NOTES ON CHAIN COMPLEXES

ANDREW BAKER

These notes are intended as a very basic introduction to (co)chain complexes and their algebra, the intention being to point the beginner at some of the main ideas which should be further studied by in depth reading. An accessible introduction for the beginner is [2]. An excellent modern reference is [3], while [1] is a classic but likely to prove hard going for a novice. Most introductory books on algebraic topology introduce the language and basic ideas of homological algebra.

1. Chain complexes and their homology

Let R be a ring and Mod_R the category of right R-modules. Then a sequence of R-module homomorphisms

$$L \xrightarrow{f} M \xrightarrow{g} N$$

is *exact* if $\operatorname{Ker} g = \operatorname{Im} f$. Of course this implies that $gf = 0$.

A sequence of homomorphisms

$$\cdots \xrightarrow{f_{n+1}} M_n \xrightarrow{f_n} M_{n-1} \xrightarrow{f_{n-1}} \cdots$$

is called a *chain complex* if for each n, $f_n f_{n+1} = 0$ or equivalently, $\operatorname{Im} f_{n+1} \subseteq \operatorname{Ker} f_n$. It is called *exact* or *acyclic* if each segment

$$M_{n+1} \xrightarrow{f_{n+1}} M_n \xrightarrow{f_n} M_{n-1}$$

is exact. We write (M_*, f) for such a chain complex and refer to the f_n as *boundary homomorphisms.*

If a chain complex is finite we often pad it out to a doubly infinite complex by adding in trivial modules and homomorphisms. In particular, if M is a R-module we can view it as the chain complex with $M_0 = M$ and $M_n = 0$ whenever $n \neq 0$. It is often useful to consider the *trivial chain complex* $0 = (\{0\}, 0)$.

Given a complex (M_*, f), we define its *homology* to be the complex $(H_*(M_*, f), 0)$ where

$$H_n(M_*, f) = \operatorname{Ker} f_n / \operatorname{Im} f_{n+1}.$$

A *morphism of chain complexes* $h\colon (M_*, f) \longrightarrow (N_*, g)$ is a sequence of homomorphisms $h_n\colon M_n \longrightarrow N_n$ for which the following diagram commutes.

$$\begin{array}{ccc} M_n & \xrightarrow{f_n} & M_{n-1} \\ {\scriptstyle h_n}\downarrow & & \downarrow{\scriptstyle h_{n-1}} \\ N_n & \xrightarrow{g_n} & N_{n-1} \end{array}$$

Notice that if $u \in \operatorname{Ker} f_n$ we have $g_n(h_n(u)) = 0$, while if $v \in M_{n+1}$,

$$h_n(f_{n+1}(v)) = g_{n+1}(h_{n+1}(v)).$$

Together these allow us to define for each n a homomorphism

$$h_*\colon H_n(M_*, f) \longrightarrow H_n(N_*, g); \quad h_*(u + \operatorname{Im} f_{n+1}) = h_n(u) + \operatorname{Im} g_{n+1}.$$

Date: [06/04/2009].

It is easy to check that if $j\colon (L_*, \ell) \longrightarrow (M_*, f)$ is another morphism of chain complexes then

$$(hj)_* = h_* j_*$$

and for the identity morphism $\mathrm{id}\colon (M_*, f) \longrightarrow (M_*, f)$ we have

$$\mathrm{id}_* = \mathrm{id}\,.$$

This shows that each H_n is a covariant *functor* from chain complexes to R-modules.

From now on we will always write a complex as (M_*, d) where the boundary d is really the collection of boundary homomorphisms $d_n\colon M_n \longrightarrow M_{n-1}$ which satisfy $d_{n-1}d_n = 0$; we often symbolically indicate these relations with the formula $d^2 = 0$.

Given a morphism of chain complexes $h\colon (L_*, d) \longrightarrow (M_*, d)$ we may define two new chain complexes $\operatorname{Ker} h = ((\operatorname{Ker} h)_*, d)$ and $\operatorname{Im} h = ((\operatorname{Im} h)_*, d)$, where

$$(\operatorname{Ker} h)_n = \operatorname{Ker} h\colon L_n \longrightarrow M_n, \quad (\operatorname{Im} h)_n = \operatorname{Im} h\colon L_n \longrightarrow M_n.$$

The boundaries are the restrictions of d to these.

A *cochain complex* is a collection of R-modules M^n together with *coboundary homomorphisms* $d^n\colon M^n \longrightarrow M^{n+1}$ for which $d^{n+1}d^n = 0$. The *cohomology* of this complex is $(H^*(M^*, d), 0)$ where

$$H^n(M^*, d) = \operatorname{Ker} d^n / \operatorname{Im} d^{n-1}.$$

2. The homology long exact sequence

Let $h\colon (L_*, d) \longrightarrow (M_*, d)$ and $k\colon (M_*, d) \longrightarrow (N_*, d)$ be morphisms of chain complexes and suppose that

$$0 \to (L_*, d) \xrightarrow{h} (M_*, d) \xrightarrow{k} (N_*, d) \to 0$$

is *short exact*, *i.e.*,

$$\operatorname{Ker} h = 0, \quad \operatorname{Im} k = (N_*, d), \quad \operatorname{Ker} k = \operatorname{Im} h.$$

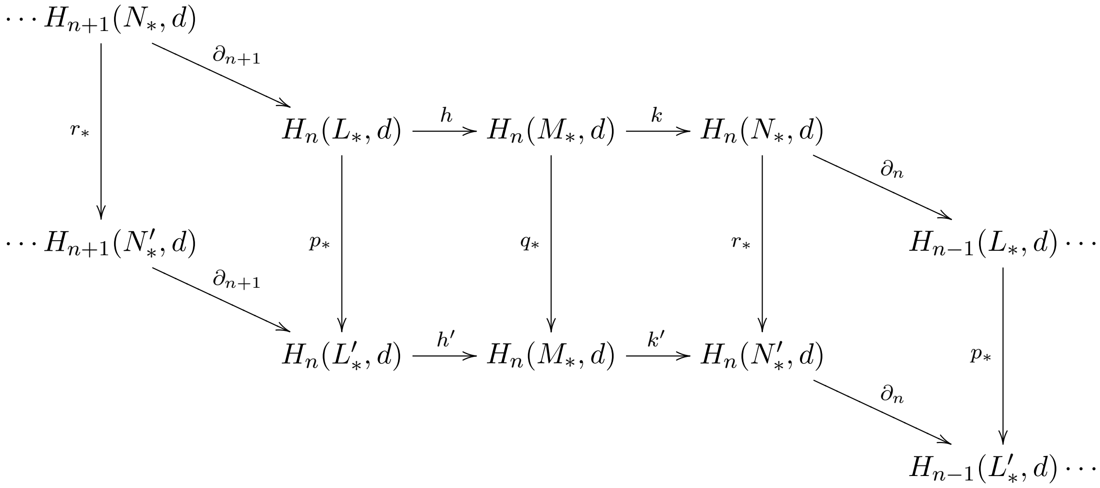
Theorem 2.1. *There is a long exact sequence of the form*

$$\cdots H_{n+1}(N_*, d) \xrightarrow{\partial_{n+1}} H_n(L_*, d) \xrightarrow{h} H_n(M_*, d) \xrightarrow{k} H_n(N_*, d) \xrightarrow{\partial_n} H_{n-1}(L_*, d) \cdots$$

Furthermore, given a commutative diagram of short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (L_*, d) & \xrightarrow{h} & (M_*, d) & \xrightarrow{k} & (N_*, d) & \longrightarrow & 0 \\
 & & \downarrow p & & \downarrow q & & \downarrow r & & \\
0 & \longrightarrow & (L'_*, d) & \xrightarrow{h'} & (M'_*, d) & \xrightarrow{k'} & (N'_*, d) & \longrightarrow & 0
\end{array}$$

there is a commutative diagram



Proof. We begin by defining $\partial_n \colon H_n(N_*, d) \longrightarrow H_{n-1}(L_*, d)$. Let

$$z + \operatorname{Im} d_{n+1} \in H_n(N_*, d) = \operatorname{Ker} d_n / \operatorname{Im} d_{n+1}.$$

Then since $k \colon M_n \longrightarrow N_n$ is epic, we may choose an element $w \in M_n$ for which $k(w) = z$. Then $d_n(w) \in M_{n-1}$ and $kd_n(w) = d_n k(w) = d_n(z) = 0$. So $d_n(w) \in \operatorname{Ker} k$. By exactness, $d_n(w) = h(v)$ for some $v \in L_{n-1}$. Notice that

$$hd_{n-1}(v) = d_{n-1}h(v) = d_{n-1}d_n(w) = 0.$$

But h is monic, so this gives $d_{n-1}(v) = 0$. Hence $v \in \operatorname{Ker} d_{n-1}$. We take

$$\partial_n(z + \operatorname{Im} d_{n+1}) = v + \operatorname{Im} d_n \in H_{n-1}(L_*, d).$$

It is now routine to check that for $z \in \operatorname{Im} d_{n+1}$ we have

$$\partial_n(z + \operatorname{Im} d_{n+1}) = 0 + \operatorname{Im} d_n,$$

hence ∂_n is a well defined function. Exactness is verified by checking on elements. □

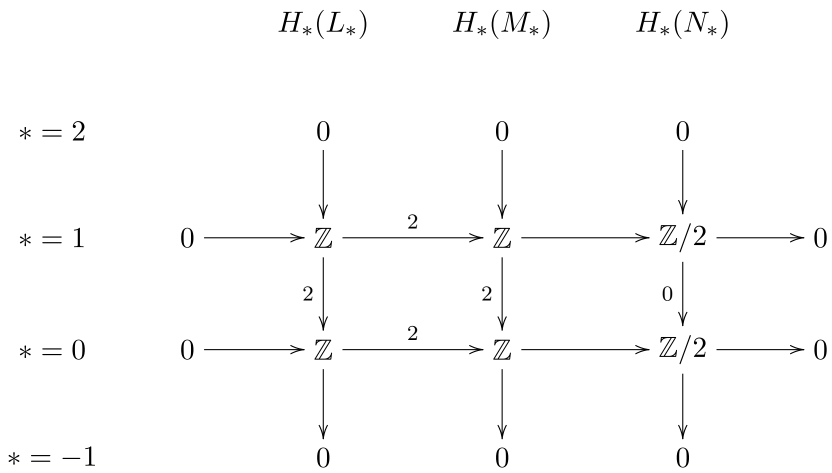
For cochain complexes we get a similar result for a short exact sequence of cochain complexes

$$0 \to (L^*, d) \xrightarrow{h} (M^*, d) \xrightarrow{k} (N^*, d) \to 0,$$

but the ∂_n are replaced by homomorphisms

$$\delta^n \colon H^n(N^*, d) \longrightarrow H^{n+1}(L^*, d).$$

Example 2.2. Consider the following exact sequence of chain complexes of $\mathbb{Z}$-modules (written vertically).



Taking homology we obtain the long exact sequence

$$\begin{array}{lccccccc}
 & H_*(L_*) & & H_*(M_*) & & H_*(N_*) & & \\
*=1 & 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}/2 & & \\
 & & & \partial & \swarrow & & & \\
*=0 & \mathbb{Z}/2 & \xrightarrow{0} & \mathbb{Z}/2 & \xrightarrow{=} & \mathbb{Z}/2 & \longrightarrow & 0
\end{array}$$

From this we conclude that

$$\partial\colon H_1(N_*) = \mathbb{Z}/2 \longrightarrow H_0(L_*) = \mathbb{Z}/2$$

is an isomorphism.

3. Tensor products, free resolutions and Tor

Let M be any right R-module and N be *left* R-module. Then we can form the *tensor product* $M \underset{R}{\otimes} N$ which is an abelian group. It is also an R-module if R is commutative. The definition involves forming the free $\mathbb{Z}$-module $F(M,N)$ with basis consisting of all the pairs (m,n) where $m \in M$ and $n \in$; then

$$M \underset{R}{\otimes} N = F(M,N)/S(M,N),$$

where $S(M,N) \leqslant F(M,N)$ is the subgroup generated by all the elements of form

$$(m_1+m_2,n)-(m_1,n)-(m_2,n), \quad (m,n_1+n_2)-(m,n_1)-(m,n_2), \quad (mr,n)-(m,rn),$$

where $m,m_1,m_2 \in M$, $n,n_1,n_2 \in N$, $r \in R$. We usually denote the coset of (m,n) by $m \otimes n$; such elements generate the group $M \underset{R}{\otimes} N$.

The tensor product $M \underset{R}{\otimes} N$ has an important *universal property* which characterizes it up to isomorphism. Write $q\colon M \times N \longrightarrow M \underset{R}{\otimes} N$ for the quotient function. Let $f\colon M \times N \longrightarrow V$ be a function into an abelian group V which satisfies

$$\begin{aligned}
f(m_1+m_2,n) &= f(m_1,n)+f(m_2,n),\\
f(m,n_1+n_2) &= f(m,n_1)+f(m,n_2),\\
f(mr,n) &= f(m,rn)
\end{aligned}$$

for all $m,m_1,m_2 \in M$, $n,n_1,n_2 \in N$, $r \in R$, there is a unique homomorphism $\widetilde{f}\colon M \underset{R}{\otimes} N \longrightarrow V$ for which $f = \widetilde{f} \circ q$.

$$\begin{array}{ccc}
M \times N & \xrightarrow{q} & M \underset{R}{\otimes} N \\
{\scriptstyle f}\downarrow & \swarrow{\scriptstyle \exists!\,\widetilde{f}} & \\
V & &
\end{array}$$

Proposition 3.1. *If $f\colon M_1 \longrightarrow M_2$ and $g\colon N_1 \longrightarrow N_2$ are homomorphisms of R-modules, there is a group homomorphism*

$$f \otimes g\colon M_1 \underset{R}{\otimes} N_1 \longrightarrow M_2 \underset{R}{\otimes} N_2$$

for which

$$f \otimes g(m \otimes n) = f(m) \otimes g(n).$$

Proposition 3.2. *Given a short exact sequence of left R-modules*

$$0 \to N_1 \xrightarrow{g_1} N_2 \xrightarrow{g_2} N_3 \to 0,$$

there is an exact sequence

$$M \underset{R}{\otimes} N_1 \xrightarrow{1\otimes g_1} M \underset{R}{\otimes} N_2 \xrightarrow{1\otimes g_2} M \underset{R}{\otimes} N_3 \to 0.$$

Because of this property, we say that $M \underset{R}{\otimes} (\ )$ is *right exact*. Obviously it would be helpful to understand $\operatorname{Ker} 1 \otimes g_1$ which measures the deviation from *left exactness* of $M \underset{R}{\otimes} (\ )$.

Let R be a ring. A right R-module F is called *free* if there is a set of elements $\{b_\lambda : \lambda \in \Lambda\} \subseteq F$ such that every element $x \in F$ can be uniquely expressed as

$$x = \sum_{\lambda \in \Lambda} b_\lambda t_\lambda$$

for elements $t_\lambda \in R$. We say that the b_λ form a *basis for F over R*. We can make a similar definition for left modules.

For example, for $n \geqslant 1$,

$$R^n = \{(t_1, \dots, t_n) : t_1, \dots, t_n \in R\}$$

is free on the basis consisting of the standard elements

$$e_1 = (1, 0, \dots, 0), \dots, e_n = (0, \dots, 0, 1).$$

This works whether we view R^n as a left or right module.

An exact complex

$$\cdots \longrightarrow F_k \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \to 0 \tag{3.1}$$

is called a *resolution of M*. Here we view M as the (-1)-term and 0 as the (-2)-term. If each F_k is also free over R then it is called a *free resolution of M*. Every M admits such a free resolution. Given such a free resolution $F_* \longrightarrow M \to 0$ of a right R-module M, and a left R-module N, we can form a new complex $F_* \underset{R}{\otimes} N \to 0$, where the boundary maps are obtained by tensoring those of $F_\bullet$ with the identity on N an taking $F_0 \underset{R}{\otimes} N \longrightarrow 0$ rather than using the original map $F_0 \longrightarrow M$. Here $F_n \underset{R}{\otimes} N$ is in degree n. We define

$$\operatorname{Tor}_n^R(M, N) = H_n(F_* \underset{R}{\otimes} N).$$

It is easy to see that

$$\operatorname{Tor}_0^R(M, N) \cong M \underset{R}{\otimes} N.$$

Of course we could also form a free resolution of N, tensor it with M and then take homology.

Theorem 3.3. Tor_*^R *has the following properties.*

i) $\operatorname{Tor}_*^R(M, N)$ *can be computed by using free resolutions of either variable and the answers agree up to isomorphism.*

ii) *Given R-module homomorphisms $f\colon M_1 \longrightarrow M_2$ and $g\colon N_1 \longrightarrow N_2$ there are homomorphisms*

$$(f \otimes g)_* = f_* \otimes g_* \colon \operatorname{Tor}_n^R(M_1, N_1) \longrightarrow \operatorname{Tor}_n^R(M_2, N_2)$$

generalizing $f_* \otimes g_* \colon M_1 \underset{R}{\otimes} N_1 \longrightarrow M_2 \underset{R}{\otimes} N_2$.

iii) *For a free right/left R-module P/Q and $n > 0$ we have*

$$\operatorname{Tor}_n^R(P, N) = 0 = \operatorname{Tor}_n^R(M, Q).$$

iv) *Associated to a short exact sequence of right R-modules*

$$0 \to M_1 \longrightarrow M_2 \longrightarrow M_3 \to 0$$

there is a long exact sequence

$$\begin{array}{ccccc}
 & & \cdots & \longrightarrow & \mathrm{Tor}^R_{n+1}(M_3, N) \\
\mathrm{Tor}^R_n(M_1, N) & \longrightarrow & \mathrm{Tor}^R_n(M_2, N) & \longrightarrow & \mathrm{Tor}^R_n(M_3, N) \\
\mathrm{Tor}^R_{n-1}(M_1, N) & \longrightarrow & \cdots & &
\end{array}$$

$$\cdots \longrightarrow M_1 \underset{R}{\otimes} N \longrightarrow M_2 \underset{R}{\otimes} N \longrightarrow M_3 \underset{R}{\otimes} N \to 0$$

and associated to a short exact sequence of left R-modules

$$0 \to N_1 \longrightarrow N_2 \longrightarrow N_3 \to 0$$

there is a long exact sequence

$$\begin{array}{ccccc}
 & & \cdots & \longrightarrow & \mathrm{Tor}^R_{n+1}(M, N_3) \\
\mathrm{Tor}^R_n(M, N_1) & \longrightarrow & \mathrm{Tor}^R_n(M, N_2) & \longrightarrow & \mathrm{Tor}^R_n(M, N_3) \\
\mathrm{Tor}^R_{n-1}(M, N_1) & \longrightarrow & \cdots & &
\end{array}$$

$$\cdots \longrightarrow M \underset{R}{\otimes} N_1 \longrightarrow M \underset{R}{\otimes} N_2 \longrightarrow M \underset{R}{\otimes} N_3 \to 0$$

Corollary 3.4. *Let Q be a left R-module for which $\mathrm{Tor}^R_n(M, Q) = 0$ for all $n > 0$ and M. Then for any exact complex (C_*, d), the complex $(C_* \underset{R}{\otimes} Q, d \otimes 1)$ is exact, and*

$$H_n(C_* \underset{R}{\otimes} Q, d \otimes 1) \cong H_n(C_*, d) \underset{R}{\otimes} Q.$$

An R-module M for which $\mathrm{Tor}^R_n(M, N) = 0$ for all $n > 0$ and left R-module N is called *flat*. Given a module M, it is always possible to find a resolution $F_* \longrightarrow M \to 0$ for which each F_k is flat. Then we also have

Proposition 3.5. *If $F_* \longrightarrow M \to 0$ is a flat resolution, then*

$$\mathrm{Tor}^R_n(M, N) = H_n(F_* \underset{R}{\otimes} N, d \otimes 1).$$

4. The Künneth Theorem

Suppose that (C_*, d) is a chain complex of free right R-modules. For any left R-module N we have another chain complex $(C_* \underset{R}{\otimes} N, d \otimes 1)$ with homology $H_*(C_* \underset{R}{\otimes} N, d \otimes 1)$. We would like to understand the connection between this homology and $H_*(C_*, d) \underset{R}{\otimes} N$.

Begin by taking a free resolution of N, $F_* \longrightarrow N \to 0$. For each n the complex

$$C_n \underset{R}{\otimes} F_* \longrightarrow C_n \underset{R}{\otimes} N \to 0$$

is still exact since C_n is free. The *double complex* $C_* \underset{R}{\otimes} F_*$ has two compatible families of boundaries, namely the 'horizontal' ones coming from the boundaries maps d tensored with the identity, $d \otimes 1$, and the 'vertical' ones coming from the identity tensored with the boundary maps δ of F_*, $1 \otimes \delta$. We can take the two types of homology in different orders to obtain

$$H^{\mathrm{v}}_*(H^{\mathrm{h}}_*(C_* \underset{R}{\otimes} F_*)) = H^{\mathrm{v}}_* = \mathrm{Tor}^R_*(H_*(C_*, d), N),$$
$$H^{\mathrm{h}}_*(H^{\mathrm{v}}_*(C_* \underset{R}{\otimes} F_*)) = H^{\mathrm{h}}_*(C_* \underset{R}{\otimes} N)) = H_*(C_* \underset{R}{\otimes} N, d \otimes 1).$$

In general, the precise relationship between these two involves a *spectral sequence*, however there are situations where the relationship is more direct.

Suppose that N has a free resolution of the form

$$0 \to F_1 \longrightarrow F_0 \longrightarrow N \to 0.$$

This will always happen when $R = \mathbb{Z}$ or any (commutative) pid and for semi-simple rings. Then for any right R-module M and $n > 1$,

$$\mathrm{Tor}^R_n(M, N) = 0.$$

Now consider what happens when we tensor C_* with such a resolution. We obtain a short exact sequence of chain complexes

$$0 \to C_* \underset{R}{\otimes} F_1 \longrightarrow C_* \underset{R}{\otimes} F_0 \longrightarrow C_* \underset{R}{\otimes} N \to 0$$

and on taking homology, an associated long exact sequence as in Theorem 2.1.

$$\cdots H_{n+1}(C_* \underset{R}{\otimes} N) \xrightarrow{\partial_{n+1}} H_n(C_* \underset{R}{\otimes} F_1) \longrightarrow H_n(C_* \underset{R}{\otimes} F_0) \longrightarrow H_n(C_* \underset{R}{\otimes} N) \xrightarrow{\partial_n} H_{n-1}(C_* \underset{R}{\otimes} F_1) \cdots$$

Because F_0 and F_1 are free, Corollary 3.4 gives in each case

$$H_n(C_* \underset{R}{\otimes} F_i) \cong H_n(C_*) \underset{R}{\otimes} F_i,$$

so our long exact sequence becomes

$$\cdots H_{n+1}(C_* \underset{R}{\otimes} N) \xrightarrow{\partial_{n+1}} H_n(C_*) \underset{R}{\otimes} F_1 \longrightarrow H_n(C_*) \underset{R}{\otimes} F_0 \longrightarrow H_n(C_* \underset{R}{\otimes} N) \xrightarrow{\partial_n} H_{n-1}(C_*) \underset{R}{\otimes} F_1 \cdots$$

The segment

$$H_n(C_*) \underset{R}{\otimes} F_1 \longrightarrow H_n(C_*) \underset{R}{\otimes} F_0$$

is part of the complex used to compute $\mathrm{Tor}^R_*(H_n(C_*),N)$ and we actually have the sequence

$$0 \to \mathrm{Tor}^R_1(H_n(C_*),N) \longrightarrow H_n(C_*)\underset{R}{\otimes}F_1 \longrightarrow H_n(C_*)\underset{R}{\otimes}F_0 \longrightarrow \mathrm{Tor}^R_0(H_n(C_*),N) \to 0$$

in which

$$\mathrm{Tor}^R_0(H_n(C_*),N) = H_n(C_*)\underset{R}{\otimes}N.$$

For $\partial_n\colon H_n(C_*\underset{R}{\otimes}N) \longrightarrow H_{n-1}(C_*\underset{R}{\otimes}F_1)$ we find that

$$\mathrm{Ker}\,\partial_n \cong \mathrm{Tor}^R_0(H_n(C_*),N), \quad \mathrm{Im}\,\partial_n \cong \mathrm{Tor}^R_1(H_{n-1}(C_*),N).$$

Theorem 4.1 (Künneth Theorem)**.** *Let (C_*,d) be a chain complex of free right R-modules and N a left R-module. For each n there is an exact sequence*

$$0 \to H_n(C_*)\underset{R}{\otimes}N \longrightarrow H_*(C_*\underset{R}{\otimes}N) \longrightarrow \mathrm{Tor}^R_1(H_{n-1}(C_*),N) \to 0.$$

5. Projective and injective resolutions, Hom and Ext

Let M,N be two right R-modules. Then we define

$$\mathrm{Hom}_R(M,N) = \{h : M \longrightarrow N : h \text{ is a homomorphism of } R\text{-modules}\}.$$

If R is commutative then $\mathrm{Hom}_R(M,N)$ is also an R-module, otherwise it may only be an abelian group.

If $f\colon M \longrightarrow M'$ and $g\colon N \longrightarrow N'$ are homomorphisms of R-modules, then there are functions

$$f^*\colon \mathrm{Hom}_R(M',N) \longrightarrow \mathrm{Hom}_R(M,N); \quad f^*h = h\circ f,$$
$$g_*\colon \mathrm{Hom}_R(M,N) \longrightarrow \mathrm{Hom}_R(M,N'); \quad g_*h = g\circ h.$$

These are group homomorphisms and homomorphisms of R-modules if R is commutative.

Proposition 5.1. *Let M,N be right R-modules.*

(a) *Given a short exact sequence of R-modules*

$$0 \to M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \to 0,$$

the sequence

$$0 \to \mathrm{Hom}_R(M_3,N) \xrightarrow{f_2^*} \mathrm{Hom}_R(M_2,N) \xrightarrow{f_1^*} \mathrm{Hom}_R(M_1,N)$$

is exact.

(b) *Given a short exact sequence of R-modules*

$$0 \to N_1 \xrightarrow{g_1} N_2 \xrightarrow{g_2} N_3 \to 0,$$

the sequence

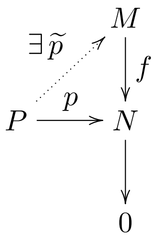
$$0 \to \mathrm{Hom}_R(M,N_1) \xrightarrow{g_{1*}} \mathrm{Hom}_R(M,N_2) \xrightarrow{g_{2*}} \mathrm{Hom}_R(M,N_3)$$

is exact.

These result show that $\mathrm{Hom}_R(\ ,N)$ and $\mathrm{Hom}_R(M,\ )$ are *left exact.*

An R-module P is called *projective* if given an exact sequence $M \xrightarrow{f} N \to 0$ and a (not usually unique) homomorphism $p\colon P \longrightarrow N$, there is a homomorphism $\widetilde{p}\colon P \longrightarrow M$ for which

$f\widetilde{p} = p$.



In particular, every free R-module is projective.

It is easy to see that if P is projective then $\mathrm{Hom}_R(P, \ )$ is right exact.

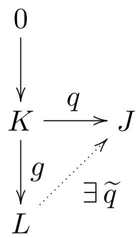
Now suppose that $P_* \longrightarrow M \to 0$ is a resolution of M by projective modules (for example, each P_n could be free). Then for any N we can form the cochain complex $\mathrm{Hom}_R(P_*, N)$ whose n-th term is $\mathrm{Hom}_R(P_n, N)$. The n-th cohomology group of this is

$$\mathrm{Ext}^n_R(M, N) = H^n(\mathrm{Hom}_R(P_*, N)).$$

It turns out that this is independent of the choice of projective resolution of M. Notice also that

$$\mathrm{Ext}^0_R(M, N) = \mathrm{Hom}_R(M, N).$$

An R-module J is called *injective* if given an exact sequence $0 \to K \xrightarrow{g} L$ and a homomorphism $q\colon K \longrightarrow J$, there is a (not usually unique) homomorphism $\widetilde{q}\colon L \longrightarrow J$ for which $\widetilde{q}g = q$.



It is easy to see that if J is injective, then $\mathrm{Hom}_R(\ , J)$ is right exact.

If $0 \to N \longrightarrow J^*$ is an exact cochain complex in which each J^n is injective (*i.e.*, an *injective resolution of* M) then we may form the cochain complex $\mathrm{Hom}_R(M, J^*)$ whose n-term is $\mathrm{Hom}_R(M, J^n)$. The n-th cohomology group of this is

$$\mathrm{rExt}^n_R(M, N) = H^n(\mathrm{Hom}_R(M, J^*)).$$

It turns out that this is independent of the choice of injective resolution of N. Notice also that

$$\mathrm{rExt}^0_R(M, N) = \mathrm{Hom}_R(M, N).$$

Proposition 5.2. *For R-modules M, N there is a natural isomorphism*

$$\mathrm{rExt}^n_R(M, N) \cong \mathrm{Ext}^n_R(M, N).$$

References

[1] H. Cartan & S. Eilenberg, Homological Algebra, Princeton University Press (1956).
[2] J. J. Rotman, An introduction to Homological Algebra, Academic Press (1979).
[3] C. A. Weibel, An Introduction to Homological Algebra, Cambridge University Press (1994).