SMOOTH, NONSYMPLECTIC EMBEDDINGS OF RATIONAL BALLS IN THE COMPLEX PROJECTIVE PLANE

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Abstract. We exhibit an infinite family of rational homology balls which embed smoothly but not symplectically in the complex projective plane. We also obtain a new lattice embedding obstruction from Donaldson’s diagonalisation theorem, and use this to show that no two of our examples may be embedded disjointly.

1. Introduction

A Markov triple is a positive integer solution \((p_1, p_2, p_3)\) to the Markov equation

\[
p_1^2 + p_2^2 + p_3^2 = 3p_1p_2p_3.
\]

Each Markov triple gives rise to an embedding

\[
\bigsqcup_{i=1}^{3} B_{p_i, q_i} \hookrightarrow \mathbb{CP}^2
\]

of a disjoint union of three rational homology balls in the complex projective plane. Here \(B_{p,q}\) is the rational homology ball smoothing of the quotient singularity \(\mathbb{P}^2(1, pq-1)\). The embedding in (2) arises by smoothing the three singular points in the weighted projective space \(\mathbb{P}(p_1^2, p_2^2, p_3^2)\), and the numbers \(q_i\) are given by

\[
q_i = \pm 3p_j/p_k \pmod{p_i},
\]

where \(i, j, k\) is a permutation of 1, 2, 3. The apparent sign ambiguity here is due to the fact that \(B_{p,q} \cong B_{p,p-q}\).

Hacking and Prokhorov proved in [5] that any projective surface with quotient singularities which admits a smoothing to \(\mathbb{CP}^2\) is \(\mathbb{Q}\)-Gorenstein deformation equivalent to some \(\mathbb{P}(p_1^2, p_2^2, p_3^2)\) as above. Evans and Smith proved in [4] that any disjoint union \(\bigsqcup_{i \in I} B_{p_i, q_i}\) which admits a symplectic embedding in \(\mathbb{CP}^2\) arises in this way, with \(|I| \leq 3\).

Let \(F(2n-1)\) denote the \(n\)th odd Fibonacci number, defined by the recursion

\[
F(2n+3) = 3F(2n+1) - F(2n-1), \quad F(1) = 1, \quad F(3) = 2.
\]

Then \((1, F(2n-1), F(2n+1))\) is a Markov triple for each \(n \in \mathbb{N}\), showing in particular that \(B_{F(2n+1), F(2n-3)}\) admits a symplectic embedding in \(\mathbb{CP}^2\) for each \(n > 1\).

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In [8] we mentioned but overlooked the significance of the following result. Here $\Delta_{p,q}$ is a properly embedded surface in the 4-ball whose double branched cover is $B_{p,q}$, and $P_+$ is the unknotted Möbius band in the 4-ball with normal Euler number 2; see [8] for further details.

**Theorem 1.** For each $n \in \mathbb{N}$, the slice surface $\Delta_{F(2n+1),F(2n-1)}$ admits a simple embedding as a sublevel surface of the unknotted Möbius band $P_+$. Taking double branched covers yields a simple smooth embedding

$$B_{F(2n+1),F(2n-1)} \hookrightarrow \mathbb{CP}^2.$$

If $n > 1$, then $B_{F(2n+1),F(2n-1)}$ does not embed symplectically in $\mathbb{CP}^2$.

Theorem 1 gives the first-known smooth embeddings of rational balls $B_{p,q}$ in the complex projective plane that do not arise from symplectic embeddings. This shows that the smooth embedding problem has an as-yet-unknown solution which differs from that to the symplectic problem solved by Evans-Smith. Bulent Tosun has informed the author that work of Nemirovski-Segal [7] implies the existence of a rational ball, bounded by a Seifert fibred space with 3 exceptional fibres, which embeds smoothly but not sympectically in $\mathbb{CP}^2$. Most of the embeddings obtained in [8], but not those given in Theorem 1, have since been reproved and generalised by different methods in [9].

A conjecture of Kollár [6] would imply that at most three rational balls $B_{p_i,q_i}$ may embed smoothly and disjointly in $\mathbb{CP}^2$. The following result gives some mild support to this conjecture.

**Theorem 2.** It is not possible to smoothly embed a disjoint union $\bigsqcup_{i \in I} B_{p_i,q_i}$ of two or more of the balls from Theorem 1 in $\mathbb{CP}^2$, where each $(p_i,q_i)$ is a consecutive pair of odd Fibonacci numbers.

This result uses a new obstruction derived from Donaldson’s diagonalisation theorem [3]. This is stated in Proposition 3.2.

**Corrigendum to [8].** In [8, sentence after Theorem 5, and Remark 4.1] we incorrectly stated that $B_{F(2n+1),F(2n-1)}$ embeds symplectically in $\mathbb{CP}^2$. I am very grateful to Giancarlo Urzúa who reminded me that the Markov triple $(1, F(2n-1), F(2n+1))$ gives rise to a symplectic embedding in $\mathbb{CP}^2$ of $B_{F(2n+1),F(2n-3)}$, and not of $B_{F(2n+1),F(2n-1)}$.

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## 2. Smooth embeddings

In this section we prove Theorem 1, using the method from [8].
We refer the reader to [1] for an excellent and readable source on Markov numbers. Suppose that \((p, a, b)\) is a solution to the Markov equation (1) with \(p > a, b\). By [1, Corollary 3.4], the integers in a Markov triple are pairwise relatively prime, so that there are unique solutions \(x = u, u'\) to
\[
b \equiv \pm xa \pmod{p}.
\]
These satisfy \(u + u' \equiv 0 \pmod{p}\), so that one of them (say \(u\)) is between 0 and \(p/2\); we call this number \(u\) the characteristic number of the Markov triple \((p, a, b)\). The Markov equation gives
\[
a^2 + b^2 \equiv 0 \pmod{p},
\]
from which it follows that
\[
u^2 \equiv -1 \pmod{p}.
\]
I am grateful to Jonny Evans for helping me to see the following result.

**Lemma 2.1.** Let \(n \in \mathbb{N}\). The rational ball \(B_{F(2n+1), F(2n-1)}\) embeds symplectically in \(\mathbb{CP}^2\) if and only if \(n = 1\).

**Proof.** From [4, Theorem 4.15] we have that \(B_{p,q}\) embeds symplectically in \(\mathbb{CP}^2\) if and only if \(p\) is the maximum of a Markov triple \((a, b, p)\), and \(q = \pm 3b/a \pmod{p}\). Then in fact \(q = \pm 3u\), where \(u\) is the characteristic number of the Markov triple.

For \(n > 1\), the odd Fibonacci number \(F(2n + 1)\) is the maximum of the Markov triple \((1, F(2n - 1), F(2n + 1))\), from which it follows that \(B_{F(2n+1), F(2n-3)}\) embeds symplectically. Also note that the characteristic number of this Markov triple is \(F(2n - 1)\), and \(F(2n - 1)^2 \equiv -1 \pmod{F(2n + 1)}\).

Then \(B_{F(2n+1), F(2n-1)}\) embeds symplectically if and only if \(F(2n+1)\) is the maximum of another Markov triple \((a, b, F(2n + 1))\), and \(F(2n - 1) = \pm 3u\), where \(u\) is the characteristic number of the triple \((a, b, F(2n + 1))\). This would imply that
\[
-1 \equiv F(2n - 1)^2 \equiv 9u^2 \equiv -9 \pmod{F(2n + 1)}.
\]
The only odd Fibonacci numbers which divide 8 are \(F(1) = 1\) and \(F(3) = 2\), so we conclude that \(n = 1\).

Finally, \(F(3) = 2\) is the maximum of the Markov triple \((1, 1, 2)\) and \(B_{F(3), F(1)} = B_{2,1}\) does embed symplectically. \(\square\)

**Proof of Theorem 1.** As noted in the proof of Lemma 2.1, the Markov triple \((1, 1, 2)\) gives rise to an embedding of \(B_{F(3), F(1)} = B_{2,1}\) in \(\mathbb{CP}^2\). Suppose now that \(n > 1\). Induction using (3) yields the Hirzebruch-Jung continued fraction expansion
\[
\frac{F(2n + 1)}{F(2n - 1)} = [3^{n-1}, 2].
\]
Now using [8, Lemma 3.1] we have
\[
\frac{F(2n + 1)^2}{F(2n + 1)F(2n - 1) - 1} = [3^{n-1}, 5, 3^{n-2}, 2].
\]
These continued fractions may be used to describe the surface $\Delta_{F(2n+1),F(2n-1)}$, as described in [8].

The proof that $\Delta_{F(2n+1),F(2n-1)}$ is a sublevel surface of $P_+$ is a minor modification of the proof of [8, Theorem 5]. We refer the reader to that source for details.

Consider the first diagram shown in Figure 1. This represents a surface $\Sigma$ bounded by the unknot, which we claim is $P_+$. Note first that the band move corresponding to the blue band labelled 0 converts the diagram to one of $\Delta_{F(2n+1),F(2n-1)}$, which is the slice disk described by Casson and Harer [2] for the two-bridge knot $S(F(2n + 1)^2,F(2n + 1)F(2n - 1) - 1)$. This shows that $\Delta_{F(2n+1),F(2n-1)}$ is a sublevel surface of the surface $\Sigma$. It remains to see that $\Sigma$ is the unknotted M"{o}bius band $P_+$ whose double branched cover is $\mathbb{C}\mathbb{P}^2$ minus a 4-ball.

![Figure 1](image-url)

**Figure 1.** The slice disk $\Delta_{F(2n+1),F(2n-1)}$ as a sublevel surface of $P_+$, and the resulting equivariant rational blow up. Numbers beside bands give the signed equivariant count of half-twists or crossings.

Figure 2 shows a sequence of isotopies and band slides converting $\Sigma$ to $P_+$ in the first case of interest which is $n = 2$. Taking double branched covers we see that $B_{5,2}$ admits a smooth embedding in $\mathbb{C}\mathbb{P}^2$. The proof for $n > 2$ follows by an induction argument involving band slides similar to those in Figure 2. The inductive step is shown in Figure 3.

Recall that an embedding of $B_{p,q}$ in a 4-manifold $Z$ is called simple if the resulting rational blow up of $Z$ is obtainable by a sequence of ordinary blow ups. The proof that the embeddings described above are simple follows as in [8, Proposition 5.1]; we again refer the reader to [8] for more details on equivariant rational blow up, and to Section 3 for a description of rational blow up. We describe here a slightly shorter version of the proof at the level of double branched covers. The second diagram in Figure 1 represents the surface in the 4-ball pushed in from the black surface of the
two-bridge diagram shown, using a chessboard colouring in which the unbounded region is white. The rational blow up of \( \mathbb{C}P^2 \), minus a 4-ball, is the double cover \( X \) of the 4-ball branched along this black surface, which in turn is the plumbing of disk bundles over \( S^2 \) corresponding to the linear graph with weights

\[
(-3)^{n-1}, -2, -1, (-3)^{n-2}, -2,
\]

where \((-3)^m\) denotes \(-3\) repeated \(m\) times. A sequence of \(-1\) blow downs reduces this to the linear plumbing with weights \(-3\) and 0, which is diffeomorphic to \( \mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \), again minus a ball. It follows that

\[
X \cong \mathbb{C}P^2 \# (2n - 1)\overline{\mathbb{C}P^2}.
\]

Together with Lemma 2.1, this completes the proof of Theorem 1. \(\square\)
3. AN OBSTRUCTION FROM DONALDSON’S DIAGONALISATION THEOREM

In this section, we derive a lattice embedding obstruction to smoothly embedding a rational homology ball bounded by a lens space, or a disjoint union of such, in $\mathbb{CP}^2$. We begin by setting some conventions and terminology.

All homology and cohomology groups in this section have integer coefficients. Recall that if $X$ is a smooth 4-manifold, possibly with boundary, then its intersection lattice $\Lambda_X$ consists of the free abelian group $H_2(X)/\text{Tors}$ together with the symmetric bilinear intersection pairing. The term lens space will be used here to refer to $L(p, q)$ with $p > q \geq 1$; in particular not $S^3$ or $S^2 \times S^1$. Given integers $a_1, \ldots, a_k$, the linear lattice $\Lambda(a_1, \ldots, a_k)$ is defined to be the free abelian group with generators $v_1, \ldots, v_n$, and with symmetric bilinear pairing given by

$$v_i \cdot v_j = \begin{cases} a_i & \text{if } i = j; \\ -1 & \text{if } |i - j| = 1; \\ 0 & \text{if } |i - j| > 1. \end{cases}$$

As this is the lattice associated to a weighted linear graph, we often refer to the generators $v_1, \ldots, v_k$ as vertices. Recall that a lens space $L(p, q)$ is the boundary of a plumbing $C$ of disk bundles over spheres determined by the weighted linear graph
with weights $a_1, \ldots, a_k \geq 2$ where

$$\frac{p}{p-q} = [a_1, a_2, \ldots, a_k] := a_1 - \frac{1}{a_2 - \ldots - \frac{1}{a_k}}.$$ 

The intersection lattice of $C$ is then $\Lambda(a_1, \ldots, a_k)$.

Let $B$ be a rational homology ball with lens space boundary. Given an embedding $B \hookrightarrow \mathbb{CP}^2$, we let $M$ be the complement $\mathbb{CP}^2 \setminus B$ and “rationally blow up” to obtain the closed positive-definite manifold $M \cup C$, where $C$ is the positive-definite plumbed manifold bounded by $\partial B$. Donaldson’s diagonalisation theorem then implies the existence of a lattice embedding

$$\Lambda_M \oplus \Lambda_C \hookrightarrow \mathbb{Z}^m,$$

where $\Lambda_M$ and $\Lambda_C$ are the intersection lattices of $M$ and $C$ respectively, and $m$ is the sum of their ranks.

The reader familiar with the use of such lattice obstructions will note that since $M$ is a submanifold of $\mathbb{CP}^2$, and since $Y = \partial B$ bounds a rational ball, each of $\Lambda_M$ and $\Lambda_C$ admit finite-index embeddings in diagonal unimodular lattices, so that an embedding as in (5) must in fact exist, with the first factor embedding in $\mathbb{Z}$ and the second in the orthogonal $\mathbb{Z}^{m-1}$. We will show that simple topological considerations place further restrictions on the lattice embedding in (5), giving rise to a useful obstruction, which also extends to the case of an embedding of a disjoint union of rational balls.

**Lemma 3.1.** Let $B_i$ be rational homology balls bounded by lens spaces for $i = 1, \ldots, n$, and suppose that the disjoint union $\bigsqcup_i B_i$ embeds smoothly in $\mathbb{CP}^2$. Then the complement $M = \mathbb{CP}^2 \setminus \bigsqcup_i B_i$ has $H_1(M; \mathbb{Z}) = 0$ and $H_2(M; \mathbb{Z}) \cong \mathbb{Z}$.

**Proof.** We use the Mayer-Vietoris sequence and induction. The base case is $n = 0$ and $M = \mathbb{CP}^2$.

Now suppose $M' = \mathbb{CP}^2 \setminus \bigsqcup_{i=1}^{n-1} B_i$ has $H_1(M'; \mathbb{Z}) = 0$ and $H_2(M'; \mathbb{Z}) \cong \mathbb{Z}$. Then

$$M' = M \cup_Y B_n,$$

where $Y = L(p_n^2, q_n)$ has $H_1(Y; \mathbb{Z}) \cong \mathbb{Z}/p_n^2\mathbb{Z}$. We have $H_2(B_n; \mathbb{Z}) = 0$, since it is a torsion subgroup of $H_2(M'; \mathbb{Z}) \cong \mathbb{Z}$; then from the long exact sequence of the pair $(B_n, Y)$, we have $H_1(B_n; \mathbb{Z}) \cong \mathbb{Z}/p_n\mathbb{Z}$. The Mayer-Vietoris sequence, with integer coefficients, shows that $H_2(M)$ is a finite-index subgroup of $\mathbb{Z}$, hence $H_2(M) \cong \mathbb{Z}$. The same sequence shows that there is a surjection from $\mathbb{Z}/p_n^2\mathbb{Z}$ to $H_1(M) \oplus \mathbb{Z}/p_n\mathbb{Z}$, from which it follows that the latter direct sum is finite cyclic and also that cyclic summands of $H_1(M)$ have orders dividing $p_n$. We conclude that $H_1(M)$ must be trivial. 

We recall the notion of rational blow up, and modify and generalise it for our convenience. If a disjoint union $\bigsqcup_i B_i$ embeds smoothly in some 4-manifold $Z$, where each $B_i$ is a rational ball bounded by a lens space $L(p_i, q_i)$, then we may excise each $B_i$.
and replace it by the positive-definite plumbed manifold $C_i$ bounded by $L(p_i, p_i - q_i)$ to obtain a new manifold

$$X = M \cup C,$$

called the positive rational blow up of $Z$. Here $M$ is the complement of $\bigsqcup_{i=1}^n B_i$ in $Z$, and $C$ is the disjoint union $\bigsqcup_{i=1}^n C_i$ of plumbed manifolds. We assume that all weights in each plumbing $C_i$ are at least 2.

**Proposition 3.2.** Let $B_i$ be rational homology balls bounded by lens spaces for $i = 1, \ldots, n$, and suppose that the disjoint union $\bigsqcup_i B_i$ embeds smoothly in $\mathbb{CP}^2$. Let $X = M \cup C$ be the resulting positive rational blow up of $\mathbb{CP}^2$. Then there exists a finite-index lattice embedding

$$\Lambda_M \oplus \Lambda_C \hookrightarrow \mathbb{Z}^m,$$

such that each unit vector $e \in \mathbb{Z}^m$ has nonzero pairing with each of $\Lambda_M$ and $\Lambda_C$. Moreover the image of the generator of $\Lambda_M$ is a primitive vector in $\mathbb{Z}^m$.

**Remark 3.3.** Let $A$ be the matrix of the embedding in (6) in terms of a basis $v_1, \ldots, v_m$ for $\Lambda_X$, where $v_1 \in \Lambda_M$ and $v_2, \ldots, v_m \in \Lambda_C$, and an orthonormal basis for $\mathbb{Z}^m$. Then the proposition states that each row of $A$ has at least two nonzero entries including one in the first column, and also that the entries of the first column of $A$, which are all nonzero, have no common divisor.

The known embeddings mentioned earlier in this section each give rise to a block diagonal matrix $A$ which does not satisfy the condition in the proposition.

**Proof of Proposition 3.2.** Let $Y$ denote the union of lens spaces which is the common boundary of $M$ and $C$. Let $e$ be a unit vector in $\Lambda_X$. We may write

$$e = e_M + e_C,$$

where $e_M \in H_2(M, Y)$ and $e_C \in H_2(C, Y)$. There are no unit vectors in $\Lambda_M$, which is a rank one lattice whose generator squared is the order of the first homology of $Y$. There are also no unit vectors in $\Lambda_C$ since we assumed all weights in each plumbing are at least 2. It follows that $e_M$ and $e_C$ are both nonzero.

Since $H_1(M) = 0$ by Lemma 3.1, all homology groups of $M$ are in fact torsion-free by standard arguments using universal coefficients, Poincaré-Lefschetz duality, and the long exact sequence of the pair. It follows that the second homology group $H_2(M)$ is the underlying group of the lattice $\Lambda_M$, and the relative homology group $H_2(M, Y)$ is the underlying group of the dual lattice $\Lambda_M^*$ via the universal coefficient theorem. Then since $\Lambda_M$ is positive definite, we see that an element of $H_2(M, Y)$ is nonzero if and only if it has nonzero intersection with some element in $H_2(M)$. Thus in particular $e_M$ and also $e$ has nonzero intersection with some element of $H_2(M)$. The same argument applies to $\Lambda_C$, so that $e_C$ and also $e$ has nonzero pairing with some element of $H_2(C)$ which is the underlying group of $\Lambda_C$. 

Finally let \( v \) denote the image in \( H_2(X) \) of the generator of \( \Lambda_M \), and suppose that \( v = kw \) for some \( k \in \mathbb{N} \) and \( w \in H_2(X) \). As above we write \( w = w_M + w_C \) and we conclude that \( w_C = 0 \) since it has zero pairing with all of \( \Lambda_C \). This implies \( w = w_M \in \Lambda_M \), but then \( k = 1 \) since \( v \) is the generator. \( \square \)

In what follows we study lattice embeddings \( \Lambda \hookrightarrow \mathbb{Z}^m \) up to lattice automorphisms of \( \mathbb{Z}^m \), or in other words, up to reordering of the orthonormal basis \( e_1, \ldots, e_m \), and/or changing signs of some orthonormal basis elements. Embeddings of linear lattices all of whose weights are 2 or 3 are very restricted, since up to \( \text{Aut}(\mathbb{Z}^m) \), vectors \( v \in \mathbb{Z}^m \) with \( v \cdot v = 2 \) or \( v \cdot v = 3 \) take the form \( v = e_1 + e_2 \) or \( v = e_1 + e_2 + e_3 \).

**Example 3.4.** The rational ball \( B_{3,1} \) does not embed smoothly in \( \mathbb{CP}^2 \).

*Proof.* The boundary of \( B_{3,1} \) is the lens space \( L(9,2) \), which also bounds the positive-definite plumbing \( C \) with weights \( [2,2,2,3] \). Let \( v_2, \ldots, v_5 \) be the generators of the linear lattice \( \Lambda_C = \Lambda(2,2,2,3) \) as in (4), and let \( v_1 \) be the generator of the rank one lattice \( \Lambda_M = \Lambda(9) \). Let \( e_1, \ldots, e_5 \) be an orthonormal basis for \( \mathbb{Z}^5 \). There is, up to lattice automorphisms of \( \mathbb{Z}^5 \), a unique embedding

\[
\Lambda_M \oplus \Lambda_C \hookrightarrow \mathbb{Z}^5;
\]

this takes \( v_1 \) to \( 3e_1 \), \( v_i \) to \(-e_i + e_{i+1}\) for \( 2 \leq i \leq 4 \), and \( v_5 \) to \( e_2 + e_3 + e_4 \). This does not satisfy the conditions of Proposition 3.2, since \( e_i \) has zero pairing with \( \Lambda_M \) for \( i > 1 \) and \( e_1 \) has zero pairing with \( \Lambda_C \). \( \square \)

**Lemma 3.5.** Up to \( \text{Aut}(\mathbb{Z}^m) \), there are precisely two ways to embed the linear lattice \( \Lambda(2,2,2) \) in \( \mathbb{Z}^m \), where \( m \geq 4 \). The first has image in a \( \mathbb{Z}^3 \) sublattice of \( \mathbb{Z}^m \), and its orthogonal complement in this sublattice is the zero sublattice. The second has image in a \( \mathbb{Z}^4 \) sublattice, and its orthogonal complement in \( \mathbb{Z}^4 \) is spanned by a vector \( w \) with \( w \cdot w = 4 \).

Let \( n > 1 \) and let \( \Lambda \) denote the linear lattice \( \Lambda(3^{n-1},2,2,3^{n-1},2) \), with rank \( r = 2n + 1 \). Up to \( \text{Aut}(\mathbb{Z}^m) \), there are precisely three ways to embed \( \Lambda \) in \( \mathbb{Z}^m \), where \( m \in \mathbb{N} \) is sufficiently large. The first has image in a \( \mathbb{Z}^r \) sublattice, and its orthogonal complement in this sublattice is the zero sublattice. The second has image in a \( \mathbb{Z}^{r+1} \) sublattice, and its orthogonal complement in \( \mathbb{Z}^{r+1} \) is spanned by a vector \( w \) with \( w \cdot w = F(2n + 1)^2 \). The third has image in a \( \mathbb{Z}^{4n} \) sublattice, and its orthogonal complement in \( \mathbb{Z}^{4n} \) contains no unit vectors.

*Proof.* For the first case, we can either map the vertices of \( \Lambda(2,2,2) \) to \(-e_1 + e_2, -e_2 + e_3, e_1 + e_2\) or to \(-e_1 + e_2, -e_2 + e_3, -e_3 + e_4\). It is straightforward to see there are no other possibilities.

In the second case we begin by embedding the two adjacent vertices of weight two. Up to automorphism of \( \mathbb{Z}^m \), these are mapped to \(-e_1 + e_2\) and \(-e_2 + e_3\). By inspection, the linear lattice \( \Lambda(3,2,2,3) \), which is a sublattice of \( \Lambda \), admits three
possible embeddings up to symmetry as follows:
\[-e_2 - e_3 - e_4, -e_1 + e_2, -e_2 + e_3, e_1 + e_2 - e_4;\]
\[-e_2 - e_3 - e_4, -e_1 + e_2, -e_2 + e_3, -e_3 + e_4 + e_5;\]
\[\text{or } e_1 + e_4 + e_5, -e_1 + e_2, -e_2 + e_3, -e_3 + e_6 + e_7.\]

The first of these does not extend to an embedding of \(\Lambda(3, 2, 2, 3, 2)\) or \(\Lambda(3, 2, 2, 3, 3)\) so we discard it. By a simple induction argument, the second of these extends uniquely to an embedding of \(\Lambda(3^{n-1}, 2, 2, 3^{n-1})\) as follows:
\[-e_{2n-2} - e_{2n-1} - e_{2n}, \ldots, -e_4 - e_5 - e_6, -e_2 - e_3 - e_4, -e_1 + e_2,\]
\[-e_2 + e_3, -e_3 + e_4 + e_5, -e_5 + e_6 + e_7, \ldots, -e_{2n-1} + e_{2n} + e_{2n+1}.\]
This can be extended to an embedding of \(\Lambda\) in precisely two ways: we may map the additional weight two vertex to \(e_{2n} - e_{2n+1}\) or to \(-e_{2n+1} + e_{2n+2}\). The first choice results in an embedding in \(\mathbb{Z}'\). The second choice results in an embedding in \(\mathbb{Z}^{r+1}\). The orthogonal complement in \(\mathbb{Z}^{r+1}\) has rank one and so is generated by a vertex \(w\). We may compute \(w\) and hence its square directly or use the fact that \(\Lambda\) is a primitive sublattice of \(\mathbb{Z}^{r+1}\) with determinant \(F(2n + 1)^2\), which is therefore also the determinant of its rank one orthogonal complement.

Finally another simple induction argument shows that the third embedding in (7) extends uniquely to \(\Lambda(3^{n-1}, 2, 2, 3^{n-1})\), and also extends uniquely up to symmetry to give the following embedding of \(\Lambda\):
\[e_{4n-7} + e_{4n-4} - e_{4n-3}, \ldots, e_5 + e_8 - e_9, e_1 + e_4 - e_5, -e_1 + e_2, -e_2 + e_3,\]
\[-e_3 + e_6 + e_7, -e_7 + e_{10} + e_{11}, \ldots, -e_{4n-5} + e_{4n-2} + e_{4n-1}, -e_{4n-1} + e_{4n}.\]
We see that each of \(e_1, \ldots, e_{4n}\) appears in (8), and therefore has nonzero pairing with the image of this embedding.

\[\square\]

Proof of Theorem 2. For the duration of this proof, we denote by \(B_n\) the rational ball \(B_{F(2n+1), F(2n-1)}\), and by \(C_n\) the positive-definite plumbed manifold with the same boundary as \(B_n\), for each \(n \in \mathbb{N}\). For \(n = 1\), the boundary of the rational ball \(B_1 = B_{2,1}\) is \(L(4, 1)\), and the plumbing \(C_1\) has weights \([2, 2, 2]\). For \(n > 1\), \(C_n\) is the plumbing with weights \([3^{n-1}, 2, 2, 3^{n-1}, 2]\), as may be seen using [8, Lemma 3.1].

Suppose first that \(B_1 \sqcup B_n\) embeds smoothly in \(\mathbb{CP}^2\). Let \(r_1 = 3\) and \(r_2\) denote the ranks of \(\Lambda_{C_1}\) and \(\Lambda_{C_n}\) respectively. By Proposition 3.2, there is a finite-index lattice embedding
\[\Lambda_M \oplus \Lambda_{C_1} \oplus \Lambda_{C_n} \hookrightarrow \mathbb{Z}^m,\]
where \(m = r_1 + r_2 + 1 = r_2 + 4\). By Lemma 3.5, the restriction of this to \(\Lambda_{C_1}\) is either contained in a \(\mathbb{Z}^3\) or is contained in a \(\mathbb{Z}^4\), spanned by \(e_1, \ldots, e_4\) say, with orthogonal complement spanned by a vector \(w\) of self-pairing 4. Since the image of the generator of \(\Lambda_M\) is orthogonal to the image of \(\Lambda_{C_1}\) and has nonzero pairing with every unit
vector in $\mathbb{Z}^m$ by Proposition 3.2, it must be the second possibility. The image of $\Lambda_{C_2}$ lies in the orthogonal complement to that of $\Lambda_{C_1}$. If it is contained in the span of $e_5, \ldots, e_m$ then this is a finite-index embedding in $\mathbb{Z}^r$ which again contradicts the fact that the image of the generator of $\Lambda_M$ has nonzero pairing with every unit vector. Thus at least one vertex of $\Lambda_{C_2}$ contains a nonzero multiple of $w$. This vertex then has self-pairing greater than that of $w$, contradicting the fact that the vertices of $\Lambda_{C_2}$ all have self-pairing 2 or 3.

We next suppose that $B_k \cup B_n$ embeds smoothly in $\mathbb{C}P^2$ with $n \geq k > 1$. Let $r_1 = 2k + 1$ and $r_2 = 2n + 1$ denote the ranks of $\Lambda_{C_k}$ and $\Lambda_{C_n}$ respectively. By Proposition 3.2, there is a finite-index lattice embedding

$$\Lambda_M \oplus \Lambda_{C_k} \oplus \Lambda_{C_n} \hookrightarrow \mathbb{Z}^m,$$

where $m = r_1 + r_2 + 1 = 2k + 2n + 3$.

Arguing as in the previous case, we see that the restriction of this embedding to $\Lambda_{C_k}$ (respectively $\Lambda_{C_n}$) cannot have image in either $\mathbb{Z}^{r_1}$ or $\mathbb{Z}^{r_1+1}$ (respectively $\mathbb{Z}^{r_2}$ or $\mathbb{Z}^{r_2+1}$). By Lemma 3.5, this leaves the possibility that the restriction to $\Lambda_{C_k}$ lies in a $\mathbb{Z}^{4k}$ sublattice, and similarly the restriction to $\Lambda_{C_n}$ lies in a $\mathbb{Z}^{4n}$ sublattice, in both cases with the orthogonal complement in said sublattice containing no unit vectors. In particular we have

$$4n \leq 2k + 2n + 3,$$

and hence $n$ is either $k$ or $k + 1$.

If $n = k + 1$, we have $m = 4n + 1$. Up to $\text{Aut}(\mathbb{Z}^m)$, we may suppose that the $\mathbb{Z}^{4n-1}$ sublattice containing the image of $\Lambda_{C_n}$ includes the vectors $-e_1 + e_2, -e_2 + e_3$ as the image of the two adjacent weight two vertices. The $\mathbb{Z}^{4n}$ sublattice of $\mathbb{Z}^{4n+1}$ containing the image of $\Lambda_{C_n}$ has to intersect the $\mathbb{Z}^3$ sublattice spanned by $e_1, e_2, e_3$ nontrivially. This means that some vertex of $\Lambda_{C_n}$ maps to a vector of the form $v + a(e_1 + e_2 + e_3)$, where $v$ is a nonzero vector in the span of $e_4, \ldots, e_m$ and $a \neq 0$, noting that the image of this vertex is orthogonal to $-e_1 + e_2, -e_2 + e_3$ and has pairing $-1$ with a neighbouring vertex. This contradicts the fact that all vertices in $\Lambda_{C_n}$ have weight 2 or 3.

Finally if $n = k$ then $m = 4n + 3$. We keep the notation $\Lambda_{C_n}$ and $\Lambda_{C_k}$ to distinguish the two copies of $\Lambda_{C_n}$. We may suppose that the $\mathbb{Z}^m$ sublattice containing the image of $\Lambda_{C_k}$ is the span of $e_1, \ldots, e_{4n}$, and that it includes the vectors $-e_1 + e_2, -e_2 + e_3$ as the image of the two adjacent weight two vertices. Arguing as in the case $n = k + 1$, the image of $\Lambda_{C_n}$ has to be orthogonal to the span of $e_1, e_2, e_3$, and so is contained in the span of $e_4, \ldots, e_m$. We may also suppose that the two adjacent weight two vertices in $\Lambda_{C_n}$ map to $-e_{4n+1} + e_{4n+2}, -e_{4n+2} + e_{4n+3}$. We consider the image of $\Lambda_{C_n}$ and $\Lambda_{C_k}$ under the projection to $\mathbb{Z}^{4n-3}$ spanned by $e_4, e_5, \ldots, e_{4n}$. From (8) we see that each of these is isomorphic to the orthogonal direct sum $\Lambda(3^{n-2}, 2) \oplus \Lambda(2, 3^{n-2}, 2)$, and has rank $2n - 1$. This leads to a contradiction, since it is not possible to orthogonally embed two lattices of rank $2n - 1$ in $\mathbb{Z}^{4n-3}$.\[\square\]

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