A TASTE OF SPECTRAL SEQUENCES

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1. Exact couples

Suppose that $\mathscr{A}$ is an abelian category; for simplicity, we assume that its objects are sets and its morphisms are functions, but this is not really necessary since everything we do with elements can be verified only using morphisms.

An exact sequence of morphisms of the form

(1.1)
$$\begin{array}{ccc} D & \xrightarrow{\quad i \quad} & D \\ {\scriptstyle k}\nwarrow & & \swarrow{\scriptstyle j} \\ & E & \end{array}$$

is called an *exact couple*. Notice that $d = jk$ is a differential, *i.e.*, it satisfies $d^2 = dd = jkjk = 0$. Therefore we can form the homology

$$\mathrm{H}(E, d) = \ker d / \operatorname{im} d.$$

Now consider the sequence

(1.2)
$$\begin{array}{ccc} D' & \xrightarrow{\quad i' \quad} & D' \\ {\scriptstyle k'}\nwarrow & & \swarrow{\scriptstyle j'} \\ & E' & \end{array}$$

where

$$D' = iD = \ker j, \quad E' = \mathrm{H}(E, d),$$

and

$$i' = i_{|D'}, \quad j' = \overline{j_{|D} i^{-1}}, \quad k' = k_{|\ker d}.$$

Here the notation $(\overline{\ })$ indicates passage to cosets.

To see what k' really does, let $z \in \mathrm{H}(E, d)$. Then $jk(z) = 0$, so $k(z) = i(z')$ for some $z' \in D$. But if $w \in D$, then $k(j(w)) = 0$, so this is well-defined on $\mathrm{H}(E, d)$. Notice also that the new differential d' works as follows: using the same notation, we have $d(z') = 0$, so $[j(z')] \in \mathrm{H}(E, d)$ and $d'([z]) = [j(z')]$.

Then (1.2) is also exact and is called the *derived exact couple* of (1.1).

Iterating this we obtain a sequence of such exact couples

(1.3)
$$\begin{array}{ccc} D^{(r)} & \xrightarrow{\quad i^{(r)} \quad} & D^{(r)} \\ {\scriptstyle k^{(r)}}\nwarrow & & \swarrow{\scriptstyle j^{(r)}} \\ & E^{(r)} & \end{array}$$

in which $E^{(r+1)} = \mathrm{H}(E^{(r)}, d^{(r)})$.

In examples it is common to have gradings on things. For example, we might have integer bigradings on D and E for which

$$i\colon D_{p,q} \longrightarrow D_{p+1,q-1}, \quad j\colon D_{p,q} \longrightarrow E_{p+1,q-1}, \quad k\colon E_{p,q} \longrightarrow D_{p-1,q}, \quad d\colon E_{p,q} \longrightarrow D_{p,q-1}.$$

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Then if we set

$$D^r = D^{(r-1)}, \quad E^r = E^{(r-1)}, \quad i^r = i^{(r-1)}, \quad j^r = j^{(r-1)}, \quad k^r = k^{(r-1)}, \quad d^r = d^{(r-1)},$$

we find that

$$i^r : D^r_{p,q} \longrightarrow D^r_{p+1,q-1}, \ j^r : D^r_{p,q} \longrightarrow E^r_{p+1,q-1}, \ k^r : E^r_{p,q} \longrightarrow D^r_{p-1,q} \ d^r : E^r_{p,q} \longrightarrow E^r_{p-r,q+r-1}.$$

2. Grothendieck spectral sequences

Let $\mathscr{A}, \mathscr{B}, \mathscr{C}$ be abelian categories and suppose that $\mathscr{A}, \mathscr{B}$ have enough projectives, so objects admit resolutions by projective objects.

Suppose that we have right exact covariant functors

$$\Phi \colon \mathscr{A} \rightsquigarrow \mathscr{B}, \quad \Theta \colon \mathscr{B} \rightsquigarrow \mathscr{C}.$$

Then the left derived functors $\mathbf{L}_s\Phi$ and $\mathbf{L}_s\Theta$ exist for $s \geqslant 0$.

Suppose also that for every projective object P of $\mathscr{A}$ we have

$$\mathbf{L}_s\Theta(\Phi(P)) = 0 \quad (s \geqslant 1).$$

Theorem 2.1 (Grothendieck spectral sequences)**.** *For every A in $\mathscr{A}$ there is a spectral sequence*

$$E^2_{p,q} = \mathbf{L}_p\Theta\mathbf{L}_q\Phi(A) \Longrightarrow \mathbf{L}_{p+q}(\Theta\Phi)(A).$$

Furthermore, this spectral sequence is functorial in A.

This is a *first quadrant spectral sequence* since $E^2_{p,q} = 0$ unless $p, q \geqslant 0$.

Example 2.2 (Change of rings for Tor)**.** Suppose that $R \longrightarrow S$ is a ring homomorphism, and let N be a right S-module. Consider the functors

$$\Phi = (\ \ ) \otimes_R S \colon \mathbf{Mod}_R \rightsquigarrow \mathbf{Mod}_S, \quad \Theta = (\ \ ) \otimes_S N \colon \mathbf{Mod}_S \rightsquigarrow \mathbf{AbGps}$$

Then for every left R-module M there is a spectral sequence

$$E^2_{p,q} = \mathrm{Tor}^S_p(\mathrm{Tor}^R_q(M, S), N) \Longrightarrow \mathrm{Tor}^R_{p+q}(M, N).$$

There are two standard ways to indicate such a first quadrant spectral sequence graphically.

In one we put $E^r_{p,q}$ at position (p, q) in the plane representing the *r-th page of the spectral sequence*, then the differentials go r steps to the left and up by $(r-1)$ steps. Notice that for each (p, q) there is a number $r(p, q)$ for which $E^r_{p,q} = E^{r(p,q)}_{p,q}$ when $r \geqslant r(p, q)$; we write

$$E^\infty_{p,q} = E^{r(p,q)}_{p,q}.$$

Then for each $n \geqslant 0$, there is a filtration

$$0 = F_{-1,n} \subseteq F_{0,n} \subseteq F_{1,n} \subseteq \cdots \subseteq F_{n,n} = \mathrm{Tor}^R_n(M, N)$$

for which

$$F_{k,n}/F_{k-1,n} \cong E^\infty_{k,n-k} \quad (0 \leqslant k \leqslant n).$$

Thus each $\mathrm{Tor}^R_n(M, N)$ arises from terms in the first quadrant lying on the line $p + q = n$ and the filtration increases with the p-value. Here the *edge homomorphism*

$$E^2_{0,n} = \mathrm{Tor}^R_n(M, S) \otimes_S N \longrightarrow E^\infty_{0,n} \xrightarrow{\cong} F_{0,n} \xrightarrow{\mathrm{inc}} \mathrm{Tor}^R_n(M, N)$$

is the obvious homomorphism induced from the associativity homomorphism

$$(M \otimes_R S) \otimes_S N \xrightarrow{\cong} M \otimes_R (S \otimes_S N) \xrightarrow{\cong} M \otimes_R N.$$

There is also another edge homomorphism

$$\mathrm{Tor}^R_n(M, N) \longrightarrow F_{n,n}/F_{n-1,n} \xrightarrow{\cong} E^\infty_{n,0} \xrightarrow{\mathrm{inc}} E^2_{n,0} = \mathrm{Tor}^S_n(M \otimes_R S, N)$$

agreeing with the obvious map.

An alternative is to plot points in the (s, t)-plane where we put $E^r_{p,q}$ at position $(p + q, p)$; then the non-trivial terms appear only in the region where $s \geqslant t \geqslant 0$. This time the differential d^r goes 1 to the left and down by r. Then $\mathrm{Tor}^R_n(M, N)$ arises from terms on the $t = n$ column and the filtration increases with the t-value.

There are variations of the Grothendieck spectral sequence 2.1 involving left (or a mixture of right and left) exact functors.

Example 2.3 (Change of rings for Ext)**.** Suppose that $R \longrightarrow S$ is a ring homomorphism, and let L be a right S-module. Consider the functors

$$\Phi = \mathrm{Hom}_R(S, \ \ ) \colon \mathbf{Mod}_R \rightsquigarrow \mathbf{Mod}_S, \quad \Theta = \mathrm{Hom}_S(L, \ \ ) \colon \mathbf{Mod}_S \rightsquigarrow \mathbf{AbGps}$$

Then for every right R-module M there is a third quadrant spectral sequence

$$E^2_{p,q} = \mathrm{Ext}_S^{-p}(L, \mathrm{Ext}_R^{-q}(S, M) \Longrightarrow \mathrm{Ext}_R^{-p-q}(L, M).$$

For each (p,q) there is an $r(p,q)$ for which

$$E^r_{p,q} = E^{r(p,q)}_{p,q} = E^\infty_{p,q}.$$

For each $n \geqslant 0$, there is a filtration

$$\mathrm{Ext}^n_R(L, M) = F^{0,n} \supseteq F^{1,n} \supseteq \cdots \supseteq F^{n,n} \supseteq F^{n+1,n} = 0$$

for which

$$F^{k,n}/F^{k+1,n} \cong E^\infty_{-k,k-n} \quad (n \geqslant k \geqslant 0).$$

Example 2.4 (Change of rings for Ext)**.** Suppose that $R \longrightarrow S$ is a ring homomorphism, and let N be a right S-module. Consider the functors

$$\Phi = (\ \ ) \otimes_R S \colon \mathbf{Mod}_R \rightsquigarrow \mathbf{Mod}_S, \quad \Theta = \mathrm{Hom}_S(\ \ , N) \colon \mathbf{Mod}_S \rightsquigarrow \mathbf{AbGps}$$

Then for every right R-module M there is a third quadrant spectral sequence

$$E^2_{p,q} = \mathrm{Ext}_S^{-p}(\mathrm{Tor}^R_{-q}(M, S), N) \Longrightarrow \mathrm{Ext}_R^{-p-q}(M, N).$$

For each (p,q) there is an $r(p,q)$ for which

$$E^r_{p,q} = E^{r(p,q)}_{p,q} = E^\infty_{p,q}.$$

For each $n \geqslant 0$, there is a filtration

$$\mathrm{Ext}^n_R(M, N) = F^{0,n} \supseteq F^{1,n} \supseteq \cdots \supseteq F^{n,n} \supseteq F^{n+1,n} = 0$$

for which

$$F^{k,n}/F^{k+1,n} \cong E^\infty_{-k,k-n} \quad (n \geqslant k \geqslant 0).$$

Remark 2.5. By switching signs, such a spectral sequencecan be regraded as a *cohomological spectral sequence* $(E_r^{p,q}, d_r)$ with $d_r \colon E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}$.

Example 2.6 (Lyndon-Hochschild-Serre spectral sequence)**.** Let G be a finite group and $N \triangleleft G$. Consider the functors

$$\Phi = (\ \ )^N \colon \mathbf{Mod}_{\mathbb{Z}[G]} \rightsquigarrow \mathbf{Mod}_{\mathbb{Z}[G/N]}, \quad \Theta = (\ \ )^{G/N} \colon \mathbf{Mod}_{\mathbb{Z}[G/N]} \rightsquigarrow \mathbf{AbGps}$$

Then for every $\mathbb{Z}[G]$-module M there is a first quadrant cohomological spectral sequence

$$E_2^{p,q} = \mathrm{H}^p(G/N; \mathrm{H}^q(N; M)) \Longrightarrow \mathrm{H}^{p+q}(G; M).$$

3. A spectral sequence for Tor and Ext of a limit

Recall that in the category of right R modules, for any small filtered category, every functor $F \colon I^{\mathrm{op}} \rightsquigarrow \mathbf{Mod}_R$ (often called an inverse system in $\mathbf{Mod}_R$) there is an inverse limit $\varprojlim_I F$. This defines a left exact covariant functor

$$\varprojlim_I \colon \mathbf{Mod}_R^{I^{\mathrm{op}}} \rightsquigarrow \mathbf{Mod}_R$$

on the abelian category of inverse systems in $\mathbf{Mod}_R$; this category has enough injectives. However, there is a weaker kind of injective object, called *flasque* or *flabby*; this is an object J for which every short exact sequence

$$0 \to J \longrightarrow A \longrightarrow B \to 0$$

gives rise to an short exact sequence

$$0 \to \varprojlim_I J \longrightarrow \varprojlim_I A \longrightarrow \varprojlim_I B \to 0.$$

For every inverse system F there is a flabby resolution $0 \to F \longrightarrow J^\bullet$ which can be defined as follows. For each $s \geqslant 0$, define J^s by

$$J^s(i) = \prod_{i_s \to i_{s-1} \xrightarrow{\neq} \cdots \xrightarrow{\neq} i_0 \xrightarrow{\neq} i} F(i_s),$$

where the product is taken over all diagrams in I where no arrow is the identity except perhaps $i_s \to i_{s-1}$. Then there is a map $F \longrightarrow J^0$ defined from the product of all the maps induced by diagrams $j \longrightarrow i$,

$$F(i) \longrightarrow J^0(i) = \prod_{j \to i} F(j).$$

Notice that this map is injective and it cokernel is $\overline{J}^0$ defined by

$$\overline{J}^0(i) = \prod_{j \xrightarrow{\neq} i} F(j).$$

Iterating this we obtain a flabby resolution $0 \to F \longrightarrow J^\bullet$. Then for $s \geqslant 0$ we have

$$\mathbf{R}^s \varprojlim_I F = \varprojlim_I{}^s F = \mathrm{H}^s(\varprojlim_I J^\bullet, \delta),$$

the cohomology of the resulting cochain complex

$$0 \to \varprojlim_I J^0 \xrightarrow{\delta} \varprojlim_I J^1 \xrightarrow{\delta} \cdots \xrightarrow{\delta} \varprojlim_I J^s \xrightarrow{\delta} \cdots$$

Theorem 3.1. *Let F be an inverse system of left R-modules for which $\varprojlim_I{}^s F = 0$ if $s > 0$.*

(a) *Let M be a right R-module which admits a resolution by finitely generated free modules. Then there is a second quadrant spectral sequence*

$$E^2_{p,q} = \varprojlim_I{}^{(-p)} \mathrm{Tor}^R_q(M, F) \Longrightarrow \mathrm{Tor}^R_{p+q}(M, \varprojlim_I F).$$

(b) *Let L be a left R-module. Then there is a first quadrant cohomological spectral sequence*

$$E_2^{p,q} = \varprojlim_I{}^p \mathrm{Ext}^q_R(L, F) \Longrightarrow \mathrm{Ext}^{p+q}_R(L, \varprojlim_I F).$$

Remark 3.2. There are some useful vanishing results for $\varprojlim_I{}^s$ due to Roos and Jensen [3, 5]. For example, if the cardinality of I is $\aleph_d$ then $\varprojlim_I{}^s F = 0$ whenever $s > d + 1$. So for example when I is countable, only $\varprojlim_I F$ and $\varprojlim_I{}^1 F$ can be non-zero.

Theorem 3.3. *Let F be an inverse system of left R-modules.*

(a) *Let M be a right R-module which admits a resolution by finitely generated free modules. Then if each F_i is flat, there is a third quadrant spectral sequence*

$$E^2_{p,q} = \mathrm{Tor}^R_p(M, \varprojlim_I{}^{(-q)} F) \Longrightarrow \varprojlim_I{}^{p+q}(M \otimes_R F).$$

(b) *Let L be a left R-module. Then there is a first quadrant cohomological spectral sequence*

$$E_2^{p,q} = \mathrm{Ext}^p_R(L, \varprojlim_I{}^q F) \Longrightarrow \varprojlim_I{}^{p+q} \mathrm{Hom}_R(L, F).$$

Example 3.4. Suppose that we have a collection of left R-modules N_i for $i \in I$ and define

$$F_i = \prod_{i' \longrightarrow i} N_{i'}.$$

Then for each $j \longrightarrow i$, we have the projection map $F_i \longrightarrow F_j$ which forgets all the factors N_k in F_i where there is no map $k \longrightarrow j$. Since each of these maps is an epimorphism, this is a flabby system and so $\varprojlim_I{}^s F = 0$ if $s > 0$. So from Theorem 3.1 we get spectral sequences of the following forms.

(a) Let M be a right R-module which admits a resolution by finitely generated free modules. Then there is a second quadrant spectral sequence

$$E^2_{p,q} = \varprojlim_I{}^{(-p)} \operatorname{Tor}^R_q(M, F) \Longrightarrow \operatorname{Tor}^R_{p+q}(M, \prod_{i \in I} N_i).$$

(b) Let L be a left R-module. Then there is a first quadrant cohomological spectral sequence

$$E_2^{p,q} = \varprojlim_I{}^{p} \operatorname{Ext}^q_R(L, F) \Longrightarrow \operatorname{Ext}^{p+q}_R(L, \prod_{i \in I} N_i).$$

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