sequence

$$\mathrm{E}_2^{s,t}(\widehat{E(n)}, M) = \mathrm{Ext}^{s,t}_{\Lambda_{K(n)_*}(\tau_j : n-1 \geqslant j \geqslant 0)}(K(n)_*, K(n)^{\widehat{E(n)}}_* M) \Longrightarrow \pi_{s+t} \mathrm{L}^{\widehat{E(n)}}_{K(n)} M.$$

For an $\widehat{E(n)}$-module spectrum M with M_* an finitely generated $\widehat{E(n)}_*$-module with no $\widehat{E(n)}_*$ torsion,

$$\pi_* \mathrm{L}^{\widehat{E(n)}}_{K(n)} M = (M_*)\widehat{_{I_n}},$$

the I_n-adic completion of M_*. When $M = \widehat{E(n)}$,

$$\mathrm{E}_2^{s,t}(\widehat{E(n)}, \widehat{E(n)}) = \mathrm{Ext}^{s,t}_{\Lambda_{K(n)_*}(\tau_j : n-1 \geqslant j \geqslant 0)}(K(n)_*, K(n)_*) \Longrightarrow \widehat{E(n)}_*.$$

These results are perhaps suggestive of interesting phenomena. The most significant consideration of localization in derived module categories to date seems to have been that of Wolbert [13, 5].

5. Some suggestive results

Given two R-modules L, M, with R not necessarily commutative, there is a function spectrum $\mathrm{F}_R(L, M)$. When $L = M$ this gives the derived endomorphism spectrum $\mathrm{REnd}_R(M)$ which is known to be an A_∞ ring spectrum by [10, 11, 5] and M is an A_∞ module over it. Dually we have the derived tensor product $M \underset{R}{\wedge} M$. If R is commutative and $M = E$ is a commutative algebra over R, then $E \underset{R}{\wedge} E$ is a commutative algebra over R with product induced by the multiplication map $\mu\colon E \underset{R}{\wedge} E \longrightarrow E$ which also induces a map

$$\mathrm{REnd}_R(E) \xrightarrow{\mu^*} \mathrm{F}_R(E \underset{R}{\wedge} E, E).$$

Then μ^* is coassociative and cocommutative in the obvious senses. These gadgets are best viewed as dual to each other in the same way that E^*E and E_*E usually are. Of course, an optimal situation occurs if $\pi_*(\mathrm{REnd}_R(E))$ and E^R_*E were truly dual. If $\pi_*(E \underset{R}{\wedge} E)$ is E_*-flat then E^R_*E is a Hopf algebroid; if we also insist it be projective then

$$\pi_*(\mathrm{REnd}_R(E)) = \mathrm{Hom}_{E_*}(E^R_*E, E_*).$$

It is well known that working over S, the Adams spectral sequence can be set up using either $E_*(\ )$ and comodules over E_*E, or using one or other of $E_*(\ )$, $E^*(\ )$ regarded as modules over E^*E. In the situation involving $E_*(\ )$, this suggests that the Adams spectral sequence is ultimately based on the action of $\mathrm{REnd}_R(E)$ on the R-module functors $E \underset{R}{\wedge}(\ )$ or $\mathrm{F}_R(\ , E)$.

In each of the examples of Sections 2 and 4, it appears that for $R' = \mathrm{REnd}_R(E)$, $\mathrm{REnd}_{R'}(E)$ is trying hard to be R at least after E-localization. In fact in either case we can replace R with the commutative ring spectrum R_E and find that

$$\mathrm{REnd}_{R'_E}(E) \simeq R_E,$$

which is reminiscent of double centralizer results for modules over simple algebras.

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