SIGNATURES, HEEGAARD FLOER CORRECTION TERMS AND QUASI–ALTERNATING LINKS

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Abstract. Turaev showed that there is a well–defined map assigning to an oriented link L in the three–sphere a Spin structure $\mathbf{t}_0$ on $\Sigma(L)$, the two–fold cover of S^3 branched along L. We prove, generalizing results of Manolescu–Owens and Donald–Owens, that for an oriented quasi–alternating link L the signature of L equals minus four times the Heegaard Floer correction term of $(\Sigma(L), \mathbf{t}_0)$.

1. Introduction

Vladimir Turaev [21, § 2.2] proved that there is a surjective map which associates to a link $L \subset S^3$ decorated with an orientation o a Spin structure $\mathbf{t}_{(L,o)}$ on $\Sigma(L)$, the double cover of S^3 branched along L. Moreover, he showed that the only other orientation on L which maps to $\mathbf{t}_{(L,o)}$ is $-o$, the overall reversed orientation. In other words, Turaev described a bijection between the set of quasi–orientations on L (i.e. orientations up to overall reversal) and the set $\mathrm{Spin}(\Sigma(L))$ of Spin structures on $\Sigma(L)$. Each element $\mathbf{t} \in \mathrm{Spin}(\Sigma(L))$ can be viewed as a Spin^c structure on $\Sigma(L)$, so if $\Sigma(L)$ is a rational homology sphere it makes sense to consider the rational number $d(\Sigma(L), \mathbf{t})$, where d is the correction term invariant defined by Ozsváth and Szabó [13]. Under the assumption that L is nonsplit alternating it was proved — in [10] when L is a knot and in [3] for any number of components of L — that

$$(*) \qquad \sigma(L, o) = -4d(\Sigma(L), \mathbf{t}_{(L,o)}) \quad \text{for every orientation } o \text{ on } L,$$

where $\sigma(L, o)$ is the link signature. For an alternating link associated to a plumbing graph with no bad vertices, this follows from a combination of earlier results of Saveliev [19] and Stipsicz [20], each of whom showed that one of the quantities in $(*)$ is equal to the Neumann-Siebenmann $\overline{\mu}$-invariant of the plumbing tree. The main purpose of this paper is to prove Property $(*)$ for the family of *quasi–alternating links* introduced in [14]:

Definition 1. The *quasi–alternating* links are the links in S^3 with nonzero determinant defined recursively as follows:

(1) the unknot is quasi–alternating;
(2) if L_0, L_1 are quasi–alternating, $L \subset S^3$ is a link such that $\det L = \det L_0 + \det L_1$ and L, L_0, L_1 differ only inside a 3–ball as illustrated in Figure 1, then L is quasi–alternating.

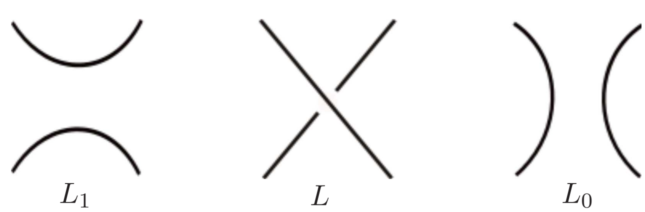

Figure 1. L and its resolutions L_0 and L_1.

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Quasi–alternating links have recently been the object of considerable attention [1, 2, 4, 5, 6, 11, 16, 17, 22, 23]. Alternating links are quasi–alternating [14, Lemma 3.2], but (as shown in e.g. [1]) there exist infinitely many quasi–alternating, non–alternating links. Our main result is the following:

Theorem 1. *Let (L, o) be an oriented link. If L is quasi–alternating then*

$$\sigma(L, o) = -4d(\Sigma(L), \mathbf{t}_{(L,o)}). \tag{1}$$

The contents of the paper are as follows. In Section 2 we first recall some basic facts on Spin structures and the existence of two natural 4–dimensional cobordisms, one from $\Sigma(L_1)$ to $\Sigma(L)$, the other from $\Sigma(L)$ to $\Sigma(L_0)$. Then, in Proposition 1 we show that for an orientation o on L for which the crossing in Figure 1 is positive, the Spin structure $\mathbf{t}_{(L,o)}$ extends to the first cobordism but not to the second one. In Section 3 we use this information together with the Heegaard Floer surgery exact triangle to prove Proposition 2, which relates the value of the correction term $d(\Sigma(L), \mathbf{t}_{(L,o)})$ with the value of an analogous correction term for $\Sigma(L_1)$. In Section 4 we restate and prove our main result, Theorem 1. The proof consists of an inductive argument based on Proposition 2 and the known relationship between the signatures of L and L_1. The use of Proposition 2 is made possible by the fact that up to mirroring L one may always assume the crossing of Figure 1 to be positive. We close Section 4 with Corollary 3, which uses results of Rustamov and Mullins to relate Turaev's torsion function for the two–fold branched cover of a quasi–alternating link L with the Jones polynomial of L.

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2. Triads and Spin structures

A Spin structure on an n–manifold M^n is a double cover of the oriented frame bundle of M with the added condition that if $n > 1$, it restricts to the nontrivial double cover on fibres. A Spin structure on a manifold restricts to give a Spin structure on a codimension–one submanifold, or on a framed submanifold of codimension higher than one. As mentioned in the introduction, an orientation o on a link L in S^3 induces a Spin structure $\mathbf{t}_{(L,o)}$ on the double–branched cover $\Sigma(L)$, as in [21]. Recall also that there are two Spin structures on $S^1 = \partial D^2$: the nontrivial or *bounding* Spin structure, which is the restriction of the unique Spin structure on D^2, and the trivial or *Lie* Spin structure, which does not extend over the disk. The restriction map from Spin structures on a solid torus to Spin structures on its boundary is injective; thus if two Spin structures on a closed 3–manifold agree outside a solid torus then they are the same. For more details on Spin structures see for example [7].

If Y is a 3–manifold with a Spin structure $\mathbf{t}$ and K is a knot in Y with framing λ, we may attach a 2–handle to K giving a surgery cobordism W from Y to $Y_\lambda(K)$. There is a unique Spin structure on $D^2 \times D^2$, which restricts to the bounding Spin structure on each framed circle $\partial D^2 \times \{\text{point}\}$ in $\partial D^2 \times D^2$. Thus the Spin structure on Y extends over W if and only if its restriction to K, viewed as a framed submanifold via the framing λ, is the bounding Spin structure. Note that this is equivalent, symmetrically, to the restriction of $\mathbf{t}$ to the submanifold λ framed by K being the bounding Spin structure. Moreover, the extension over W is unique if it exists.

Let L, L_0, L_1 be three links in S^3 differing only in a 3–ball B as in Figure 1. The double cover of B branched along the pair of arcs $B \cap L$ is a solid torus $\widetilde{B}$ with core C. The boundary of a properly embedded disk in B which separates the two branching arcs lifts to a disjoint pair of meridians of $\widetilde{B}$. The preimage in $\Sigma(L)$ of the curve λ_0 shown in Figure 2 is a pair of parallel framings for C; denote one of these by $\tilde\lambda_0$. Similarly, let $\tilde\lambda_1$ denote one of the components of the preimage in $\Sigma(L)$ of λ_1. Since λ_0 is homotopic in $B - L$ to the boundary of a disk separating the two components of $L_0 \cap B$, we see that $\Sigma(L_0)$ is obtained from $\Sigma(L)$ by $\tilde\lambda_0$–framed surgery on C. Similarly, λ_1 is homotopic in $B-L$ to the boundary of a disk separating the two components of $L_1 \cap B$, and $\Sigma(L_1)$ is obtained from $\Sigma(L)$ by $\tilde{\lambda}_1$–framed surgery on C.

The two framings $\tilde{\lambda}_0$ and $\tilde{\lambda}_1$ differ by a meridian of C. In the terminology from [14], the manifolds $\Sigma(L)$, $\Sigma(L_0)$ and $\Sigma(L_1)$ form a *triad* and there are surgery cobordisms

$$V : \Sigma(L_1) \to \Sigma(L), \quad \text{and} \quad W : \Sigma(L) \to \Sigma(L_0). \tag{2}$$

The surgery cobordism W is built by attaching a 2–handle to $\Sigma(L)$ along the knot C with framing $\tilde{\lambda}_0$. The cobordism V is built by attaching a 2–handle to $\Sigma(L_1)$. Dualising this handle structure, V is obtained by attaching a 2–handle to $\Sigma(L)$ along the knot C with framing $\tilde{\lambda}_1$ (and reversing orientation).

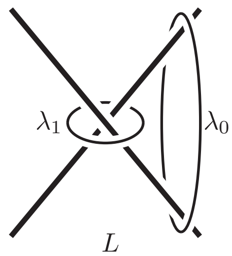

FIGURE 2. The loops λ_0 and λ_1.

Proposition 1. *For any orientation o on L such that the crossing shown in Figure 1 is positive, the Spin structure $\mathbf{t}_{(L,o)}$ extends to a unique Spin structure $\mathbf{s}_o$ on the cobordism V and does not admit an extension over W. The restriction of $\mathbf{s}_o$ to $\Sigma(L_1)$ is the Spin structure $\mathbf{t}_{(L_1,o_1)}$, where o_1 is the orientation on L_1 induced by o.*

Proof. Let $\pi : \Sigma(L) \to S^3$ be the branched covering map. The Spin structure $\mathbf{t}_{(L,o)}$ is the lift $\tilde{\mathbf{s}}$ of the Spin structure restricted from S^3 to $S^3 - L$, twisted by $h \in H^1(\Sigma(L) - \pi^{-1}(L); \mathbb{Z}/2\mathbb{Z})$, where the value of h on a curve γ is the parity of half the sum of the linking numbers of $\pi \circ \gamma$ about the components of L (following Turaev [21, §2.2]). Suppose that the crossing in Figure 1 is positive as, for example, illustrated in Figure 3, so that the orientation o induces an orientation o_1 on L_1.

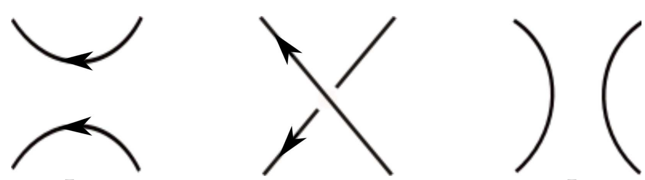

FIGURE 3. The oriented link (L, o) together with the oriented resolution (L_1, o_1) and the unoriented resolution L_0.

Then, we can compute from Figure 2 that $h(\tilde{\lambda}_1) = 0$ and $h(\tilde{\lambda}_0) = 1$. The Spin structure on S^3 restricts to the bounding structure on each of λ_0 and λ_1 using the 0–framing. The map π restricts to a diffeomorphism on neighbourhoods of $\tilde{\lambda}_0$ and $\tilde{\lambda}_1$. Therefore, the restriction of $\tilde{\mathbf{s}}$ to each of $\tilde{\lambda}_0$ and $\tilde{\lambda}_1$ using the pullback of the 0–framing is also the bounding structure. Also note that the preimage under π of a disk bounded by λ_i is an annulus with core C, so the framing of $\tilde{\lambda}_i$ given by C is the same as the pullback of the 0–framing.

The spin structure $\mathbf{t}_{(L,o)}$ is equal to $\tilde{\mathbf{s}}$ twisted by h. Since $\tilde{\mathbf{s}}$ restricts to the bounding spin structure on $\tilde{\lambda}_1$, and $h(\tilde{\lambda}_1) = 0$, we see that $\mathbf{t}_{(L,o)}$ restricts to the bounding Spin structure on $\tilde{\lambda}_1$ using the framing given by C. On the other hand since $h(\tilde{\lambda}_0) = 1$, $\mathbf{t}_{(L,o)}$ restricts to the Lie Spin structure on $\tilde{\lambda}_0$, again using the framing given by C. It follows that $\mathbf{t}_{(L,o)}$ admits a unique extension $\mathbf{s}_o$ over the 2–handle giving the cobordism V, and does not extend over the cobordism W.

The restriction of $\mathbf{s}_o$ to $\Sigma(L_1)$ coincides with $\mathbf{t}_{(L_1,o_1)}$ outside of the solid torus $\widetilde{B}$, and therefore also on the closed manifold $\Sigma(L_1)$. □

3. Relations between correction terms

By [14, Proposition 2.1] we have the following exact triangle:

$$\begin{array}{ccc} \widehat{HF}(\Sigma(L_1)) & \xrightarrow{F_V} & \widehat{HF}(\Sigma(L)) \\ & \nwarrow \quad \swarrow {\scriptstyle F_W} & \\ & \widehat{HF}(\Sigma(L_0)) & \end{array}$$

where the maps F_V and F_W are induced by the surgery cobordisms of (2). (All the Heegaard Floer groups are taken with $\mathbb{Z}/2\mathbb{Z}$ coefficients.)

By [14, Proposition 3.3] (and notation as in that paper), if $L \subset S^3$ is a quasi–alternating link and L_0 and L_1 are resolutions of L as in Definition 1 then $\Sigma(L)$, $\Sigma(L_0)$ and $\Sigma(L_1)$ are L–spaces. Moreover, by assumption we have

$$|H^2(\Sigma(L);\mathbb{Z})| = |H^2(\Sigma(L_0);\mathbb{Z})| + |H^2(\Sigma(L_1);\mathbb{Z})|. \tag{3}$$

Since for every L–space Y we have $|H^2(Y;\mathbb{Z})| = \dim \widehat{HF}(Y)$, the Heegaard Floer surgery exact triangle reduces to a short exact sequence:

$$0 \to \widehat{HF}(\Sigma(L_1)) \xrightarrow{F_V} \widehat{HF}(\Sigma(L)) \xrightarrow{F_W} \widehat{HF}(\Sigma(L_0)) \to 0. \tag{4}$$

The type of argument employed in the proof of the following proposition goes back to [9] and was also used in [20].

Proposition 2. *Let L be a quasi–alternating link and let L_0, L_1 be resolutions of L as in Definition 1. Let o be an orientation on L for which the crossing of Figure 1 is positive, and let o_1 be the induced orientation on L_1. Then, the following holds:*

$$-4d(\Sigma(L),\mathbf{t}_{(L,o)}) = -4d(\Sigma(L_1),\mathbf{t}_{(L_1,o_1)}) - 1.$$

Proof. Since $\Sigma(L)$, $\Sigma(L_1)$ and $\Sigma(L_0)$ are L–spaces, we may think of the Spin^c structures on these spaces as generators of their $\widehat{HF}$–groups, and we shall abuse our notation accordingly. Let $V : \Sigma(L_1) \to \Sigma(L)$ be the surgery cobordism of (2), and let $\mathbf{s}_o$ be the unique Spin structure on V which extends $\mathbf{t}_{(L,o)}$ as in Proposition 1. Recall that, by definition, the map F_U associated to a cobordism $U : Y_1 \to Y_2$ is given by

$$F_U = \sum_{\mathbf{s}\in \mathrm{Spin}^c(U)} F_{U,\mathbf{s}},$$

where $F_{U,\mathbf{s}} : \widehat{HF}(Y_1,\mathbf{t}_1) \to \widehat{HF}(Y_2,\mathbf{t}_2)$ and $\mathbf{t}_i = \mathbf{s}|_{Y_i}$ for $i = 1, 2$. We claim that

$$F_{V,\mathbf{s}_o}(\mathbf{t}_{(L_1,o_1)}) = \mathbf{t}_{(L,o)}. \tag{5}$$

The Heegaard Floer $\widehat{HF}$–groups admit a natural involution, usually denoted by $\mathcal{J}$. The maps induced by cobordisms are equivariant with respect to the $\mathbb{Z}/2\mathbb{Z}$–actions associated to conjugation on Spin^c structures and the $\mathcal{J}$–map on the Heegaard Floer groups, in the sense that, if $\overline{x} := \mathcal{J}(x)$ for an element x, we have

$$F_{W,\overline{\mathbf{s}}}(\overline{x}) = \overline{F_{W,\mathbf{s}}(x)} \tag{6}$$

for each $\mathbf{s} \in \mathrm{Spin}^c(W)$. Since by Proposition 1 there are no Spin structures on the surgery cobordism $W : \Sigma(L) \to \Sigma(L_0)$ of (2) which restrict to $\mathbf{t}_{(L,o)}$, the element $F_W(\mathbf{t}_{(L,o)}) \in \widehat{HF}(\Sigma(L_0))$ has no Spin component. In fact, since $\mathbf{t}_{(L,o)}$ is fixed under conjugation and we are working over $\mathbb{Z}/2\mathbb{Z}$, (6) implies that the contribution of each non–Spin $\mathbf{s} \in \mathrm{Spin}^c(W)$ to a Spin component of $F_W(\mathbf{t}_{(L,o)})$ is cancelled by the contribution of $\overline{\mathbf{s}}$ to the same component. Therefore we may write

$$F_W(\mathbf{t}_{(L,o)}) = x + \overline{x}$$

for some $x \in \widehat{HF}(\Sigma(L_0))$. By the surjectivity of F_W there is some $y \in \widehat{HF}(\Sigma(L))$ with $F_W(y) = x$, therefore $F_W(\mathbf{t}_{(L,o)} + y + \overline{y}) = 0$, and by the exactness of (4) we have $\mathbf{t}_{(L,o)} + y + \overline{y} = F_V(z)$ for some $z \in \widehat{HF}(\Sigma(L_0))$. Since $F_V(\overline{z}) = \overline{F_V(z)} = F_V(z)$, the injectivity of F_V implies $z = \overline{z}$. Moreover, z must have some nonzero Spin component, otherwise we could write $z = u + \overline{u}$ and

$$F_V(u + \overline{u}) = \overline{F_V(u)} + \overline{F_V(\overline{u})} = \overline{F_V(u)} + F_V(u)$$

could not have the Spin component $\mathbf{t}_{(L,o)}$. This shows that there is a Spin structure $\mathbf{t} \in \widehat{HF}(\Sigma(L_1))$ such that $F_V(\mathbf{t}) = \mathbf{t}_{(L,o)}$. But, as we argued before for $F_W(\mathbf{t}_{(L,o)})$, in order for $F_V(\mathbf{t})$ to have a Spin component it must be the case that there is some Spin structure $\mathbf{s}$ on V such that $F_{V,\mathbf{s}}(\mathbf{t}) = \mathbf{t}_{(L,o)}$. Applying Proposition 1 we conclude $\mathbf{s} = \mathbf{s}_o$ and therefore $\mathbf{t} = \mathbf{t}_{(L_1,o_1)}$. This establishes Claim (5).

Using Equation (3) and the fact that $\det(L_1) > 0$ it is easy to check that V is negative definite. The statement follows immediately from Equation (5) and the degree–shift formula in Heegaard Floer theory [15, Theorem 7.1] using the fact that $c_1(\mathbf{s}_o) = 0$, $\sigma(V) = -1$ and $\chi(V) = 1$. □

4. The main result and a corollary

Theorem 1. *Let (L, o) be an oriented link. If L is quasi–alternating then*

$$\sigma(L, o) = -4d(\Sigma(L), \mathbf{t}_{(L,o)}). \tag{1}$$

Proof. The statement trivially holds for the unknot, because the unknot has zero signature and the two–fold cover of S^3 branched along the unknot is S^3, whose only correction term vanishes. If L is not the unknot and L is quasi–alternating, there are quasi–alternating links L_0 and L_1 such that $\det(L) = \det(L_0) + \det(L_1)$ and L, L_0 and L_1 are related as in Figure 1. To prove the theorem it suffices to show that if the statement holds for L_0 and L_1 then it holds for L as well.

Denote by L^m the mirror image of L, and by o^m the orientation on L^m naturally induced by an orientation o on L. The orientation–reversing diffeomorphism from S^3 to itself taking L to L^m lifts to one from $\Sigma(L)$ to $\Sigma(L^m)$ sending $\mathbf{t}_{(L,o)}$ to $\mathbf{t}_{(L^m,o^m)}$. Thus by [8, Theorem 8.10] and [13, Proposition 4.2] we have

$$\sigma(L^m, o^m) = -\sigma(L, o) \quad \text{and} \quad 4d(\Sigma(L^m), \mathbf{t}_{(L^m,o^m)}) = 4d(-\Sigma(L), \mathbf{t}_{(L,o)}) = -4d(\Sigma(L), \mathbf{t}_{(L,o)}),$$

therefore Equation (1) holds for (L, o) if and only if it holds for (L^m, o^m). Hence, without loss of generality we may now fix an orientation o on L so that the crossing appearing in Figure 1 is positive.

Denote by o_1 the orientation on L_1 naturally induced by o. By [11, Lemma 2.1]

$$\sigma(L, o) = \sigma(L_1, o_1) - 1. \tag{7}$$

Since we are assuming that the statement holds for L_1, we have

$$\sigma(L_1, o_1) = -4d(\Sigma(L_1), \mathbf{t}_{(L_1,o_1)}). \tag{8}$$

Equations (7) and (8) together with Proposition 2 immediately imply Equation (1). □

Corollary 3. *Let (L, o) be an oriented, quasi–alternating link. Then,*

$$\tau(\Sigma(L), \mathbf{t}_{(L,o)}) = -\frac{1}{12} \frac{V'_{(L,o)}(-1)}{V_{(L,o)}(-1)},$$

where τ is Turaev's torsion function and $V_{(L,o)}(t)$ is the Jones polynomial of (L, o).

Proof. By [18, Theorem 3.4] we have

$$d(\Sigma(L), \mathbf{t}_{(L,o)}) = 2\chi(HF^+_{\text{red}}(\Sigma(L))) + 2\tau(\Sigma(L), \mathbf{t}_{(L,o)}) - \lambda(\Sigma(L)), \tag{9}$$

where λ denotes the Casson–Walker invariant, normalized so that it takes value -2 on the Poincaré sphere oriented as the boundary of the negative E_8 plumbing. Moreover, since L is quasi–alternating $\Sigma(L)$ is an L–space, therefore the first summand on the right–hand side of (9) vanishes. By [12, Theorem 5.1], when $\det(L) > 0$ we have

$$\lambda(\Sigma(L)) = -\frac{1}{6}\frac{V'_{(L,o)}(-1)}{V_{(L,o)}(-1)} + \frac{1}{4}\sigma(L,o)), \tag{10}$$

Therefore, when (L, o) is an oriented quasi–alternating link, Theorem 1 together with Equations (9) and (10) yield the statement. □

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