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Three-strand pretzel knots, knot Floer homology and concordance invariants

by

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Abstract

This thesis is concerned with determining the knot Floer homology and concordance invariants of pretzel knots, in particular three-strand pretzel knots. Knot Floer homology is a package of knot invariants developed by Ozsváth and Szabó, and despite the invariants being known for simple classes of knots — for example quasi-alternating, two-bridge and L-space knots — there are still many simple families for which knot Floer homology and the associated concordance invariants are not known.

Recent work by Ozsváth-Szabó in [49] developed a construction of an algebraic invariant $\mathcal{C}(D)$ conjectured by them to be equal to a variant of knot Floer homology. This complex $\mathcal{C}(D)$ is a bigraded, bifiltered chain complex whose filtered chain homotopy type is an invariant of a knot [49]. Their construction — which has also been implemented in a C++ program, see [47] — is a divide and conquer method which decomposes knot diagrams in a certain form into smaller pieces, to which algebraic objects are then associated. These algebraic objects are themselves invariants (up to appropriate equivalence) of partial knot diagrams, and are pieced together to form the full invariant. As with classical knot Floer homology, one can study the homology of this complex $\mathcal{C}(D)$, or the homology of subcomplexes and quotient complexes, which are also invariants of a knot.

Even more recent work of Ozsváth-Szabó in [48] confirms that this conjectured equivalence between the theories holds. Hence, like the well-known grid homology of a knot [30, 35], this algebraic method provides a combinatorial construction of knot Floer homology — or in this case some slightly modified version of classical knot Floer homology, like that presented by Dai-Hom-Stroffregen-Truong in [4]. The benefit of such combinatorial constructions is that they do not rely on computation of the counts of pseudo-holomorphic representatives of Whitney disks in some high-dimensional space, unlike classical knot Floer homology.

The grid homology developed in [30, 35] has the disadvantage that although one need not calculate these counts — since by construction all Whitney disks considered in this theory have a single pseudo-holomorphic representative — this is at the expense of computing the homology of chain complexes with a very large number of generators (relative to crossing number).

However, the algebraic invariant $\mathcal{C}(D)$ of Ozsváth-Szabó has the form of a chain complex whose generators are in one-to-one correspondence with the Kauffman states of a knot diagram. Kauffman states are decorated, oriented knot projections, and the bigrading of the corresponding generators can be determined from the Kauffman states. Similarly, classical knot Floer homology can also be calculated from a chain complex generated by Kauffman states, as demonstrated in [37].

Adapting the work of Eftekhary in [5], the Kauffman states for a three-strand pretzel knot P can be placed into three families, based upon the positions of the decorations on each of the three strands. These families have grading information that is determined by the positions of the decorations on each strand — see Table 2.1 and Table 2.2 for explicit calculations of these gradings. Using the grading information associated to these Kauffman states, one can restrict the possible differentials within the knot Floer chain complex $CFK^\infty(P)$, as demonstrated by Lemma 2.10. Furthermore, the classification of the Kauffman states into these three families with well-understood grading information makes three-strand pretzel knots particularly amenable to study using the divide and conquer construction of [49].

After an introduction to knot Floer homology and the current knowledge for pretzel knots and links provided in Chapter 1, this thesis will present in Chapter 2 a definition of Kauffman states, their grading information, and in particular the possible Kauffman states for three-strand pretzel knots of the form $P(2a, -2b-1, 2c+1)$ and $P(2a, -2b-1, -2c-1)$. Moreover, in Chapter 2, it will be demonstrated how the grading information of the Kauffman states for these pretzel knots can be used to restrict the possible Maslov disks between generators of the classical knot Floer homology. In so doing, one can read off certain knot Floer homology groups directly from the combinatorial information, see for example Lemma 2.7 and Lemma 2.9.

Chapter 3 defines many of the simpler concordance invariants extracted from classical knot Floer homology, and in particular Section 3.3 describes how the concordance invariants of some families of pretzel knots can be bounded by using the sharper slice-Bennequin inequality of [18, 19]. In particular, the family of three-strand pretzel knots described by $P(2a, -2b-1, -2c-1)$ for $a, b, c \in \mathbb{N}$ are quasipositive, and so have concordance invariants ν and τ equal to their Seifert genus. Furthermore, one can place bounds upon the τ and ν-invariants of the family $P(2a, -2b-1, 2c+1)$ using the sharper slice-Bennequin inequality and work of [18], and what is more, these bounds are strong enough to deremine these concordance invariants the case of $b \geq c$, as demonstrated by Lemma 3.19.

Before describing the construction of the algebraic invariant $\mathcal{C}(D)$ defined by Ozsváth-Szabó, it is first necessary in Chapter 4 to define the algebraic objects used in the construction: namely A_∞-algebras, associated to every horizontal level of a knot diagram in the required form; DA-bimodules, associated to every Morse event such as crossings, maxima and minima; Type D structures, associated to upper knot diagrams; and A_∞-modules, associated to lower knot diagrams. In this chapter, the specific algebraic objects used in the construction of $\mathcal{C}(D)$ are defined over the required differential graded algebras. Furthermore, because all three-strand pretzel knots admit knot diagrams in a certain form — see Figure 5.1 — a new A_∞-module associated to the minima in these special knot diagrams will be defined in Section 4.6.2. This new A_∞-module greatly simplifies the calculation of the invariant $\mathcal{C}(P(2a, -2b-1, 2c+1))$, allowing the inductive proofs presented in Chapter 5 determining this invariant to be more closely motivated by the Heegaard diagrams for this family of knots used by Eftekhary in [5].

Using the DA-bimodules defined by Ozsváth-Szabó in [49], and introduced in Chapter 4, the Type D structure for upper knot diagrams of three-strand pretzel knots can be determined inductively. Under certain conditions, the tensor product between a DA-bimodule and a Type D structure can be taken to yield another Type D structure. This process is outlined in Section 4.5. Intuitively, since Type D structures are associated to upper knot diagrams, and DA-bimodules to Morse events (such as crossings or maxima), attaching a Morse event to an upper knot diagram yields another upper knot diagram.

The generators of Type D structures are in bijection with upper Kauffman states, and for three-strand pretzel knots the upper Kauffman states can also be separated into distinct families based upon the decorations on each strand. This separation of upper Kauffman states into families allows one to determine the Type D structure after an arbitrary number of crossings in each strand. In the proofs in Chapter 5, much use is made of both the truncation of the A_∞-algebras explained in Chapter 4, and the diagrammatic representation of Type D structures: see for example Figure 5.3.

For D a three-strand pretzel knot in the family $P(2a, -2b-1, 2c+1)$, the structure of $\mathcal{C}(D)$ — and the associated homology theories recently proven in [48] to be equivalent to $\widehat{HFK}(D)$ and $HFK^-(D)$ — will be determined in Chapter 6, relying on the inductive computations of Chapter 5 and the construction of a new A_∞-module associated to the minima of a special knot diagram for these knots outlined in Section 4.6.2.

From these homology theories, the invariants $\underline{\nu}$ and $\underline{\tau}$ will be determined. These were defined by Ozsváth-Szabó in [49], and although they are now proven to be equivalent to the familiar concordance invariants ν and τ, they are themselves invariants of the local equivalence class of the bigraded complex $\mathcal{C}(D)$. In Section 6.2.3 the invariants $\underline{\nu}$ and $\underline{\tau}$ are demonstrated to also be additive under connected sum. This is as a corollary of the fact that the complex $\mathcal{C}(D_1 \# D_2)$ satisfies the Künneth relation, see Proposition 6.16.

Theorem 6.6, determining the homology theory $H(\mathcal{C}^-(D))$, is also sufficient to determine the infinite family of concordance invariants $\{\varphi_i\}_{i\in\mathbb{N}}$, introduced by Dai-Hom-Stroffregen-Truong in [4]. This is a linearly independent family of concordance invariants, extracted from what they call a reduced knot-like complex. Since the complex $\mathcal{C}(D)$ is equivalent to the complex $CFK_{\mathcal{R}'}(D)$, defined by [4], one could also simplify $\mathcal{C}(D)$ to a reduced knot like complex. However, in the case of the three-strand pretzel knots $P(2a, -2b-1, 2c+1)$, this is not needed to compute the invariants $\{\varphi_i\}_{i\in\mathbb{N}}$, as demonstrated by Lemma 6.14.

Chapter 6 finishes by suggesting new areas where the techniques outlined within this thesis might be employed, and open problems in the study of three-strand pretzel knots. In particular, the remaining examples of three-strand pretzel knots whose slice genus is not known will be discussed. The concordance invariants defined in Chapter 6 are insufficient to answer these open questions; it is hoped, however, that since $\mathcal{C}(D)$ provides more information that $HFK^-(D)$ and $\widehat{HFK}(D)$, Theorem 6.1 determining $\mathcal{C}(P(2a, -2b-1, 2c+1))$ might prove useful for answering these questions in the future.

Figures within this thesis have been constructed by the author using the vector drawing package [3]. Where these have been adapted from existing figures in other works, this has been appropriately cited.

I acknowledge with thanks the love, support, and dedication of my partner Christina Sunneklep, without whom I would have been lost. I also gratefully acknowledge the help and guidance of my supervisor Dr. Brendan Owens, who has always been supportive and understanding. Lastly, I acknowledge the many stimulating discussions I have had with Tom Hockenhull, and I wish him well for the future.

Declaration of originality

I certify that the thesis presented here for examination for the degree of Doctor of Philosophy at the University of Glasgow is solely my own work other than where I have clearly indicated that it is the work of others (in which case the extent of any work carried out jointly by me and any other person is clearly identified in it) and that the thesis has not been edited by a third party beyond what is permitted by the University's PGR Code of Practice.

The copyright of this thesis rests with the author. No quotation from it is permitted without full acknowledgement.

I declare that the thesis does not include work forming part of a thesis presented successfully for another degree.

I declare that this thesis has been produced in accordance with the University of Glasgow's Code of Good Practice in Research.

I acknowledge that if any issues are raised regarding good research practice based on review of the thesis, the examination may be postponed pending the outcome of any investigation of the issues.

Contents

Chapter 1

Overview of knots and knot Floer homology

The primary object of study through this thesis will be links, embeddings of $S^1 \sqcup \cdots \sqcup S^1$ into S^3, and the invariants used to distinguish them. Knots are one-component links, and are thus embeddings of S^1 into S^3. It will often be useful and convenient to present these as subsets of S^3 up to ambient isotopy in S^3. Moreover, the distinction between knots, and knot diagrams (their representation on the plane) will only be made when it is explicitly needed. For expositional material on knots and knot diagrams, the author refers the reader to [55].

1.1 Pretzel links

Pretzel links are a class of links that are amongst the most well-studied, having a simple presentation as a link diagram and many useful symmetries. A pretzel link $L = P(a_1, a_2, \cdots, a_n)$ with $a_i \in \mathbb{Z}\backslash\{0\}$ admits a standard link diagram with a_i half-twists attached in the manner shown in Figure 1.1. The sign of a_i denotes whether the twists on the strand are positive half twists, or negative half-twists, see Figure 1.2.

Through rotation by 180 degrees, it is simple to see that there is an isotopy between pretzel links $P(a_1, a_2, \cdots, a_n)$ and $P(a_n, a_{n-1}, \cdots, a_1)$. Moreover, an isotopy exists between $P(a_1, a_2, \cdots, a_n)$ and $P(a_n, a_1, a_2, \cdots, a_{n-1})$, and so the integers a_i only determine a diagram for a pretzel link up to cyclic permutation and reversing the order.

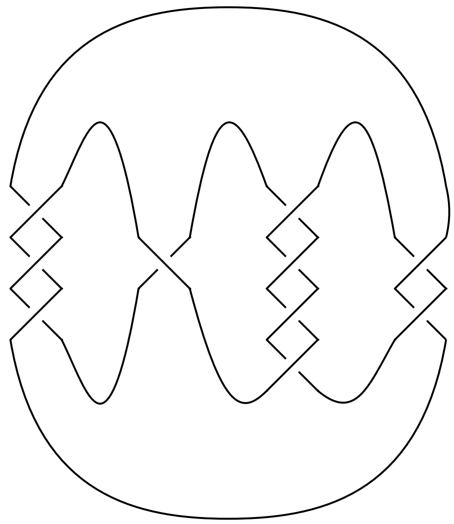


Figure 1.1: *An diagram of the two-component pretzel link* $P(3, -1, 4, 2)$

Another useful symmetry is that the mirror of the link $P(a_1, a_2, \cdots, a_n)$, denoted by $\overline{P(a_1, \cdots, a_n)}$ is isotopic to $P(-a_1, -a_2, \cdots, -a_n)$. As a consequence, since the invariants studied in this thesis react to mirroring in a known way, often only one of these pairs will be considered.

Note, if all of the a_i are of the same sign, the link is alternating. In fact, this condition is necessary and sufficient when $n \geq 3$, see for example [8]. Since pretzel links are a subset of Montesinos links, for $n \geq 3$ the standard pretzel link diagrams following Figure 1.1 are reduced Montesinos diagrams, which achieve their minimum crossing number by [22, Thm. 10]. Moreover, it is easily demonstrated that the pretzel link $P(a_1, \cdots, a_n)$ is a knot if and only if there is at most one even coefficient a_i when n is odd, and if and only if one of the a_i is even for even n.

Using Figure 1.1, since a_3 and a_4 are even, $P(3, -1, 4, 2)$ can be seen to have more than one component. Furthermore, this diagram is *almost-alternating*, that is a single crossing change (to the negative half twist) would change this into an alternating diagram, with all the coefficients a_i being of the same sign.

A larger class of links than alternating links is that of quasi-alternating links, introduced in [42, Def. 3.1]. Denote the class of quasi-alternating links as $\mathcal{Q}$. Then, $\mathcal{Q}$ is the smallest set of links such that

- The unknot belongs to $\mathcal{Q}$.

Figure 1.2: *The convention for positive and negative half twists in diagrams of pretzel links. Note, that in an oriented pretzel link, the sign of the half twists does not necessarily correspond to the sign of the oriented crossing.*

- If L has a projection with a crossing c such that the 0 and 1 resolutions at c yield links L_0 and L_1 with $det(L) = det(L_0) + det(L_1)$, and $L_0, L_1 \in \mathcal{Q}$, then $L \in \mathcal{Q}$.

Note that all alternating links are contained within $\mathcal{Q}$. Quasi-alternating pretzel links are of particular interest because they are homologically thin with respect to both Khovanov and knot Floer homology theories [31]. This means that the homology theory is supported entirely in a single diagonal, and for quasi-alternating links this diagonal is determined by the (classical) signature of the knot. In [8, Theorem. 1.4], Greene determines which of the Montesinos links $M\left(-e, ; (p_1, 1), \ldots, (p_n, 1), (q_1, -1), \ldots, (q_m, -1)\right)$ are quasi-alternating, which when $e = 0$ specialises to determine the quasi-alternating pretzel links, stated here in Theorem 1.1.

Theorem 1.1 *The pretzel link $P(p_1, p_2, \cdots, p_n, -q_1, \cdots, -q_m)$, with $p_i \geq 2, q_j \geq 3$ is quasi-alternating if and only if one of the following holds.*

1. *$n = 1$, and $p_1 > \min\{q_1, \cdots, q_m\}$ or $m \leq 1$.*
2. *$m = 1$, and $q_1 > \min\{p_1, \cdots, p_n\}$ or $n \leq 1$.*

The pretzel knots with three strands take one of the following forms: $P(2a+1, 2b+1, 2c+1)$ for $a, b, c \in \mathbb{Z}$, or $P(2a, -2b-1, -2c-1)$ and $P(2a, -2b-1, 2c+1)$, or the corresponding mirrors, with $a, b, c \in \mathbb{N}$.

In the first case, $P(2a+1, 2b+1, 2c+1)$, the knot Floer homology has been studied in [41, Sec. 5]. As described above, when all of the coefficients have the same sign, the knots are alternating, and hence are homologically thin. However, if $b < 0$ and $a, c > 0$, the knots are non-alternating. The knot Floer homology of this family of pretzel knots has been determined in [41, Sec. 5].

Theorem 1.2 *[41, Theorem 1.3] For $a, b, c \in \mathbb{N}$, the three strand pretzel knot $K =$*

$P(2a+1, -2b-1, 2c+1)$ *has* $\widehat{HFK}(K,1)$ *isomorphic to*

$$\widehat{HFK}_d(K,1) = \begin{cases} \mathbb{Z}_{(1)}^{ab+bc+b-ac} & \textit{if } b \geq min(a,c) \\ \mathbb{Z}_{(1)}^{b(b+1)} \oplus \mathbb{Z}_{(2)}^{(b-a)(b-c)} & \textit{if } b < min(a,c) \end{cases}.$$

The proof of this theorem uses the long exact sequence in knot Floer homology associated to the oriented skein relation presented in [39, Thm. 10.2]. In particular, for this family of pretzel knots, resolving using the oriented Skein relation relates the knot Floer homology of $P(2a+1, -2b-1, 2c+1)$ to the knot Floer homology of the torus link $T_{2,2|a-b|}$, which is well understood.

Note, if $b > 0$, and $a, c < 0$, one can mirror the knot to put it into the above form. This is not an isotopy, but mirroring changes the knot Floer homology in a known way. Namely, from [39], one has that

$$\widehat{HFK}_d(K,i) \cong \widehat{HFK}^{-d}(\overline{K}, -i).$$

One can then use the universal coefficient theorem to relate this cohomology group to homology.

The remaining cases of pretzel knots have been considered by Eftekhary in [5], with the 'hat' version of knot Floer homology $\widehat{HFK}$ calculated using Kauffman states. However, as the author has determined, if the conjectural equivalence between the bordered invariant of Ozsváth-Szabó and classical knot Floer homology holds, then $\widehat{HFK}\,(P(2a, -2b-1, 2c+1))$ is more complicated than as presented in [5, Lem. 1, Thm. 1,2]. Using Theorem 6.3, examples of three strand pretzel knots can be presented that have $\widehat{HFK}$ that disagrees with the calculation in [5]. In particular, this can also be verified using the computer implementation of [47], as described in Remark 6.4.

Both of these cases considered in [5] are non-alternating when $a, b, c \geq 1$. They can be quasi-alternating, precisely when the conditions of Theorem 1.1 are satisfied, but are not always.

1.2 Classical knot Floer homology

Knot Floer homology is a family of homology theories providing invariants of unoriented knots and links, originally outlined by Oszváth and Szabó in [39], and Rasmussen in [53].

This family consists of three ‘flavours’ of knot Floer homology associated to a knot K, generalised to links in [43], taking the form of $\mathbb{Z} \oplus \mathbb{Z}$-bigraded theories. These homology theories have a chain complex generated by the same basis, but differing in the definition of the differential and differing in the ring over which the chain complex is defined. The two gradings associated to the chain complex (and homology theory) are the *Maslov grading* and *Alexander grading*.

1.2.1 Heegaard diagrams and the knot Floer complex

As summarised by [29, Def. 3.1], a multi-pointed Heegaard diagram $H = (\Sigma, \alpha, \beta, w, z)$ for a knot K is defined by the following.

- A closed surface $\Sigma \subset S^3$ of genus $g \geq 0$ splitting S^3 into handlebodies U_0 and U_1. Typically, one orients Σ as the boundary of U_0.
- A collection $\alpha = \{\alpha_1, \cdots, \alpha_{g+k-1}\}$ of pairwise disjoint, simple closed curves on Σ, such that each α_i bounds a properly embedded disk D_i^α in U_0, and such that the complement of these disks in U_0 is a union of k balls. On Σ, by convention α-curves are coloured red.
- A collection $\beta = \{\beta_1, \cdots, \beta_{g+k-1}\}$ with the same properties in the handlebody U_1. On Σ, by convention β-curves are coloured blue.
- Two collections of points on Σ, $w = \{w_1, \cdots, w_k\}$ and $z = \{z_1, \cdots, z_k\}$, all disjoint from each other and the α and β curves. Give the points w_i a positive orientation, and z_j a negative orientation.
- The knot K is then the isotopy class of simple, closed curve formed by tracing a path disjoint from the disks in each handlebody through the points $w_1, z_1, w_2, \ldots, z_k, w_1$ agreeing with the orientation.

In this thesis, Heegaard diagrams for knots are commonly doubly pointed, so one has unique basepoints z and w. There are further admissibility conditions on Heegaard diagrams for knots, as described in [43, Sec. 3.1], and the reader is referred there for further detail.

With a doubly pointed Heegaard diagram for a knot K, one can then form the generators of the knot Floer complex $CFK^\infty(K)$ as follows. Let $Sym^g(\Sigma)$ denote the smooth, real

$2g$-dimensional manifold

$$Sym^g(\Sigma) := \Sigma^g / S_g,$$

the g-fold product of Σ, quotiented out by the action of the symmetric group S_g.

The α and β curves then form two half-dimensional submanifolds $T_\alpha = \alpha_1 \times \cdots \times \alpha_g$ and $T_\beta = \beta_1 \times \cdots \times \beta_g \subset Sym^g(\Sigma)$. Following [51], these tori are said to be real: fixing a complex structure on Σ induces a complex structure on $Sym^g(\Sigma)$, and in relation to this complex structure the tori T_α and T_β have tangent spaces containing no complex lines, and so are totally real.

The intersection of the two submanifolds T_α and T_β provides the basis for the modules in knot Floer homology, namely: $\widehat{CFK}(K)$, $CFK^-(K)$ and $CFK^\infty(K)$.

Definition 1.3 *Where $\mathbb{F}$ is a commutative ring, the module $\widehat{CFK}(K)$ is defined as the $\mathbb{F}$-module freely generated by the intersection points $T_\alpha \cap T_\beta$. Each of these intersection points can be thought of as unordered g-tuples of points on Σ, with one point on each α curve and each β curve.*

These same intersection points also freely generate two more modules, $CFK^-(K)$ and $CFK^\infty(K)$, which for a doubly pointed Heegaard diagram are $\mathbb{F}[U]$ and $\mathbb{F}[U, U^{-1}]$ modules respectively.

Unless otherwise specified, in this thesis $\mathbb{F} \cong \mathbb{Z}/2$. It is also common to take $\mathbb{F} \cong \mathbb{Z}$, although this adds the complication of counting with sign, which is absent when $\mathbb{F} \cong \mathbb{Z}/2$.

An example of a suitable Heegaard diagram for the right-hand trefoil (the $(2,3)$-torus knot), is presented in Figure 1.3.

These generators admit a Maslov grading and an Alexander grading, $M(x)$ and $A(x)$ respectively for $x \in T_\alpha \cap T_\beta$. A relative bigrading is defined using Whitney disks between intersection points.

From [40, Sec. 2.4], define the following for any Heegaard diagram $\mathcal{H} = (\Sigma, \alpha, \beta, z, w)$.

Definition 1.4 *For intersection points $x, y \in T_\alpha \cap T_\beta$, denote by $\pi_2(x, y)$ the set of homotopy classes of Whitney disks*

$$\left\{ u : D^2 \to Sym^g(\Sigma) \,\middle|\, \begin{smallmatrix} u(-i)=x,\ u(i)=y \\ u(S^1 \cap \{Re<0\}) \subset T_\beta \\ u(S^1 \cap \{Re>0\}) \subset T_\alpha \end{smallmatrix} \right\}.$$

Such disks and are pseudo-holomorphic if they satisfy the non-linear Cauchy-Riemann equations for a generic one-parameter family of almost-complex structures on $Sym^g(\Sigma)$, see [40, Sec. 3.2].

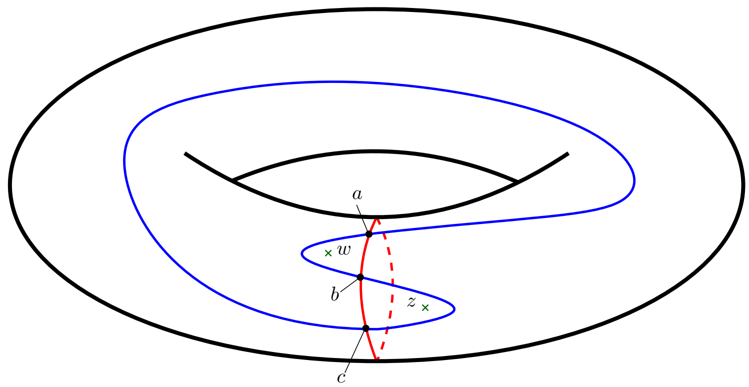


Figure 1.3: *A Heegaard diagram for the right-hand trefoil. Note that $Sym^g(\Sigma) \cong \Sigma$, since this is a genus 1 Heegaard diagram. So, there are three generators for the knot Floer complex, denoted a, b and c.*

Define the space $\mathcal{M}(\phi)$ as the moduli space of pseudo-holomorphic representatives of ϕ. One can associate to this moduli space an integer $\mu(\phi)$ known as the Maslov index. This index is such that $\mathcal{M}(\phi)$ is a smooth manifold of dimension $\mu(\phi)$. Moreover, this index is calculable in a combinatorial way following the work of [53, Sec. 9] and [23, Cor. 4.10], through an examination of domains on the surface Σ associated to Whitney disks.

Definition 1.5 *[40, Def. 2.13] Denote by $D_1, D_2, \ldots, D_m \subset \Sigma$ the closures of the regions $\Sigma \setminus (\alpha \cup \beta)$. A domain $D(u)$ associated to Whitney disk $u \in \pi_2(x, y)$ is then*

$$D(u) = \sum_{i=1}^{m} n_{z_i}(u) D_i,$$

where $z_i \in int(D_i)$, and $n_{z_i}(\phi)$ is the intersection number

$$\# u^{-1} \left(\{z_i\} \times Sym^{g-1}(\Sigma) \right).$$

Remark 1.6 *Every Whitney disk $\phi \in \pi_2(x, y)$ determines a domain, i.e. the formal sum of components of $\Sigma \setminus (\alpha \cup \beta)$, but the interior of these domains does not uniquely determine the class $\phi \in \pi_2(x, y)$. There are examples of formal sums of regions in $\Sigma \setminus (\alpha \cup \beta)$ that represent Whitney disks between different pairs of intersection points in $T_\alpha \cap T_\beta$. In Section 6.3.2, this will be discussed in greater depth in relation to punctured polygonal domains, as defined by [9, Def. 6.4]. Figure 6.3 provides an example where the formal sum of regions does not uniquely determine the corners of the domain.*

However, as discussed in [40, Sec. 3.5], Whitney disks between generators are uniquely defined by their domains when one specifies the corners of the domain for Heegaard diagrams of genus greater than one.

For a Whitney disk $\phi \in \pi_2(x, y)$ with corresponding domain $D(\phi) \subset \Sigma$, the corners of the domain are points of intersection between α curves and β curves on Σ. These are points on Σ, not elements of $T_\alpha \cap T_\beta \subset Sym^g(\Sigma)$. As described by [53, Sec. 9], for a formal sum of regions in Σ, there are admissibility conditions on the multiplicities of the regions incident with the α and β curves at a corner in order for this intersection point to be a valid corner of a domain, see [53, Fig. 32].

The existence of an $\mathbb{R}$-action on $\mathcal{M}(\phi)$ by automorphisms of the disks fixing the endpoints means that one can form a smooth, compact manifold $\widehat{\mathcal{M}}(\phi) = \mathcal{M}(\phi)/\mathbb{R}$, which, when $\mu(\phi) = 1$, means that $\widehat{\mathcal{M}}(\phi)$ is a finite set of points. This action by $\mathbb{R}$ upon the moduli space is most easily seen in terms of the cylindrical reinterpretation of [23], where Whitney disks are now interpreted as strips $\mathbb{R} \times [0, 1]$, with the action by $\mathbb{R}$ being translation.

For each $\phi \in \pi_2(x, y)$, each basepoint $v \in w \cup z$ has a corresponding codimension 2 manifold $R_v = \{v\} \times Sym^{g-1}(\Sigma) \subset Sym^g(\Sigma)$. One can then define $n_v(\phi) \in \mathbb{Z}$ as the intersection number between ϕ (more formally the image of ϕ) and R_v.

Then, relative Maslov and Alexander gradings are defined as follows.

Definition 1.7 *For $x, y \in T_\alpha \cap T_\beta$, with $\phi \in \pi_2(x, y)$:*

$$\begin{aligned} M(x) - M(y) &= \mu(\phi) - 2n_w(\phi), \\ M(U^k x) &= M(x) - 2k, k \in \mathbb{Z}, \\ A(x) - A(y) &= n_z(\phi) - n_w(\phi), \\ A(U^k x) &= A(x) - k, k \in \mathbb{Z}. \end{aligned}$$

This relative bigrading can further be fixed to an absolute grading, but first it is useful to introduce the differential maps.

Definition 1.8 *For $x \in T_\alpha \cap T_\beta$, define the differentials*

$$\widehat{\partial} x = \sum_{y \in T_\alpha \cap T_\beta} \sum_{\substack{\phi \in \pi_2(x,y) \\ \mu(\phi)=1 \\ n_z(\phi)=0=n_w(\phi)}} \left(\# \widehat{\mathcal{M}}(\phi) \right) \cdot y,$$

$$\partial^- x = \sum_{y \in T_\alpha \cap T_\beta} \sum_{\substack{\phi \in \pi_2(x,y) \\ \mu(\phi)=1 \\ n_z(\phi)=0}} \left(\# \widehat{\mathcal{M}}(\phi) \right) \cdot U^{n_w(\phi)} y,$$

$$\partial^\infty x = \sum_{y \in T_\alpha \cap T_\beta} \sum_{\substack{\phi \in \pi_2(x,y) \\ \mu(\phi)=1}} \left(\# \widehat{\mathcal{M}}(\phi) \right) \cdot U^{n_w(\phi)} \cdot y.$$

Then, $\widehat{CFK}(K)$, $CFK^-(K)$ and $CFK^\infty(K)$ are respectively the bigraded $\mathbb{F}$-module, $\mathbb{F}[U]$-module and $\mathbb{F}[U,U^{-1}]$-module generated by the points $T_\alpha \cap T_\beta$, and one defines $\widehat{HFK}(K)$, $HFK^-(K)$ and $HFK^\infty(K)$ as the respective homology theories.

These complexes, up to filtered chain homotopy equivalence, are invariants of knots, not merely the Heegaard diagram associated to a knot.

Theorem 1.9 *[39, Thm. 3.1] For K an oriented knot in S^3, the filtered chain homotopy type of the complex $CFK^\infty(K)$ is a topological invariant of the knot. Moreover, the associated homology groups $\widehat{HFK}(K)$ and $HFK^-(K)$ associated to subcomplexes of $CFK^\infty(K)$ are topological invariants of the knot.*

The key element in the proof of this statement is demonstrating that two admissible Heegaard diagrams for the same knot can be related by a sequence of moves, namely stabilisation of the Heegaard diagram, destabilisation, isotopies of the α and β curves, and handleslides. For three-manifolds, this is a classical result of Reidemeister and Singer, but in [39, Sec. 3.2] Ozsváth and Szabó prove that each of these moves on a Heegaard diagram for a *knot* does not change the filtered chain homotopy type of the complex, so yielding the result.

1.2.2 Bifiltration of the full knot Floer complex

The full knot Floer complex $CFK^\infty(K)$ can be thought of as a $\mathbb{Z}\oplus\mathbb{Z}$-filtered complex over $\mathbb{F}$, following [39, Sec. 3.1]. Then, elements of CFK^∞ are triples $[x,i,j]$ with $x \in T_\alpha \cap T_\beta$, $i,j \in \mathbb{Z}$. These generators are then representable on an (i,j) grid as points, see Figure 1.4. Furthermore, one defines the action of U such that $U \cdot [x,i,j] = [x,i-1,j-1]$.

The triple $[x, i, j]$ then corresponds to generator $U^{-i}x$, where x is has Alexander grading j. This gives a useful diagram for the complex, with the complex lying in the (i, j) plane. The differential is then defined as

$$\partial^\infty([x, i, j]) = \sum_{\substack{y \in T_\alpha \cap T_\beta \\ \phi \in \pi_2(x,y) \\ \mu(\phi)=1}} \left(\#\widehat{\mathcal{M}}(\phi)\right) [y, i - n_w(\phi), j - n_z(\phi)].$$

An example of this calculation is presented in Figure 1.4 for the Heegaard diagram in Figure 1.3.

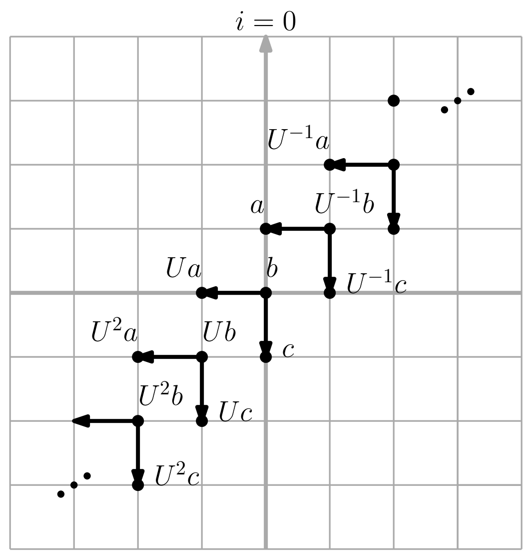


Figure 1.4: *The full knot Floer complex for the right hand trefoil pictured in Figure 1.3. Note that there are two bigons in the genus 1 Heegaard diagram, one containing the basepoint w, and the other z. Each of these corresponds to a Whitney disk: $\phi \in \pi_2(b, c)$ with $n_z(\phi) = 1$, and the other $\psi \in \pi_2(b, a)$, with $n_w(\psi) = 1$.*

The $\mathbb{Z} \oplus \mathbb{Z}$-filtration of the complex is then given by the coordinates, i.e. there exists a filtration $\mathcal{F} : (T_\alpha \cap T_\beta) \times \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ defined as $[x, i, j] \mapsto (i, j)$. As described by [39, Sec. 3.1], this allows the definition of subcomplexes since the differential is non-increasing in both i and j, and a partial ordering can be placed on the filtration by defining $(i, j) \leq (i', j')$ when $i \leq i'$ and $j \leq j'$.

1.2.3 Subcomplexes and absolute gradings

As summarised by [13, Sec. 2.2], placing conditions upon i and j allows one to extract from this interpretation for $CFK^\infty(K)$ the other complexes $CFK^-(K)$ and $\widehat{CFK}(K)$, and associated concordance invariants.

Define the subcomplex $C\{i=0\} := \{[x,0,j]\}_{x \in T_\alpha \cap T_\beta}$, with the differential only counting those ϕ with $n_w(\phi) = 0$, whose homology is isomorphic to $\widehat{HF}(S^3)$. This last group is the hat version of the Heegaard Floer homology for the 3-manifold S^3, as defined in [39]. This homology group has only a single generator, which is by convention in grading zero. The corresponding generator in $H_*\left(C\{i=0\}\right)$ is given a Maslov grading of 0, making the relative grading into an absolute grading.

Then, the relative Alexander grading is made absolute by the fact that the Alexander grading of $\widehat{HFK}(K)$ is symmetric about 0, and as stated by [53], the filtered Euler characteristic of $\widehat{HFK}(K)$ is equal to the Alexander polynomial $\Delta_K(t)$.

In fact, $\widehat{CFK}(K)$ is equal to the complex $C\{i=0\}$, equipped with the differential

$$\widehat{\partial}([x,0,j]) = \sum_{\substack{y \in \pi_2(x,y) \\ \mu(\phi)=1 \\ n_z(\phi)=0=n_w(\phi)}} \#\left(\widehat{\mathcal{M}}(\phi)\right) \cdot [y,0,j].$$

This can be thought of as setting all vertical and horizontal arrows to 0, so only those that are between generators with the same (i,j) coordinates contribute non-trivially to the differential. Using Figure 1.4, one thus has that

$$\widehat{HFK}_d(T_{2,3},s) = \begin{cases} \mathbb{F} & \text{if } (d,s) = (0,1) \\ \mathbb{F} & \text{if } (d,s) = (-1,0) \\ \mathbb{F} & \text{if } (d,s) = (-2,-1) \\ 0 & \text{otherwise.} \end{cases}$$

One can also define $(CFK^-(K), \partial^-)$ as the complex $\bigoplus_{s \in \mathbb{Z}} C\left(\{i \leq 0, j = s\}\right)$, equipped with the differential

$$\partial^-([x,i,j]) = \sum_{\substack{y \in \pi_2(x,y) \\ \mu(\phi)=1 \\ n_z(\phi)=0}} \#\left(\widehat{\mathcal{M}}(\phi)\right) \cdot [y, i - n_w(\phi), j].$$

Each of the flavours of knot Floer homology then decomposes into graded parts, for example $HFK^-(K) = \bigoplus_{d,s} HFK_d^-(K,s)$. Here d is the Maslov grading, and s is the Alexander grading.

1.3 Zemke's reformulation

In [60, 62], Zemke provided a reformulation of the full knot Floer complex in terms of an $\mathcal{R}$-module, where $\mathcal{R} = \mathbb{F}[U, U^{-1}, V, V^{-1}]$. The two settings, i.e. the classical knot Floer homology defined in [39, 53], and the setting presented in [60] are equivalent, as proved in [60, Sec. 8]. They both take the form of complexes equipped with a $\mathbb{Z} \oplus \mathbb{Z}$-bigrading, and a $\mathbb{Z} \oplus \mathbb{Z}$-filtration, and one can translate between the two formulations.

The $\mathcal{R}$-module complex is of particular interest in this thesis because the bordered invariant presented by Ozsváth-Szabó in [49], and concordance invariants defined by [4] are both adapted from this formulation. The cut-and-paste methods of [47, 49] are central to the determination of invariants for three-strand pretzel knots presented in this thesis, hence Zemke's formulation is the most convenient to use.

Definition 1.10 *[4, 62] For a doubly pointed Heegaard diagram $\mathcal{H} = (\Sigma, \alpha, \beta, z, w)$ associated to a knot K as defined in [39, Def. 2.2], define $CFK_{\mathcal{R}}(K)$ to be the $\mathcal{R} = \mathbb{F}[U, U^{-1}, V, V^{-1}]$ module freely generated by the intersection points $T_\alpha \cap T_\beta$.*

The complex $CFK_{\mathcal{R}}(K)$ admits two integer valued gradings, gr_U and gr_V that have the following relative grading formula for $\phi \in \pi_2(x, y)$.

$$gr_U(x) - gr_U(y) = \mu(\phi) - 2n_w(\phi)$$
$$gr_V(x) - gr_V(y) = \mu(\phi) - 2n_z(\phi)$$
$$gr_U(Ux) = gr_U(x) - 2$$
$$gr_V(Vy) = gr_V(y) - 2.$$

The differential in the complex $CFK_{\mathcal{R}}(K)$ is then

$$\partial(x) = \sum_{\substack{y \in \pi_2(x,y) \\ \mu(\phi)=1}} \# \left(\widehat{\mathcal{M}}(\phi) \right) \cdot U^{n_w(\phi)} V^{n_z(\phi)} y.$$

The grading gr_U is known as the homological or Maslov grading, and one can check using Definition 1.7 that this satisfies the same relative grading formula as the Maslov grading as presented in [39]. However gr_V is not equal to the familiar Alexander grading, but it can be recovered.

$$\begin{aligned} A(x) - A(y) &= n_z(\phi) - n_w(\phi) \\ &= \frac{1}{2} \left((gr_U(x) - gr_U(y)) - (gr_V(x) - gr_V(y)) \right). \end{aligned}$$

These gradings can be lifted to absolute gradings, as is proved in [60, Sec. 5.5]. Moreover, as proven by in [60, Sec. 8], the Ozsváth-Szabó and Zemke formulations are equivalent, which is demonstrated by showing that the absolute gradings agree in the two formulations, since both complexes have the same basis and same relative gradings.

It is worth noting the effect of the differential upon the bigrading, as it is not the same as the classical knot Floer setting, which leaves the Alexander grading unchanged.

$$gr_U(\partial(x)) = gr_U(x) - 1$$
$$gr_V(\partial(x)) = gr_V(x) - 1.$$

In the complex $CFK^\infty(K)$ in the Ozsváth-Szabó setting, one has an action on the complex of multiplication by U. This drops the Alexander grading by 1, and the Maslov grading by 2. This has an analogous action in $CFK_\mathcal{R}(K)$, which is multiplication by UV. So, taking this product corresponds to multiplication by U in the Ozsváth-Szabó setting.

Furthermore, as stated in [62, Def. 2.1], there is a filtration $\mathcal{G}_{i,j}$, defined as the subset of the complex $CFK_\mathcal{R}(K)$ generated by elements $U^m V^n \cdot x$ for $x \in T_\alpha \cap T_\beta$ and $m \geq i, n \geq j$. So, both the Zemke formulation and Ozsváth-Szabó formulation are $\mathbb{Z} \oplus \mathbb{Z}$-bifiltered and bigraded.

This filtration is demonstrated in Figure 1.5, which exhibits Zemke's reformulation of the knot Floer homology for the Heegaard diagram of the right hand trefoil in Figure 1.3. One key difference in diagrammatic representations of these two formulations is that the gr_U-grading and gr_V-grading cannot be read off from the bifiltered diagram in the Zemke setting, whereas one can read off the Alexander grading of states in the diagram for the complex in the classical setting, see Figure 1.4.

Remark 1.11 *It is not immediately apparent that the complex is indeed a knot invariant, and also that the differential squares to zero. However, both of these are true. In a generalisation to the link setting, it was proven in [61, Prop. 3.5] and [60, Prop. 2.1] that if a link L has two admissible Heegaard diagrams $\mathcal{H}$ and $\mathcal{H}'$, then there is a filtered, $\mathcal{R}$-equivariant chain homotopy equivalence between the associated complexes $CFK_\mathcal{R}(\mathcal{H})$ and $CFK_\mathcal{R}(\mathcal{H}')$. Together with [39, Prop. 3.5, Thm. 3.1], this equivalence between Heeagaard diagrams is enough to prove that the complex is a knot invariant.*

Although the generalisation of the complex to links does not have $\partial^2 = 0$, as demonstrated

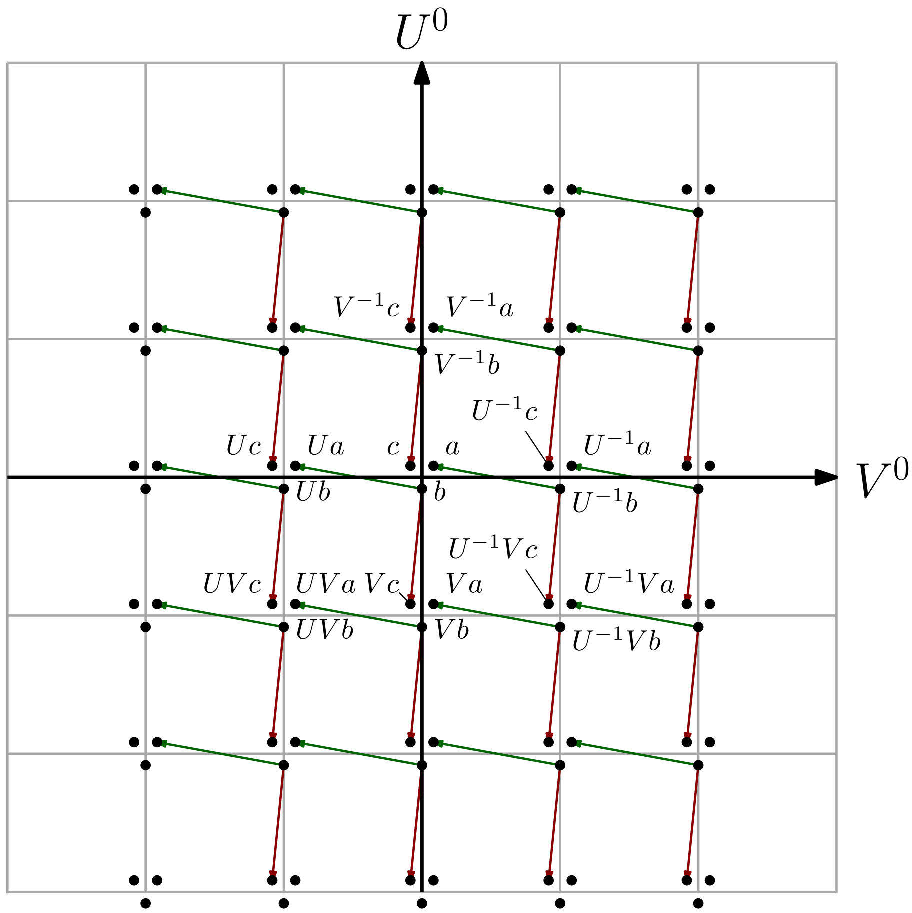


Figure 1.5: *Diagrammatic representation of the complex $CFK_{\mathcal{R}}(T_{2,3})$, from the Heegaard diagram represented in Figure 1.3. Note, the green arrows represent ∂_U, and the red ∂_V, as introduced in Section 1.4.1. The product UV in this setting moves the generators down and left in the filtration, as multiplying by U does in the Ozsváth-Szabó setting.*

by [59, Lem. 2.1] in the case of doubly pointed Heegaard diagrams for knots, or indeed $2n$-pointed diagrams for links, ∂^2 is equal to 0, and so $CFK_{\mathcal{R}}(K)$ is a chain complex.

As noted by [4], useful properties of the full knot Floer complex $CFK^\infty(K)$ are echoed in Zemke's setting.

Remark 1.12

- *There is a filtered, $\mathcal{R}$-equivariant, chain homotopy equivalence between the complexes $CFK_{\mathcal{R}}(K\#J)$ and $CFK_{\mathcal{R}}(K)\otimes CFK_{\mathcal{R}}(J)$, see [62, Thm. 1.1].*
- *There is a filtered, $\mathcal{R}$-equivariant, chain homotopy equivalence between the complexes $CFK_{\mathcal{R}}(-K)$ and $CFK_{\mathcal{R}}(K)^*$, where $*$ denotes the dual complex, see [62, Lem 2.17, 2.18].*

1.3.1 Local equivalence

There is a notion of equivalence of complexes $CFK_{\mathcal{R}}(K)$ which is useful in the definition of the concordance invariants in [62] and [4]. This is local equivalence.

Definition 1.13 *[4, Def. 2.4] Two (gr_U, gr_V)-bigraded complexes C_1 and C_2, both $\mathcal{R}$-modules, are defined to be locally equivalent if there are filtered, grading-preserving, $\mathcal{R}$-equivariant chain maps $f : C_1 \to C_2$ and $g : C_1 \to C_2$ such f and g induce isomorphisms on homology.*

This also has a refinement to a setting over a different ring: $\mathcal{R}' = \mathbb{F}[U, V]/(UV)$. This is the setting used by Dai *et al* in [4] to define concordance invariants from Zemke's reformulation. The properties in Remark 1.12 also hold when one uses the ring $\mathcal{R}'$ in place of $\mathcal{R}$. Define the complex $CFK_{\mathcal{R}'}(K)$ in the same way as $CFK_{\mathcal{R}}(K)$ is defined in Definition 1.3, except now taking $UV = 0$.

As remarked above, multiplication by UV in $CFK_{\mathcal{R}}(K)$ corresponds directly with multiplication by U in the $CFK^{\infty}(K)$ setting. The differential in $CFK_{\mathcal{R}'}(K)$ now has coefficients determined by counting pseudo-holomorphic representatives of those Maslov index one disks in the Heegaard digram passing over at most one of the basepoints.

The advantage of this is twofold. Firstly, by keeping track of less information, the calculation of the complex and associated invariants is easier: as evidenced by the utilisation of $\mathcal{R}'$ in the algorithmic construction of [47]. Secondly, as noted by [4, Thm. 1.3] and [11, Prop. 4.1], the local equivalence classes of complexes in this ring $\mathcal{R}'$ admit a total ordering. As proved in [4], the notion of local equivalence classes of $CFK_{\mathcal{R}'}(K)$ are identical to the ε-equivalence classes defined in [11], and using the total order available on the equivalence classes allows one to prove linear independence results in the topological concordance group.

Remark 1.14 *It is important to note the distinction between the knot-like complexes defined as local equivalence classes of $CFK_{\mathcal{R}'}(K)$ by [4] and the chain complex defined by Ozsváth-Szabó in [49].*

The first construction is a modification of Zemke's reformulation of classical knot Floer homology, and so the differential maps in the complex necessarily count pseudo-holomorphic representatives of Whitney disks. The advantage is that the properties of knot Floer homology pass to the new perspective, so invariants of local equivalence classes of $CFK_{\mathcal{R}'}(K)$

are concordance invariants of the knot.

Because the setting in [47, 49] is purely the association of an algebraic invariant to a knot, when the invariant $\mathcal{C}(D)$ was first constructed there was only a conjectural equivalence between this bordered invariant and the classical knot Floer setting. Hence, invariants determined from equivalence classes of the bordered invariant $\mathcal{C}(D)$ defined in [49] were not necessarily concordance invariants, as one could not utilise information on behaviour under connect sums and mirror reflection from knot Floer homology.

The recent proof of this conjectural equivalence in [48] does demonstrate that invariants of local equivalence classes of complexes $\mathcal{C}(D)$ are also invariants of the local equivalence classes of the corresponding complexes $CFK_{\mathcal{R}'}(D)$, and so are invariants of concordance classes. One can still examine the effect of taking connect sums of knots in the algebraic invariant of Ozsváth-Szabó without using this equivalence, as will be discussed in Section 6.2.3.

1.4 Knot-like complexes

As introduced by [4] using their modification of Zemke's reformulation of knot Floer homology, one can undertake the abstract study of (gr_U, gr_V)-bigraded complexes over the ring $\mathcal{R}'$. From these complexes, one can extract numerical invariants of the local equivalence class of complex without reference to the fact that these complexes arise from a knot.

Definition 1.15 *For C a (gr_U, gr_V)-graded chain complex over $\mathcal{R}' \cong \mathbb{F}[U,V]/UV$, a V-nontorsion tower is said to be a generator of $\mathbb{F}[V]$ in $H_*(C/U)$, and a U-nontorsion tower is said to be a generator of $\mathbb{F}[U]$ in $H_*(C/V)$.*

Definition 1.16 *[4, Def. 3.1] C is defined to be a knot-like complex if C is a free, finitely generated bigraded chain complex over $\mathcal{R}'$, such that*

1. *The differential ∂ of the chain complex affects the bigrading by $(-1,-1)$.*
2. *$H_*(C/U)$ has a single V-nontorsion tower lying in $gr_U = 0$.*
3. *$H_*(C/V)$ has a single U-nontorsion tower lying in $gr_V = 0$.*

The U and V-tower classes defined above will be important in the determination of local equivalence for knot-like complexes. This is defined slightly differently than for the local equivalence classes of $CFK_{\mathcal{R}'}(K)$ in Definition 1.13, as C_1 and C_2 knot-like complexes are defined to be locally equivalent if there are maps $f : C_1 \to C_2$ and $g : C_2 \to C_1$ that are absolutely U-graded, *relatively* V-graded $\mathcal{R}'$-equivariant chain maps that are isomorphisms on $H_*(C_i/U)/V-$torsion.

In fact, it is proven in [4, Lemma 6.9] that two locally equivalent knot-like complexes do in fact have absolutely graded isomorphisms between their homologies, but the original notion introduced is weaker.

Remark 1.17 *The complex $CFK_{\mathcal{R}}(T_{2,3})$ shown in Figure 1.5 has the differential ∂_U marked in green, and ∂_V marked in red. Modding out by UV would not modify the axes of the diagram, as these would not contain UV-product terms. Then, the corresponding complex $CFK_{\mathcal{R}'}$ would be a knot-like complex, with $gr_U(a) = 0$ as a V-nontorsion tower, and $gr_V(c) = 0$ as a U-nontorsion tower.*

1.4.1 Reduced knot-like complexes

In the construction of concordance invariants in knot Floer homology, it is convenient to treat the horizontal differential (non-decreasing in U) or vertical differential (non-decreasing in V) separately. More specifically, constructions like [14, Sec. 2.2] make use of the fact that one can find equivalent complexes that are 'vertically' or 'horizontally' simplified. A similar notion exists for knot-like complexes, that of reduced knot-like complexes and standard complexes, and much use is made of this in the determination of the family of concordance invariants in [4].

Definition 1.18 *[4, Def. 3.7] Recall, a knot-like complex is a freely-generated chain complex over $\mathcal{R}'$. Let $\{x_i\}$ be an $\mathcal{R}'$-basis for the knot-like complex C.*

C is said to be a reduced knot-like complex if for every x_i, one has that

$$\partial(x_i) = \sum_{j \neq i} P_j(U, V)x_j,$$

where each polynomial $P_j(U, V)$ is either zero, or a polynomial in either *U or V of degree ≥ 1 with no constant term.*

Hence, as no constant term appears in each $P_j(U, V)$, the differential ∂ can be subdivided into $\partial = \partial_U + \partial_V$, where ∂_U and ∂_V have image with polynomials in U or V respectively.

By definition all reduced knot-like complexes are knot-like complexes. Every non-zero differential has an image with polynomial terms in U or V with strictly positive powers, and so in the $\mathbb{Z} \oplus \mathbb{Z}$-filtered picture no arrow representing the differential remains at the same coordinate. With the reformulation of $\widehat{\partial}$ in Zemke's setting, a reduced knot-like complex is thus generated by elements that all lie in $ker(\widehat{\partial})$, and since ∂_U and ∂_V decrease either the horizontal or vertical coordinate, no generator lies in $im(\widehat{\partial})$. Consequently, generators of a reduced knot-like complex are in bijective correspondence with generators of $\widehat{HFK}(K)$.

Using a similar construction to that of horizontally and vertically simplified bases in classical knot Floer homlogy presented in [12], one has the following lemma. This is of use in the definition of the concordance invariants $\{\varphi_j\}_{j\in\mathbb{N}}$, introduced in Section 6.2.2.

Lemma 1.19 *[4, Lemma 3.8] Every knot-like complex C is locally equivalent to a reduced knot-like complex C'.*

The proof of the above lemma presents an algorithm for the determination of a locally equivalent reduced knot-like complex, based upon the fact that for a basis element x of C such that $\partial(x)$ contains a term x_i without a U or V polynomial, then one can construct a split short exact sequence

$$0 \to \langle x, \partial(x)\rangle \to C \to C' \to 0,$$

such that the projection $p : C \to C'$ and section $s : C' \to C$ are isomorphisms on homology, since $\langle x, \partial(x)\rangle$ is by construction acyclic. These maps then provide a local equivalence between C and C', a knot-like complex with at least one fewer generator not satisfying the conditions of a reduced knot-like complex. Proceeding in this manner yields a locally equivalent reduced knot-like complex.

There is, however, no guarantee that the algorithm presented in [4, Lemma 3.8] yields a unique result, nor whether the generators of the reduced knot-like complex C' correspond to single generators of C. Generators of C' can be the sum of generators of C, and so if the generators of C are in correspondence with Kauffman states (introduced in Chapter 2), one can lose this correspondence when working with a locally equivalent reduced knot-like complex.

Chapter 2

Kauffman states and three strand pretzel knots

In order to define the knot Floer complex associated to a knot, it is necessary to specify a Heegaard diagram $(\Sigma, \alpha, \beta, z, w)$, and the intersection points $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$.

The choice of Heegaard diagram does not alter the filtered chain homotopy type of the knot Floer complex, as noted in Theorem 1.2.1. But, certain choices of Heegaard diagram can ease the computation of the knot Floer homology. In some fortunate cases it is possible to choose a Heegaard diagram for which the counts of pseudo-holomorphic representatives of Whitney disks is relatively simple.

As demonstrated by [35], using an arc-presentation of a knot one can use 'grid diagrams' to define a genus one Heegaard diagram for any knot. Grid diagrams are computationally useful, because they are constructed in such a way that not only can Whitney disks with Maslov index equal to 1 be read off combinatorially from the diagram — see [30] — but also that these disks have a known count of pseudo-holomorphic representatives. However, this comes at the cost of drastically increasing the number of intersection points in the Heegaard diagram, yielding a more unwieldy complex.

Another method for producing the knot Floer complex for a knot comes from thickening a projection of the knot to form a genus $(c+1)$ Heegaard diagram, where c is the number of crossings in a knot diagram. This process was originally introduced in [36], and the generating intersection points $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ are in bijective correspondence with *Kauffman states* of the diagram.

2.1 Ozsváth-Szabó's definition of Kauffman states

The use of Kauffman states in the calculation of knot Floer homology started with Ozsváth and Szabó's paper [36]. Kauffman states were originally introduced by Kauffman in [16, Ch. 2], and from these states the Alexander polynomial of a knot is calculable. Indeed, in [17] it is proven that the Kauffman states model for knot Floer homology introduced by [36] directly categorifies the Alexander polynomial in the setting of [16].

An early application of the Kauffman state model in [36] is the determination that the knot Floer homology of an alternating knot K has $\widehat{HFK}_{s+\frac{\sigma}{2}}(K,s) \cong \mathbb{F}^{|a_s|}$ [36, Thm. 1.3]. Here σ is the signature of the knot, and a_s is the coefficient of the Alexander polynomial in degree s.

Definition 2.1 *[36, Def. 1.1] For a knot K, consider the 4-valent graph G from projection of this knot into the plane $z = 0$. This cuts the plane into regions. Choose two regions separated by a single edge of the graph G. Denote these regions by A and B, and mark this edge. Note, this edge corresponds to an arc in a knot diagram of K, and this is a decorated projection for the knot K.*

Then, each vertex of the graph has 4 quadrants, each of which is a corner of some region of the graph G. Assign a decoration in one of the quadrants by each vertex, such that no decoration is present in regions A or B, and such that each region other than A and B has only a single marked corner.

Such a decorated diagram is a Kauffman state for the knot K. In place of 4-valent graphs G, one can consider the corresponding picture in the original knot diagram. The 4-valent graph G can then be yielded by joining the arcs at each crossing. An example can be seen in Figure 2.1. The Kauffman state for a knot diagram should have a marked point on one arc, a decoration quadrant at each crossing, and every region (excepting A and B) should contain exactly one decoration.

To summarise, a Kauffman state for a knot is a decorated knot projection associated to a diagram for the knot. From this collection of Kauffman states — namely the different ways to decorate this knot projection — one can extract information like the Alexander polynomial for the knot. If one takes a different projection for the knot, for example one from a different, isotopic knot diagram, the projection has similarly defined Kauffman states from which the same information about the knot can be drawn. However, this is a different collection of Kauffman states, and there need not be a bijection between the two sets of states.

Ozsváth and Szabó associate two integer values to each Kauffman state, determined through the total of local contributions across all crossings in a knot diagram. To each crossing c in a knot diagram, half-integer values $M(c)$ and $A(c)$ can be assigned, based upon which quadrant is occupied by the marked point of that crossing, following Figure 2.2.

Definition 2.2 *Using the gradings depicted in Figure 2.2, let $C(K)$ be the crossings in a knot diagram of K. Then, define the Maslov grading of a Kauffman state x as*

$$M(x) := \sum_{c \in C(K)} M(c),$$

and the Alexander grading of a Kauffman state x as

$$A(x) := \sum_{c \in C(K)} A(c).$$

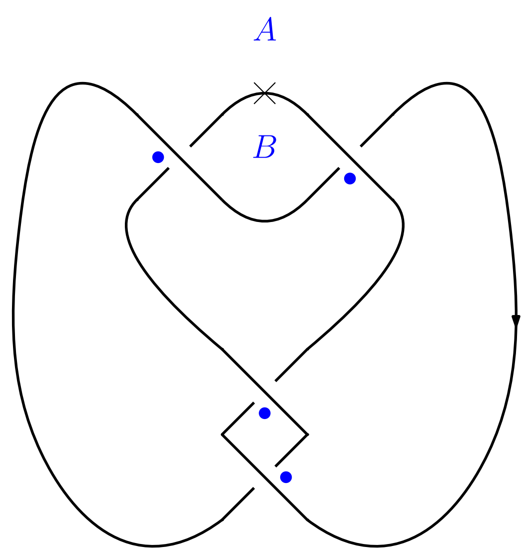


Figure 2.1: *A Kauffman state for the oriented figure 8 knot is shown here. Note, the regions A and B neighbour the marked arc, and all conditions for Kauffman states are satisfied.*

Then, Ozsváth and Szabó prove the following theorem, detailing the correspondence between Kauffman states and the knot Floer complex.

Theorem 2.3 *[36, Thm. 1.2] For K a knot in S^3, choose a decorated projection for K. Then there is a Heegaard diagram $\mathcal{H} = (\Sigma, \alpha, \beta, z, w)$ for K such that $\widehat{CFK}(K)$ is freely generated by Kauffman states, and there is equality between the Maslov and Alexander indices of Kauffman states, and the Maslov and Alexander gradings of $T_\alpha \cap T_\beta$.*

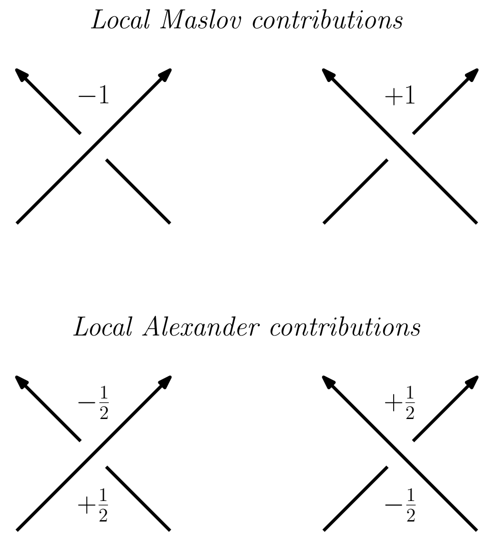


Figure 2.2: *The contributions to M and A at each crossing, as defined by [36, Figs. 2,3].*

A Heegaard diagram can be associated to a knot diagram as follows. Take the boundary of a tubular neighbourhood of the graph G, which for a knot diagram with c crossings is a $(c+1)$ genus surface, Σ. Add an α corresponding to the boundary of each region in the decorated projection, excepting B. At the marked point of the decorated projection, place a meridional curve β_0, and the two basepoints either side of this curve. Place the remaining β curves corresponding to the crossing of the knot diagram. This is detailed more fully in [36, Sec. 2.2], but should be clear from Example 2.4.

Example 2.4 *Consider the three crossing unknot, with two positive crossings and one negative crossing. With a marked edge as in Figure 2.3, there are three possible Kauffman states, denoted a, b and c.*

A Kauffman state marks each crossing in a knot diagram with a point. In the Heegaard diagram this marked point corresponds to the choice of an intersection point between an α-curve and β-curve at the tubular neighbourhood of this crossing. This choice then dictates that no other points may be put on these curves, just as marking a point at a crossing forces that there are no other marked points in that region, or at that crossing. See Figure 2.4 for generators in the Heegaard diagram associated to two of the Kauffman states in Figure 2.3.

Also, note that at the distinguished edge, there is only one α curve intersecting the β curve in the tubular neighbourhood of this edge. Hence, all generators of $T_\alpha \cap T_\beta \subset Sym^{c+1}(\Sigma)$ have this intersection point as part of the unordered $(c+1)$-tuple.

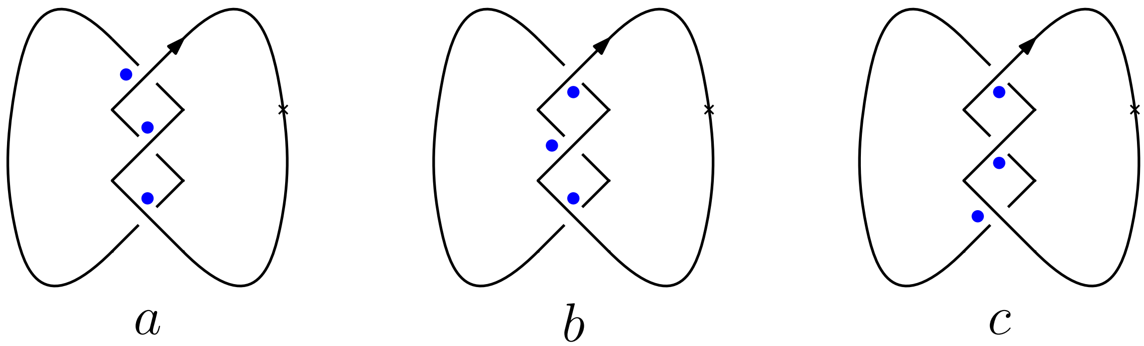


Figure 2.3: *The three possible Kauffman states associated to a knot diagram for an unknot with three crossings. Using Figure 2.2, the bigrading (M, A) of state a is $(0,0)$, the bigrading of b is $(1,1)$, and the bigrading of c is $(0,1)$.*

The equivalence in the grading information is then proved by Ozsváth-Szabó using an action relating two Kauffman states called transpositions, and how these correspond to domains, defined earlier in Definition 1.5.

2.1.1 An observation on this model and the grading information

When trying to compute knot Floer homology using this model, there is a slight nuance that should be noted. The algorithm given in [36, Sec. 2.2] associating a Heegaard diagram to a decorated knot projection is detailed in the position of the α and β curves, as exemplified in Figures 2.3 and 2.4.

Using the gradings for Kauffman states in Definition 2.2, one has that the bigradings of the states shown in Figure 2.3 are $M(a) = 0$, $A(a) = 0$; $M(b) = 1$, $A(b) = 1$; $M(c) = 0$, $A(c) = 1$.

From the definition of a Whitney disk given in Definition 1.4, the rectangular domain $D(\phi)$ highlighted in Figure 2.4 might thus be thought of as corresponding to a Whitney disk $\phi \in \pi_2(c, b)$. Since Whitney disks map the interior of the disk $D^2 \subset \mathbb{C}$ to the corresponding region in $Sym^g(\Sigma)$, and $S^1 \cap \{Re < 0\}$ to T_β. Hence, the domain seems to represent a Whitney disk from c to b.

A helpful way to think of this is using [40, Lem. 3.6]. The disk $\phi \in \pi_2(x, y)$ is described

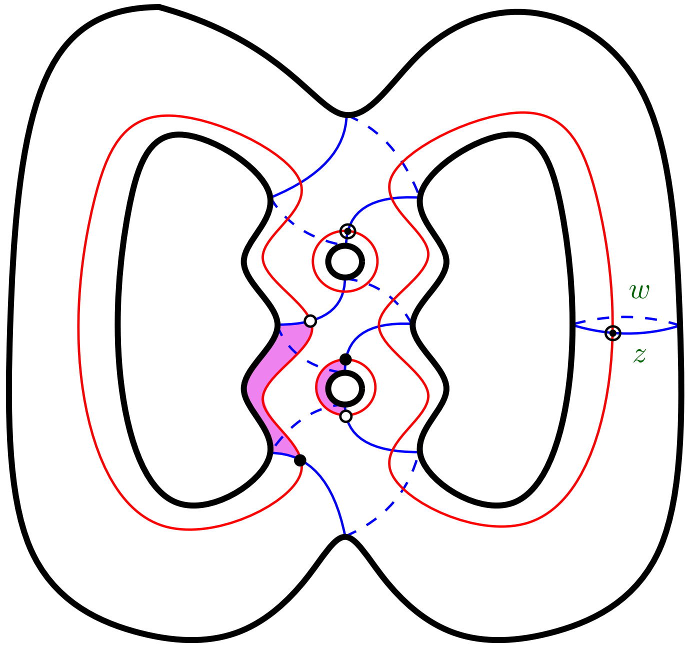


Figure 2.4: *A Heegaard diagram constructed using the algorithm of [36, Sec. 2.2] for the three crossing unknot in Figure 2.3. Shown are the generators of $\widehat{CFK}(U)$ corresponding to the Kauffman states c (solid disk) and b (open disk).*

by a domain $D(\phi)$, and the spaces $S^1 \cap \{Re < 0\} \subset D^2$ and $S^1 \cap \{Re > 0\} \subset D^2$ are carried by ϕ to a path in $Sym^g(\Sigma)$ with preimage in $\Sigma^{\times g}$ of g arcs in the β and α curves respectively. The arcs start at corners of the domain corresponding to the intersection point x and terminate at corners for intersection point y. Travelling along one of these β arcs from an x-corner to a y-corner should keep the interior of the domain on the right. Hence, the domain pictured in Figure 2.4 is seemingly one representing a Whitney disk from c to b.

The Maslov index and pseudo-holomorphic count of the Whitney disk ϕ is then calculable using a result of [53]. A domain is said to be polygonal if it is an embedded disk in Σ with only acute corners, following [23, 53].

Lemma 2.5 *[53, Lemma. 9.11] If $D(\phi)$, the domain of a homotopy class $\phi \in \pi_2(x, y)$, is a polygonal domain, then $\mu(\phi) = 1$, and $\#\widehat{\mathcal{M}}(\phi) = \pm 1$.*

Applying this lemma, one has that $\mu(\phi) = 1, n_z(\phi) = 0 = n_w(\phi)$. Using the relative grading information in Definition 1.7, one should have that $M(c) - M(b) = 1 - 0 = 1$. But, using Ozsváth-Szabó's gradings from Figure 2.2, $M(c) - M(b) = 0 - 1 = -1$.

Consequently, one might thus deduce that there is either an inconsistency with the grading information provided by Definition 2.2, or in the construction of the Heegaard diagram from a knot projection, since this is a Heegaard diagram shows a Maslov index 1 domain that should contribute to the differential $\widehat{\partial}$.

In fact, this is not the case, and the description of Heegaard diagrams for knots arising from Kauffman states does not lead to an inconsistency. The definition of a Heegaard diagram for a knot as presented in Section 1.2.1 and [36, Sec. 2.2], specifies that the closed surface Σ be oriented **as the boundary of $\mathbf{U_0}$**, the handlebody in which each α-curve bounds a properly embedded disk. Looking at Figure 2.4, this would mean that the handlebody U_0 is the one including the point at infinity.

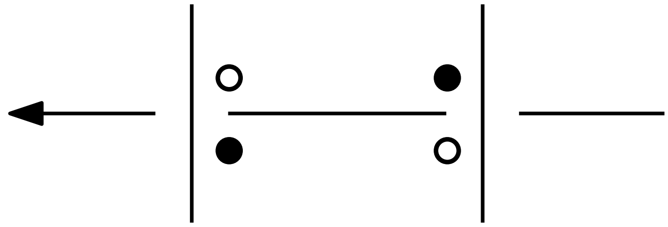
Thus, the appropriate way to orient the surface Σ is as the boundary of this surface, rather than as one might think as the boundary of the thickened up knot projection. This offers a rectification of the seeming inconsistency, as departing from the corner of b in Σ along the blue β-curve, the interior of the disk is kept on the right of the observer, and hence the domain $D(\phi)$ shown corresponds to $\phi \in \pi_2(b, c)$, as required by the grading information[1].

This further makes sense of the seeming inconsistency in the paper introducing the use of Kauffman states by Ozsváth-Szabó, namely in [36, Lem. 2.3 and Fig. 5]. In said figure, copied in Figure 2.5 the Heegaard diagram for the (partial) knot diagram is constructed as described, and a rectangular domain between the two generators. The dark circles represent the generator x, and the light represent the generator y. For the domain shown to represent a Whitney disk $\phi \in \pi_2(x, y)$, one must use the somewhat unintuitive orientation described above.

2.2 Kauffman states for three strand pretzel knots

As noted in Section 1.1, the three-strand pretzel knots that do not have well-understood knot Floer homology fall into one of two categories, $P(2a, -2b-1, -2c-1)$ or $P(2a, -2b-1, 2c+1)$ where $a, b, c \in \mathbb{N}$, since the knot Floer homology of three-strand pretzel knots with all odd coefficients has already been studied in [41].

[1]The author extends his thanks to Dr. Owens for discussions on clarifying this matter, who extends his own thanks to Matthew Hedden.



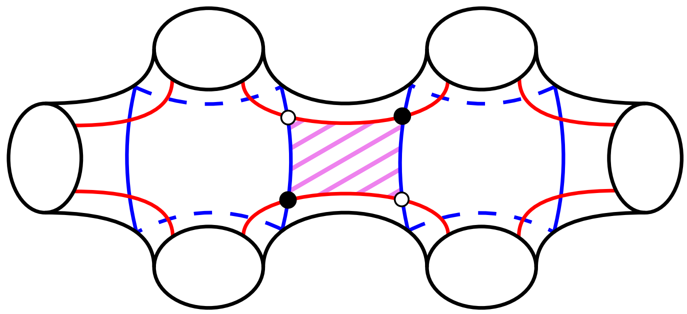


Figure 2.5: *A copy of [36, Fig. 5] demonstrating the seeming inconcistency between the Kauffman states for a knot diagram and the associated Heegaard diagram for the knot. The shaded region is the support of a class $\phi \in \pi_2(x, y)$, where y is the solid black dot, and x the white dot. As in the construction of [36], the β curves are around each crossing.*

Following the definitions of [32], the family of three strand pretzel knots $P(2a, -2b-1, -2c-1)$, with $a, b, c, \in \mathbb{N}$, the knots are all negative.

Definition 2.6 *[32, Def. 2.1] An oriented knot K is defined to be positive if it admits an oriented knot diagram with only positive crossings, as in Figure 2.6. Likewise, if it admits a diagram with only negative crossings, it is defined to be negative.*

From Section 1.1, one thus has that the knots $P(-2a, 2b+1, 2c+1)$ are positive (as the mirrors of a negative knot). Positive knots, and the slightly more general family of *quasipositive knots* have well-understood concordance invariants and 4-genus g_4, with corresponding results stated in Section 3.3.

Kauffman states are particularly useful in the study of the knot Floer homology of three strand pretzel knots (and the bordered invariant of Ozsváth-Szabó [49]) because the rigid structure of the Kauffman states means the generators are well understood, and moreover lend themselves to inductive arguments.

Because the algebraic construction of [49] places the distinguished edge at the global

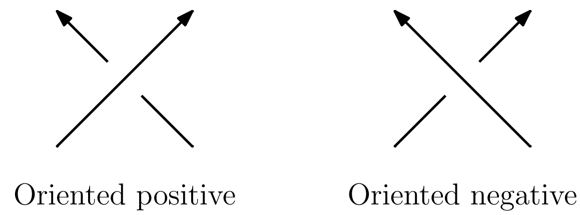


Figure 2.6: *The classification of crossings as positive or negative in an oriented knot diagram.*

minimum, Figures 2.7 and 2.8 describe the possible types of Kauffman states for the pretzel knots $P(-2c-1,-2b-1,2a) \cong P(2a,-2b-1,-2c-1)$ and $P(2c+1,-2b-1,2a) \cong P(2a,-2b-1,2c+1)$ respectively, slightly changing the notation of [5]. Using Definition 2.2, the bigradings of the Kauffman states are displayed in Tables 2.1 and 2.2, using the indicated orientations.

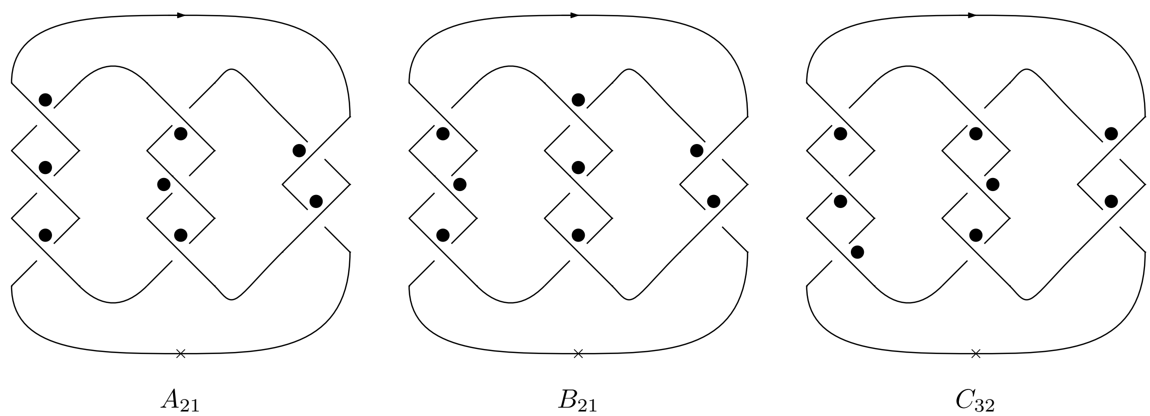


Figure 2.7: *The three types of Kauffman states for the pretzel knots $P(-2c-1,-2b-1,2a)$, shown here with $a=b=c=1$. The indices dictate the position of the marked points in the interior regions, read from left to right. The left hand set of half-twists are those described by the coefficient $-2c-1$, the middle are those described by $-2b-1$, and the right by $2a$.*

The grading information displayed in Tables 2.1 and 2.2 can enable one to determine topological information about the knot and the knot Floer homology, using the definitions above, and properties of knot Floer homology.

First, it is simple to see that in each of the cases, the knot Floer homology is contained in at most two diagonals. Define the Δ-grading as $\Delta = M - A$, then since the differential in $\widehat{CFK}(K)$ drops M by 1 and keeps A constant, $\Delta(\widehat{\partial}(x)) = \Delta(x) - 1$.

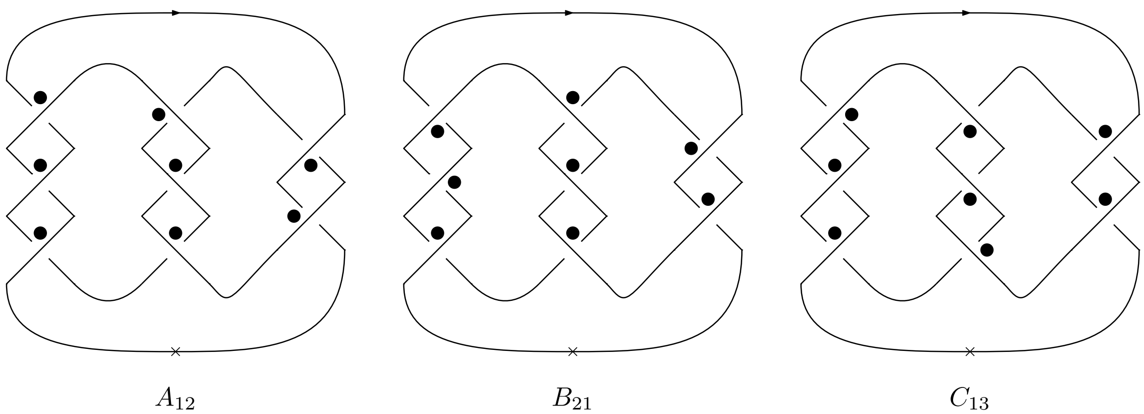


Figure 2.8: *The three types of Kauffman states for the pretzel knots $P(2a, -2b-1, 2c+1)$, shown here with $a = b = c = 1$. The indices dictate the position of the marked points in the interior regions, read from left to right.*

For the knots $P(2a, -2b-1, -2c-1)$, the homology has possible support in the diagonals $\Delta \in \{c+b, c+b+1\}$. Furthermore, the only generators in the $c+b$ diagonal must be C_{ij} states. Using the restrictions on i, j, and k, one has that

$$\begin{aligned} c-b \leq& A(A_{jk}) \leq b+c+1 \\ -b-c-1 \leq& A(B_{ik}) \leq c-b \\ -b-c \leq& A(C_{ij}) \leq b+c. \end{aligned}$$

By the fact that the only states in the maximal Alexander grading $b+c+1$ are A_{jk} states, and that these are in the least Δ-grading, they cannot be in the image of $\widehat{\partial}$, since this would require C_{ij} states in the same Alexander grading (a contradiction). Furthermore, they must be in the kernel of $\widehat{\partial}$, since $\widehat{\partial}$ drops Δ-grading by 1, yet there are no states in Δ-grading $c+b-1$. Consequently, one has the following.

Lemma 2.7 *For $K = P(2a, -2b-1, -2c-1)$, with $a, b, c \in \mathbb{N}$, the $\widehat{HFK}(K)$ has*

$$\widehat{HFK}(K, b+c+1) \cong \mathbb{F}^a_{(2b+2c+2)},$$

with no non-trivial homology groups in higher Alexander gradings.

Proof The only states in the maximal Alexander grading are $A_{2b+1,odd}$, of which there are a examples. Using the above observations on the Δ-grading then gives the results. ∎

Lemma 2.7 then provides more information about the knots using results from knot Floer homology. From [38, Thm. 1.2], the Seifert genus g_3 of a knot K is determined

State	Maslov	Alexander
A_{jk}	$j+\epsilon(k)+2c$	$j+\epsilon(k)+c-b-1$
B_{ik}	$-i+\epsilon(k)+2c+1$	$-i+\epsilon(k)+c-b$
C_{ij}	$-i+j+2c$	$-i+j+c-b$

Table 2.1: *Table giving the Maslov and Alexander gradings of the Kauffman states displayed in Figure 2.7 for the knot $P(2a,-2b-1,-2c-1)$. Here, $\epsilon(t)=t \mod (2)$, and $1 \leq i \leq 2c+1$, $1 \leq j \leq 2b+1$, $1 \leq k \leq 2a$.*

State	Maslov	Alexander
A_{jk}	$j+\epsilon(k)-2c-2$	$j+\epsilon(k)-b-c-2$
B_{ik}	$i+\epsilon(k)-2c-1$	$i+\epsilon(k)-b-c-2$
C_{ij}	$i+j-2c-2$	$i+j-b-c-2$

Table 2.2: *Table giving the Maslov and Alexander gradings of the Kauffman states displayed in Figure 2.8 for the knot $P(2a,-2b-1,2c+1)$.*

by $\widehat{HFK}(K)$, namely

$$g_3(K)=\max_s\left\{rk\left(\widehat{HFK}(K,s)\right)>0\right\}.$$

Moreover, [33, Thm. 1.1] determines that

$$rk\left(\widehat{HFK}(K,g_3)\right)=1 \Leftrightarrow K \text{ is fibred.}$$

So, one has the following easy corollary.

Corollary 2.8 *The knot $K=P(2a,-2b-1,-2c-1)$ has Seifert genus $b+c+1$, and is fibred only when $a=1$.*

These facts are more easily seen in other ways, as these knots are negative (so have positive mirrors), but this will be discussed in terms of concordance invariants later.

Similar information can be seen for the family $P(2a,-2b-1,2c+1)$, using the information in Table 2.2.

Lemma 2.9 *The knot $K=P(2a,-2b-1,2c+1)$, for $a,b,c \in \mathbb{N}$ is fibred, and has Seifert genus $g_3(K)=b+c$.*

Proof Using the inequalities for i, j and k in Table 2.1, the grading information in Table 2.2 yields

$$\begin{aligned} -b-c-1 \leq & A(A_{jk}) \leq b-c \\ -b-c-1 \leq & A(B_{ik}) \leq c-b \\ -b-c \leq & A(C_{ij}) \leq b+c. \end{aligned}$$

Since b and c are both positive, one there is a unique state with maximal Alexander grading, namely $C_{2c+1,2b+1}$, with $A(C_{2c+1,2b+1}) = b+c$.

The knot Floer homology of this knot has a support in at most two Δ diagonals, $\Delta = b-c$ spanned by A_{jk} and C_{ij} states, and $\Delta = b-c+1$, spanned by B_{ik} states. Hence, since $\widehat{\partial}$ drops the Δ grading by one, the states C_{ij} lie in $\ker(\widehat{\partial})$ as they occupy the least Δ grading. Furthermore, $b+c > c-b = \max_{x \in B_{ik}} \{A(x)\}$, and so since the states B_{ik} are the only states in the $\Delta = b-c+1$ diagonal, one has that any state C_{ij} with $A(C_{ij}) > c-b$ must be a generator of $\widehat{HFK}(K)$. So, $\widehat{HFK}(K, b+c) \cong \mathbb{F}_{(2b)}$, and the result follows from [33, Thm. 1.1] and [38, Thm 1.2]. ■

More information about possible Maslov index 1 disks in the corresponding Heegaard diagram is also extractable from Table 2.2, as displayed in the following lemma.

Lemma 2.10 *If $\phi \in \pi_2(x, y)$ is a Whitney disk for the Heegaard diagram corresponding to the decorated knot diagrams in Figure 2.8 for the knot $K = P(2a, -2b-1, 2c+1)$, then if $\mu(\phi) = 1$, one has the following restriction on $n_z(\phi) + n_w(\phi)$.*

$x \backslash y$	A_{jk}	B_{ik}	C_{ij}
A_{jk}	1	2	1
B_{ik}	0	1	0
C_{ij}	1	2	1

Table 2.3: *Table describing the quantity $n_w(\phi) + n_z(\phi)$ for $\phi \in \pi_2(x, y)$ with $\mu(\phi) = 1$ in the Heegaard diagram for $P(2a, -2b-1, 2c+1)$ from the Kauffman states in Figure 2.8.*

Proof Only the calculation for $x = A_{jk}$ will be presented here, as the other cases are very similar.

Assume that one has a disk $\phi \in \pi_2(A_{jk}, A_{pq})$, with $\mu(\phi) = 1$. Then, using Definition 1.7 yields

$$\begin{aligned} M(A_{jk}) - M(A_{pq}) &= (j + \epsilon(k)) - (p + \epsilon(q)) \\ &= 1 - 2n_w(\phi) \\ A(A_{jk}) - A(A_{pq}) &= (j + \epsilon(k)) - (p + \epsilon(q)) \\ &= n_z(\phi) - n_w(\phi). \end{aligned}$$

Hence, one has that $1 - 2n_w(\phi) = n_z(\phi) - n_w(\phi)$, and so $1 = n_z(\phi) + n_w(\phi)$.

Likewise, with $x = A_{jk}$, $y = B_{pq}$, one has that

$$\begin{aligned} M(A_{jk}) - M(B_{pq}) &= (j + \epsilon(k)) - (p + \epsilon(q)) - 1 \\ &= 1 - 2n_w(\phi) \\ A(A_{jk}) - A(B_{pq}) &= (j + \epsilon(k)) - (p + \epsilon(q)) \\ &= n_z(\phi) - n_w(\phi). \end{aligned}$$

So, one has that $2 - 2n_w(\phi) = n_z(\phi) - n_w(\phi)$, and so $2 = n_z(\phi) + n_w(\phi)$.

Setting $x = A_{jk}$, $y = C_{pq}$, one as that

$$\begin{aligned} M(A_{jk}) - M(C_{pq}) &= (j + \epsilon(k)) - (p + q) \\ &= 1 - 2n_w(\phi) \\ A(A_{jk}) - A(C_{pq}) &= (j + \epsilon(k)) - (p + q) \\ &= n_z(\phi) - n_w(\phi). \end{aligned}$$

which yields $n_z(\phi) + n_w(\phi) = 1$. ■

By the principle of positivity of domains, see [53, Sec. 2.4], no domain $D(\phi)$ with pseudo-holomorphic representative can have negative coefficients in the formal sum of regions presented in Definition 1.5, and so to be counted in any differential $\widehat{\partial}$, ∂^- or ∂^∞, both n_z and n_w are strictly non-negative.

So, guided by Lemma 2.10, one could then determine that the differential $\partial^-(C_{ij})$ would have states $U^2 B_{pq}$ as possible elements in the image. This would then require that $M(B_{pq}) = M(C_{ij}) + 3$, restricting the possible states in the image of ∂^-. Hence, the gradings of the Kauffman states can inform the search for Whitney disks that contribute to the differentials in knot Floer homology.

Chapter 3

Concordance invariants

The package of knot Floer homology, and the reformulations by [4, 60, 62] allow the extraction of concordance invariants associated to subcomplexes of $CFK^{\infty}(K)$. Recall the following definition from [55].

Definition 3.1 *Oriented knots $K_1, K_2 \subset S^3$ are smoothly concordant if there is some smooth embedding of the cylinder $S^1 \times [0,1]$ into $S^3 \times [0,1]$ such that $\partial(S^1 \times [0,1]) = (K_1 \times \{0\}) \cup (-K_2 \times \{1\})$.*

A knot K is said to be smoothly slice if $K \subset S^3$ is the boundary of a smoothly embedded $D^2 \subset B^4$. It is well known (see [55]) that the connect sum $K \# -K$ is slice, where $-K$ is the mirror-reverse of the oriented knot K. Every slice knot is thus concordant to the unknot.

Concordance of knots defines an equivalence relation on the set of knots, and hence one can define the group

$$\mathcal{C} = \left\{ K \subset S^3 \text{ a knot } \right\} / \sim,$$

whose elements are concordance classes of knots, with the operation of connect sum. The identity element is then the class of slice knots, and the inverse element of the class $[K]$ is $[-K]$.

Any function $f : \mathcal{C} \to G$, where G is some algebraic object is then said to be a concordance invariant. If G is a group, f need not necessarily be a homomorphism, although many concordance invariants are, for example the τ-invariant defined by Ozsváth-Szabó is a group homomorphism from $\mathcal{C}$ to $\mathbb{Z}$.

3.1 Ozsváth-Szabó's τ-invariant.

In [37], a surjective group homomorphism $\tau : \mathcal{C} \to \mathbb{Z}$ was defined by examining maps of subcomplexes into $C\{i = 0\}$, introduced in Section 1.2.3. This integer-valued invariant has two equivalent formulations, with this equivalence demonstrated in [50, App. A].

Definition 3.2 *Following [37], define the subquotient complex*

$$C\{i = 0\} \cong C\{i \leq 0\}/C\{i < 0\},$$

with the differential induced by this quotient. Namely, one has

$$\partial_{vert}([x, 0, j]) = \sum_{y \in T_\alpha \cap T_\beta} \sum_{\substack{\phi \in \pi_2(x,y) \\ \mu(\phi)=1 \\ n_w(\phi)=0}} \# \left(\widehat{\mathcal{M}}(\phi) \right) [y, 0, j - n_z(\phi)].$$

From Section 1.2.3 and [37, Sec. 2.2],

$$H_* \left(C\{i = 0\} \right) \cong \widehat{HF}(S^3) \cong \mathbb{F}_{(0)}.$$

Using the filtration on $CFK^\infty(K)$, one has the natural inclusion map

$$\iota^s : C\{i = 0, j \leq s\} \to \widehat{CF}(S^3),$$

that has the induced map on homology

$$\iota^s_* : H_* \left(C\{i = 0, j \leq s\} \right) \to \widehat{HF}(S^3) \cong \mathbb{F}_{(0)}.$$

The concordance invariant $\tau(K)$ is then defined as

$$\tau(K) = \min \left\{ s \in \mathbb{Z} \text{ s.t. } \iota^s_* \text{ is non trivial } \right\}.$$

It is useful to note that the generator of $\widehat{HF}(S^3) \cong H_*(C\{i = 0\})$ is in Maslov index 0. It does not mean that the complex $C\{i = 0\}$ has a single generator in Maslov index 0 that lies in $ker(\partial_{vert})$ and also $ker(\widehat{\partial})$, as is the case for the complex in Figure 1.4. Take for example Figure 3.1, where the sum of two generators would give the required generator of the homology group.

The equivalent formulation of $\tau(K)$ as defined in [50] uses the idea of non-torsion elements in $HFK^-(K)$.

Definition 3.3

$$\tau(K) = - \max_{s \in \mathbb{Z}} \left\{ \theta \in HFK^-(K, s) \ \mid \ U^n \theta \neq 0 \ \ \forall n \in \mathbb{N} \right\}.$$

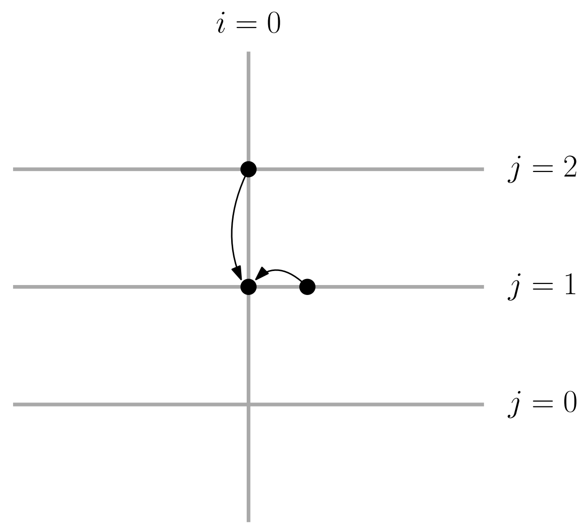


Figure 3.1: *An example diagram for part of a chain complex, demonstrating that a generator of homology in $H_*(C\{i=0\})$ might not lie in $ker(\widehat{\partial})$.*

In particular, this formulation is also applicable in Zemke's reformulation, and calculable from the complex $CFK_{\mathcal{R}'}(K)$ defined by [4]. This is because this formulation from [50] uses information from HFK^-, which is calculable from only horizontal information in $CFK^\infty(K)$, and so from ∂_U as defined in Definition 1.18. In particular, $\tau(K)$ would then be equal to the the negative of the Alexander grading of the U-nontorsion tower.

As summarised by [26], the τ-invariant as defined above has the following properties. They are not proven here, but the fact that τ is a concordance invariant is proven in [37, Thm. 1.2], with other results being corollaries in [26, 37].

Theorem 3.4 *Properties of τ [37]*

- *$\tau : \mathcal{C} \to \mathbb{Z}$ is a group homomorphism, with $\tau(U) = 0$.*
- *Defining $g_4(K)$ as the minimal genus of surface D bounded by K where D is smoothly embedded in B^4, one has the bound $|\tau(K)| \leq g_4(K)$. The quantity $g_4(K)$ is called the slice genus of K.*
- *If K_+ is the oriented knot with a marked positive crossing, and K_- is the oriented knot yielded by changing this marked crossing to a negative crossing, then*

$$\tau(K_+) - 1 \leq \tau(K_-) \leq \tau(K_+).$$

The above crossing change inequality and lower bound on the slice genus will be useful in the later discussion of bounds on concordance invariants for three strand pretzel knots.

By convention, τ is positive for positive knots such as the right hand trefoil, so $\tau(T_{2,3}) = 1$ corresponding to the generator a from Figure 1.4.

For an oriented knot K, the quantity $\tau(K)$ is insensitive to reversing the orientation, see [37, Sec. 3.2]. However, the concordance class the mirror-reverse $-K$ is the inverse of class $[K]$, and since τ is a group homomorphism, one thus has that $\tau(-K) = -\tau(K)$.

The properties demonstrated are similar to properties of the signature $\sigma(K)$ of a knot, and the Rasmussen s-invariant from Khovanov homology. It was conjectured in [54] that $\tau(K) = -\frac{\sigma(K)}{2} = -\frac{s}{2}$ for all knots K. This was demonstrated to be false by [10], but the following is true.

Theorem 3.5 *[37, Thm. 1.4] For K an alternating knot in S^3, $\tau(K) = -\dfrac{\sigma(K)}{2} = -\dfrac{s}{2}$.*

This statement is more generally true for all homologically thin knots, since the entire knot Floer homology $\widehat{HFK}$ is supported in a single diagonal $\Delta = M - A$.

3.2 Concordance invariants ν and ε

One can consider the inclusion of other subcomplexes into $C\{i = 0\}$ to extract similar concordance invariants.

Define $C\{\max(i, j - s = 0)\}$ as the quotient $C\{i = 0\}/C\{i = 0, j < s\}$. As a quotient complex, this includes into $C\{i = 0\}$ with the following family of maps:

$$\widehat{\nu}^s : C\{\max(i, j - s = 0)\} \to C\{i = 0\} \cong \widehat{CF}(S^3).$$

In the context of $CFK^\infty(K)$, this can be thought of as the complex lying on the horizontal line $j = s, i \leq 0$ and the vertical line $i = 0, j \leq s$.

The complex $C\{\max(i, j - s = 0)\}$ is isomorphic to the chain complex of $\widehat{CF}(S^3_N(K), s)$ for $|s| \leq \frac{N}{2}$ [44, Thm. 2.3], which is the Heegaard Floer homology of N surgery along the knot $K \subset S^3$.

As in Definition 3.2, this map gives an induced map on homology

$$\widehat{\nu}^s_* : \widehat{HF}(S^3_N(K), s) \to \widehat{HF}(S^3),$$

which is necessarily trivial for $s < \tau(K)$ from Definition 3.2. Likewise, for $s > \tau(K)$ this map is non-trivial.

Definition 3.6 *[45, Defn. 9.1]*

$$\nu(K) = \min\left\{s \in \mathbb{Z} \textit{ s.t. } \widehat{\nu}^s_* \textit{ is surjective}\right\}.$$

Hence, $\nu(K) \in \{\tau(K), \tau(K)+1\}$, and moreover from [15, Prop. 2.3], one has that

$$\tau(K) \leq \nu(K) \leq g_4(K)^1.$$

Again, since the only information from CFK^∞ needed to define $\nu(K)$ came from the horizontal and vertical information separately — namely on the lines $\{i = 0, j \leq s\}$ and $\{j = s, i \leq 0\}$ — this invariant is calculable from $CFK_{\mathcal{R}'}(K)$, and also from the reduced complexes discussed in Section 1.4.1.

Unlike $\tau(K)$, $\nu(K)$ is not a group homomorphism from $\mathcal{C} \to \mathbb{Z}$. This can be seen from that observation that for any knot K, one has that $\nu(K)$ is either equal to $\tau(K)$ or $\tau(K)+1$. Hence, if one has that $\nu(K) = \tau(K)+1$, then $\nu(-K) \in \{\tau(-K), \tau(-K)+1\}$, and thus

$$\nu(-K) > -\tau(K) - 1 = -\nu(K).$$

Since there exist knots K with $\nu(K) \neq \tau(K)$ — an example of which is the left-hand trefoil $-T_{2,3}$, as demonstrated in [29, Fig. 6] — taking an inverse in $\mathcal{C}$ does not correspond to taking the inverse in $\mathbb{Z}$, implying this is not a group homomorphism.

3.2.1 Simplified bases and the ε invariant

As mentioned previously, the notion of a reduced knot-like complex from Definition 1.18 is similar to notions previously introduced in knot Floer homology, namely that of horizontally and vertically simplified bases from [12].

Definition 3.7 *Let $C = CFK^-(K)$ be the subcomplex $C\{i \leq 0\}$ of CFK^∞. The complex C is then* reduced *if the differential ∂^∞ drops the i-filtration level, j-filtration level or both. A reduced complex represented in the familiar $\mathbb{Z} \oplus \mathbb{Z}$ grid then has no arrows that begin and end at the same coordinate. Hence, each generator of a reduced complex lies in $\widehat{HFK}(K)$.*

Further, define $C_{a,b}$ to be the subcomplex $C\{i \leq a, j \leq b\}$. A basis $\{x_k\}$ for $(CFK^-(K), \partial^\infty)$ is then a filtered basis *if for every pair (a, b) the set*

$$\{x_k \mid x_k \in C_{a,b}\}$$

[1] This inequality is as stated in the proposition, which holds when $\tau(K) \geq 0$. In the case where τ is negative, then taking the mirror of the knot would yield a similar inequality with τ positive.

is a basis for $C_{a,b}$.

Using this definition, for a reduced complex (C, ∂^∞), one can separate the differential ∂^∞ into $\partial^\infty = \partial_{vert} + \partial_{horz}$, where each drops the i and j filtration respectively. As a consequence, a reduced complex is amenable to finding a basis that simplifies one (or both) of these differentials.

Definition 3.8 *A filtered basis* $\{x_k\}$ *for a reduced complex* C *is then a vertically simplified basis if for every element* x_i*, one has the following trichotomy.*

- $x_i \in ker(\partial_{vert})$ *but* $x_i \notin im(\partial_{vert})$.
- $x_i \notin ker(\partial_{vert})$, *and* $\partial_{vert}(x_i) = x_{i+1}$.
- $x_i \in ker(\partial_{vert})$ *and there is a* unique x_{i-1} *such that* $x_i = \partial_{vert}(x_{i-1})$.

An equivalent notion then exists for the horizontal differential and horizontally simplified bases. The property of having a vertically or horizontally simplified basis is universal, as demonstrated by then following.

Lemma 3.9 *[12, Lem. 2.1] Every* $CFK^-(K)$ *is filtered chain homotopy equivalent to a reduced complex with a vertically (or horizontally) simplified basis.*

If such a basis can be found, and the appropriate equivalent reduced complex, the calculation of the concordance invariants ν and τ becomes easier. For example, the generator of $\widehat{HF}(S^3)$ determining τ would be the basis element of the vertically simplified basis that falls into the first category.

The trichotomy and correspondence of $C\{i = 0\}$ with $\widehat{CF}(S^3)$ dictates that there is some filtered chain homotopy equivalent complex admitting a basis such that ∂_{vert} cancels generators in pairs, except for a single *distinguished generator*. Using this, Hom defines the following concordance invariant [12].

Definition 3.10 *Let* x_j *be the distinguished generator of the vertically simplified basis for* C*, a reduced complex filtered chain homotopy equivalent to* $CFK^-(K)$*. Using this basis for* C*, one can then find a horizontally simplified basis by [12, Lem. 3.2,3.3]. Then, define*

$\varepsilon(K) \in \{-1, 0, 1\}$ *by*

$$\varepsilon(K) = \begin{cases} -1 & \text{if } x_j \notin ker(\partial_{horz}) \\ 0 & \text{if } x_j \in ker(\partial_{horz}), x_j \notin im(\partial_{horz}) \\ 1 & \text{if } x_j \in im(\partial_{horz}) \end{cases}.$$

Informally, this corresponds to the distinguished generator of the vertically simplified basis then being at the start of a horizontal arrow, the distinguished horizontal generator, or at the end of a horizontal arrow.

In fact, ε can be defined purely in terms of τ and ν a knot K and its mirror.

Definition 3.11 *[12, Rmk. 3.5]*

$$\varepsilon(K) = (\tau(K) - \nu(K)) - (\tau(-K) - \nu(-K)).$$

Hence, as τ and ν are both concordance invariants, it then follows immediately that ε is an invariant of not only the filtered chain homotopy class of $CFK^-(K)$, but also the concordance class $[K] \in \mathcal{C}$.

Remark 3.12 *Since τ and ν are both determinable from $CFK_{\mathcal{R}'}(K)$, Zemke's formulation over the reduced ring $\mathcal{R}'$, so is $\varepsilon(K)$. In fact, such a numerical invariant is extractable from a reduced knot-like complex, a fact that is useful later on when considering the bordered invariant of Ozsváth-Szabó.*

3.3 Kawamura bounds and concordance results

From the Kauffman states, and observations on the standard diagrams of three-strand pretzel knots, one can place bounds on the concordance invariants defined above. In particular, the concordance invariants defined above provide lower bounds for the slice genus, see [15, Prop. 2.3].

Consequently, if one can bound these concordance invariants from below, then one restricts the possible values for the slice genus. In the case of quasipositive knots, this restriction is enough to determine $\nu(K)$, $\tau(K)$ and $g_4(K)$.

Definition 3.13 *[2, Ch. 2] From an oriented knot diagram for K, one can form a decorated set of Seifert circles by smoothing each crossing agreeing with the orientation (to* *form Seifert circles), and at each crossing decorating the region with a signed arc, positive or negative, as exemplified by Figure 3.2.*

A knot diagram is defined to be quasipositive *if it admits smoothing to a decorated set of Seifert circles such that the signed arcs can be partitioned into single crossings and pairs of crossings, such that*

- *Each single crossing is positive.*
- *Each pair of crossings consists of a positive and negative crossing joining the same two Seifert circles.*
- *Traversing a Seifert circle from one element of a pair to another, one does not meet both elements of any other pair.*

Quasipositive knots (and more generally quasipositive links) bound Seifert surfaces corresponding to the above set of decorated Seifert circles called quasipositive surfaces. As demonstrated by Rudolph [57, Thm. 90], quasipositive Seifert surfaces are ambient isotopic to subsurfaces of the fibre F of a torus knot $T_{p,q}$. Using this fact, and that from [37, Cor. 1.7] one has

$$\tau(T_{p,q}) = \frac{pq - p - q + 1}{2} = g_4(T_{pq}),$$

Livingston proves the following.

Theorem 3.14 *[26, Thm. 4] If K is a quasipositive knot, bounding a quasipositive Seifert surface, then*

$$\tau(K) = g_3(K) = g_4(K).$$

Idea of proof Since quasipositive surfaces are subsurfaces of the fibre F of a torus knot T, the connect sum $T\#\overline{K}$ bounds a surface in a cobordism between $T_{p,q}$ and $\overline{K}$. Using properties of slice genus, and properties of τ proved in [37], one can bound $g_4(K)$ and $g_3(K)$ on each side by τ, yielding the result. ■

As a simple corollary, since the three-strand pretzel knots $P(-2a, 2b+1, 2c+1)$ are positive knots, and clearly the corresponding set of decorated Seifert circles has purely positive decorated arcs one has the following.

Corollary 3.15 *For $K = P(-2a, 2b+1, 2c+1)$, the concordance invariants τ and ν are*

$$\tau(K) = \nu(K) = g_4(K) = g_3(K) = b + c + 1.$$

Proof K is positive, hence quasipositive, then the result is implied by Theorem 3.14 and Lemma 2.7, together with the fact that $\tau(K) \leq \nu(K) \leq g_4(K)$. ■

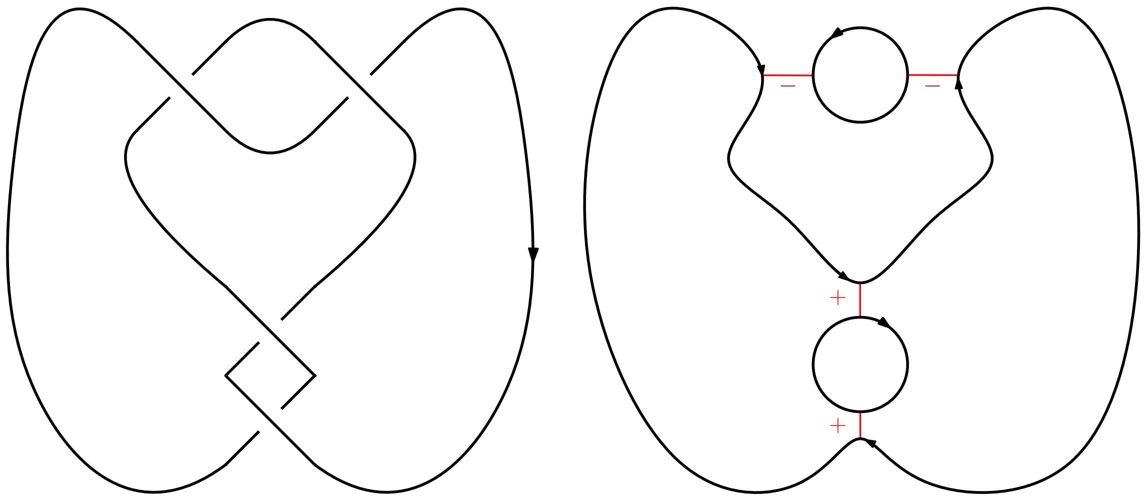


Figure 3.2: *The decorated Seifert circles from an oriented knot diagram for the figure eight knot 4_1. Note, since there is a Seifert circle with two negative crossings, this is not a quasipositive diagram.*

3.3.1 Sharper slice Bennequin inequality

For $S \subset B^4$ with $\partial S = K \subset \partial B^4$, in [56] Rudolph examines the slice Bennequin inequality for quasipositive knots, which places bounds on the Euler characteristic $\chi(S)$ for quasipositive knots K. Since S is a surface bounding a knot smoothly embedded in B^4 one has that $\chi(S) = 1 - g(S)$, and hence such bounds also bound $g_4(K)$.

Motivated by this, in [18, 19] Kawamura uses a closer examination of the Seifert circles as pictured in Figure 3.2 to bound the concordance invariant τ.

Definition 3.16 *A Seifert circle S as in Figure 3.2 is* strictly negative *if there are no positive arcs incident with this circle in the decorated collection of Seifert circles. If a Seifert circle is not strictly negative, then it is defined to be* positive.

Furthermore, associated to a knot diagram D_K for a knot K, define the quantities

$$w(D_K) = \#(\text{ positive crossings }) - \#(\text{ strictly negative crossings })$$
$$O_{\geq}(D_K) = \#(\text{ positive Seifert circles })$$
$$O_{<}(D_K) = \#(\text{ strictly negative Seifert circles }).$$

Using the above definitions, the concordance invariant τ can be bounded as follows.

Theorem 3.17 *[18, Thm. 5.3] For K any knot with diagram D_K, such that the associated set of Seifert circles has $O_{\geq}(D_K) \geq 1$, then*

$$\tau(K) \geq \frac{1}{2}\left(w(D_K) - O_{\geq}(D_K) + O_{<}(D_K) + 1\right).$$

Although the knots in the family $P(2a, -2b-1, 2c+1)$ are clearly not positive or negative, as can be seen through the examination of Figure 3.3, using Theorem 3.17 one can place bounds on $\tau(P(2a, -2b-1, 2c+1))$ as follows.

Proposition 3.18

$$\begin{aligned} c-b-1 \;\; &\leq \;\; \tau(P(2a, -2b-1, 2c+1)) \;\; \leq \;\; c-b \\ b-c \;\; &\leq \;\; \tau(P(-2a, 2b+1, -2c-1)) \;\; \leq \;\; b-c+1. \end{aligned}$$

Proof From Figure 3.3, abusing notation slightly, it is clear that

$$\begin{aligned} w(P(2a, -2b-1, 2c+1)) &= 2c - 2b - 2a \\ O_{\geq}(P(2a, -2b-1, 2c+1)) &= 2 \\ O_{<}(P(2a, -2b-1, 2c+1)) &= 2a - 1. \end{aligned}$$

Hence, applying the bound from Theorem 3.17, one has that

$$\tau(P(2a, -2b-1, 2c+1)) \geq \frac{1}{2}\left(2c - 2b - 2a - 2 + 2a - 1 + 1\right) = c - b - 1.$$

Then, noting that the mirror reverse $P(-2a, 2b+1, -2c-1)$ has the same set of Seifert circles with opposite decorations on the arcs, one has that

$$\begin{aligned} w(P(-2a, 2b+1, -2c-1)) &= 2b + 2a - 2c \\ O_{\geq}(P(-2a, 2b+1, -2c-1)) &= 2a + 1 \\ O_{<}(P(-2a, 2b+1, -2c-1)) &= 0. \end{aligned}$$

Applying the same theorem, one has that $\tau(P(-2a, 2b+1, -2c-1)) \geq b - c$. But, as $\tau(P(-2a, 2b+1, -2c-1)) = -\tau(P(2a, -2b-1, 2c+1))$, this implies that $\tau(P(2a, -2b-1, 2c+1) \leq c - b$. ■

This limits the possible values of τ to two values for the family of pretzel knots $P(2a, -2b-1, 2c+1)$. But by using simple band moves, and the crossing change formula presented in Theorem 3.4, one can determine τ and ν for subfamilies of these knots.

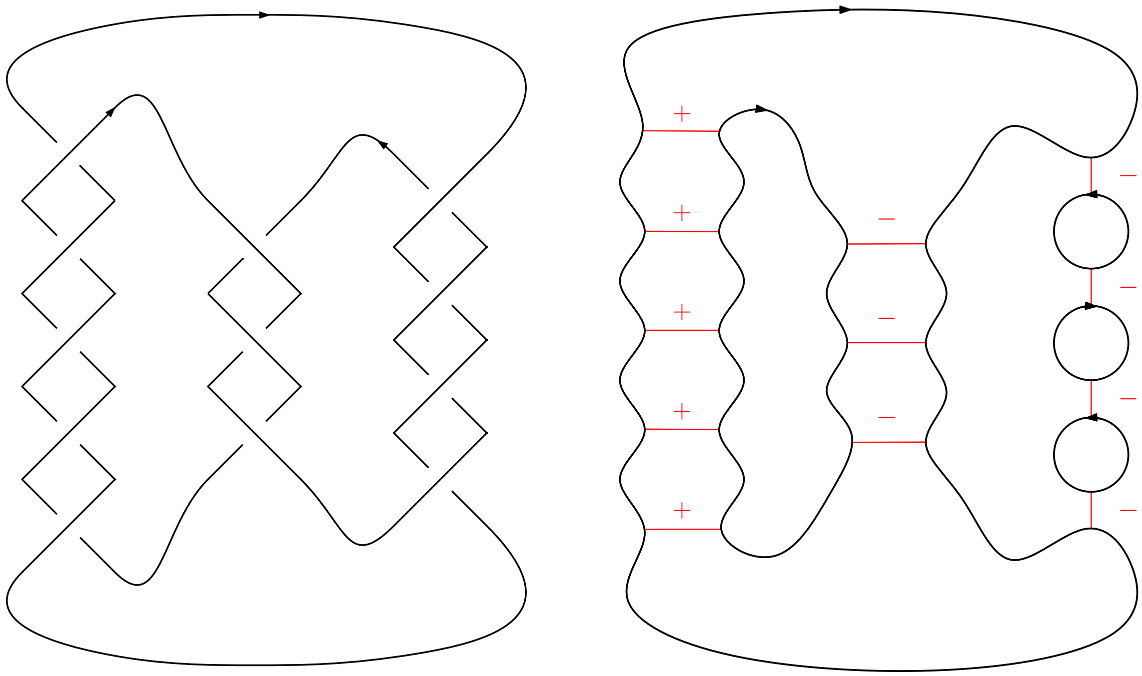


Figure 3.3: *The associated Seifert circles for the standard diagram of* $P(2c+1,-2b-1,2a)$. *The corresponding Seifert circles for the knot* $P(-2c-1,2b+1,-2a)$ *can be found by multiplying all of the decorations on the arcs by* -1.

When $b=c$, it is simple to see that the $P(2a,-2b-1,2b+1)$ is slice. In particular, for any knot $P(2a,-2b-1,2c+1)$, Figure 3.4 demonstrates the placement of a band that, after surgery along this oriented band yields a link isotopic to $T_{2,2|b-c|}$. If $b=c$, this link is isotopic to the two-component unlink.

In the familiar movie format describing cobordisms between surfaces bounding knots, band moves between arcs on the same knot correspond to saddle points in a cobordism. If there is a surface $F \subset B^4$ such that $\partial F = J \subset S^3$, and there is some oriented band move between K and the two-component link J, then there is a genus 0 cobordism between K and J, and so $g_4(K) \leq g(F)$.

Lemma 3.19 *For* $K = P(-2a, 2b+1, -2c-1)$, *with* $b \geq c$, *one has that*

$$\tau(K) = b - c = g_4(K).$$

Proof For $b=c$, the band move demonstrated in Figure 3.4 yields an unlink, which is slice. The surface corresponding to this cobordism, together with the slice disks for each component of the unlink imply that $P(2a,-2b+1,2c+1)$ is slice, since the knot bounds a genus 0 surface with only one saddle point and two minima. Applying Theorem 3.4, this

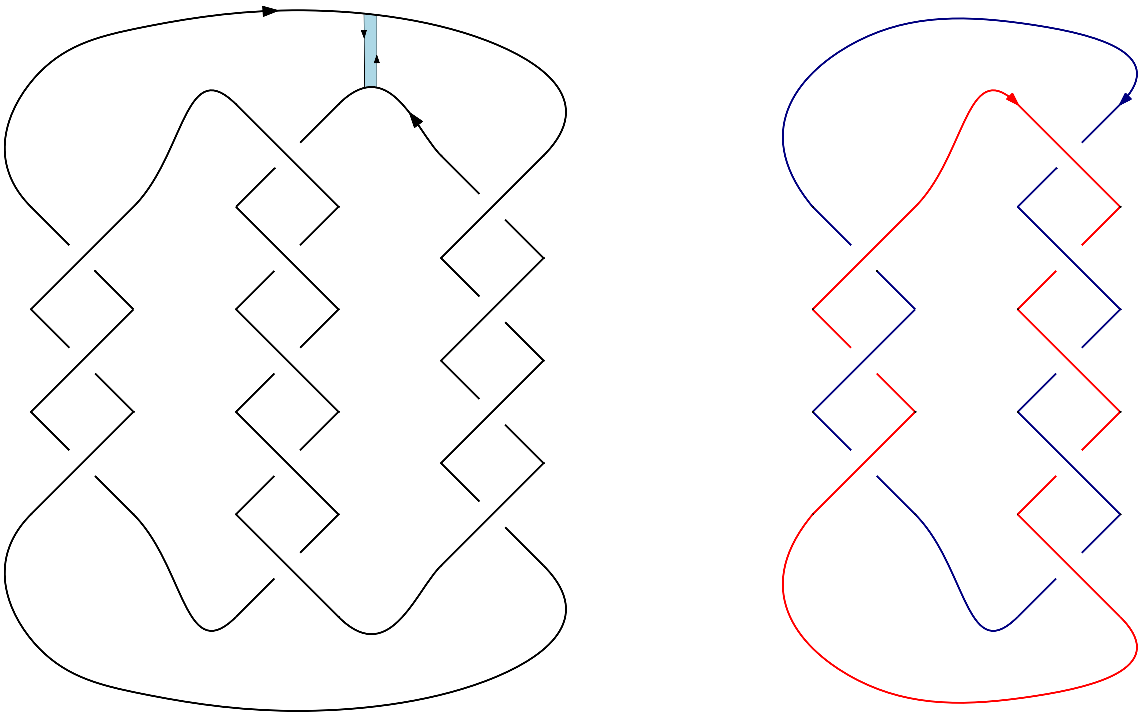


Figure 3.4: *Diagram demonstrating the band move for the pretzel* $P(2c+1, -2b-1, 2a)$ *that results in a link isotopic to* $T_{2,2|b-c|}$. *Here,* $b=2$, $c=1$ *and* $a=2$. *There is an isotopy of the above diagram to the standard form of* $P(2a, -2b-1, 2c+1)$ *by rotating the entire diagram about the third strand.*

implies that $\tau(P(2a, -2b-1, 2b+1)) = 0$, and hence the reverse knot $P(-2a, 2b+1, -2b-1)$ also has $\tau(P(-2a, 2b+1, -2b-1)) = 0$.

Restrict now to the case $b > c$. Since the standard diagram for the pretzel knot $P(2a, -2b-1, 2c+1)$ is isotopic to the diagram in Figure 3.4, and the suggested band move gives a genus zero cobordism between $K = P(2a, -2b-1, 2c+1)$ and $T_{2,2(b-c)}$. This cobordism and then implies that $g_4(K) \leq g_4(T_{2,2(b-c)}) = b - c$. It is a well known fact that taking the reverse of the knot does not affect the slice genus, hence it is also true that $g_4(P(-2a, 2b+1, -2c-1)) \leq b - c$.

Using Proposition 3.18, one thus has that

$$b - c \leq \tau(K) \leq g_4(K) \leq b - c,$$

which yields the result. ■

Remark 3.20 *One can also recover this lemma using the crossing change formula from Theorem 3.4, in a similar way to the proof of the statement below.*

Lemma 3.21 *If* $\tau(P(2a,-2b-1,2c+1)) = c-b-1$ *, then for any* $A > a$*, one has that* $\tau(P(2A,-2b-1,2c+1)) = c-b-1$.

Proof As depicted in Figure 3.5, the encircled crossing is positive. So, in the crossing change formula depicted in Theorem 3.4, denote the knot $P(2c+1,-2b-1,2a) = P(2a,-2b-1,2c+1)$ as K_+.

Changing the circled crossing to negative yields the knot $K_- = P(2(a+1),-2b-1,2c+1)$. If $\tau(K_+) = c-b-1$, then the crossing change inequality implies that

$$\begin{aligned} \tau(K_+) - 1 \ &\leq\ \tau(K_-) \ \leq\ \tau(K_+) \\ c-b-2 \ &\leq\ \tau(K_-) \ \leq\ c-b-1. \end{aligned}$$

Applying the bounds from Proposition 3.18, one thus has that $\tau(K_-) = c-b-1$, since the bounds presented in the proposition do not depend on a. Repeating this process with the Reidemeister two move, and relabelling K_- as K_+ yields the result. ■

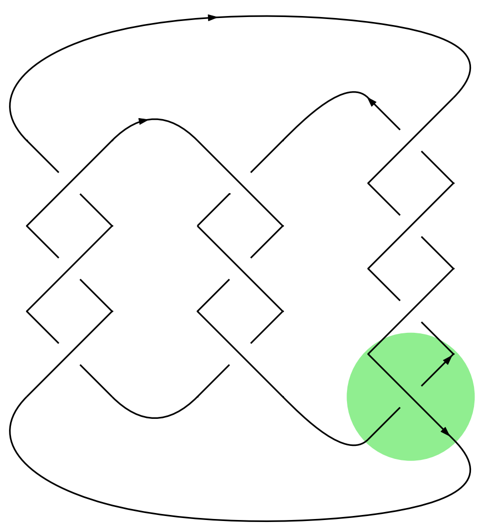


Figure 3.5: *An example of the oriented knot* $P(2a,-2b-1,2c+1)$*, where* $a=1$*. Changing the circled crossing from positive to negative yields the knot* $P(2(a+1),-2b-1,2c+1)$*.*

Chapter 4

Algebraic objects in the construction

As described during the description of Zemke's reformulation of classical knot Floer homology in Section 1.3, in [46, 49], Ozsváth-Szabó defined an algebraic invariant associated to a knot which takes the form of a chain complex over $\mathcal{R}' \cong \mathbb{F}[U, V]/(UV)$. Note, that this is the same ring over which Dai et al defined the complex $CFK_{\mathcal{R}'}(K)$ and reduced knot-like complexes.

In [49], Ozsváth-Szabó use a cut-and-paste construction to associate a $(\mathbb{Z} \oplus \mathbb{Z})$-bigraded, bifiltered chain complex $\mathcal{C}(D)$ to an oriented knot diagram D for the knot K such that every Morse event projects onto the (x, y)-plane at a different y-coordinate. This construction is described in more detail in Section 4.1 and Section 4.5. As a cut-and-paste construction, to smaller pieces of the knot diagram D one associates algebraic objects that can be 'pasted' together in the appropriate algebraic sense to form the larger algebraic invariant $\mathcal{C}(D)$. The filtered chain homotopy type of the complex $\mathcal{C}(D)$ is invariant under Reidemeister moves, as proven in [49, Thm. 1.1].

Theorem 4.1 (Ozsváth-Szabó) *If D and D' are isotopic oriented knot diagrams for the oriented knot K, such that in both D and D' every maximum, minimum and crossing appears in a projection to the (x, y)-plane at a different y-coordinate, then there is a filtered chain homotopy equivalence $\mathcal{C}(D) \cong \mathcal{C}(D')$. Hence, the filtered chain homotopy type $\mathcal{C}(D)$ for D a diagram of K is an invariant of the oriented knot K.*

In [49, Thm. 1.1], it is stated that only the homology of the complex $\mathcal{C}(D)$ is an invariant of the oriented knot. However as remarked in [1, Thm. 5], in [49, Sec. 8] the filtered chain homotopy type of the complex does not change under the application of bridge moves and Reidemeister moves to the corresponding knot diagram. More specifically, each Reidemeister move or bridge move applied to the knot diagram corresponds to a different sequence of algebra elements used in the cut-and-paste construction: however in the proof of [49, Thm. 1.1] it is verified that the corresponding DA-bimodules associated to the partial knot diagrams before and after these moves are equivalent.

As a consequence, the filtered chain homotopy type of the entire complex is a knot invariant, not just the homology of the complex. This is because the complex is constructed from the box-tensor product of the algebraic pieces corresponding to subsets of the knot diagram (see Section 4.5), and if the algebraic pieces from which the complex is determined are invariant under the application of Reidemeister and bridge moves, then so is the full complex.

The cut-and-paste method lends itself to a computer implementation, and indeed Ozsváth-Szabó in [47] have developed *C++* code to implement the calculation of the invariant $\mathcal{C}(D)$ from a given PD-code for a knot diagram. This has been adapted by the author in [58] for the simple calculation of this invariant for three strand pretzel knots without the need to manually determine a PD code for a specific example. Moreover, [58] includes the ability to terminate the algorithm at any point in order to examine intermediate invariants.

The algebraic pieces into which the full knot invariant $\mathcal{C}(K)$ is decomposed are Type D structures, DA-bimodules and A_∞-algebras. The specific examples of the objects associated to crossings, maxima and minima in the construction of [49] are presented in this chapter, after first defining the objects following [24, 25, 46]. Furthermore, since the main consideration in this thesis is the determination of $\mathcal{C}(D)$, for D a diagram of a three strand pretzel knot, when adaptations have been made of the objects used in the constructions of [47, 49], this will be highlighted.

4.1 The complex $\mathcal{C}(K)$

For D an oriented knot diagram for the knot K, the invariant $\mathcal{C}(D)$ defined by Ozsváth-Szabó in [49] is a bigraded chain complex over the ring $\mathcal{R}'$. This chain complex has a generating set over $\mathcal{R}'$ that is in one to one correspondence with Kauffman states for the oriented knot diagram D.

The reformulation of knot Floer homology from Dai et al described in Section 1.3.1 and [4] is determined from a Heegaard diagram associated to a knot. Theorem 2.3 implies that the Kauffman states for a knot diagram are in one to one correspondence with the intersection points of a Heegaard diagram for this knot, and so provide an $\mathcal{R}'$-basis for the complex $CFK_{\mathcal{R}'}(D)$, the local equivalence class of which is a knot invariant.

In [49, Sec. 1.3], Ozsváth and Szabó conjectured that the two formulations are equivalent, with this equivalence to be proven in a forthcoming paper. Very recently, this conjectural equivalence was proven in [48]. However, the two chain complexes are defined in two completely independent ways. In [4], Dai et al adapt Zemke's reformulation of knot Floer homology to be taken over the ring $\mathcal{R}'$, and the definition of the differential involves counting pseudo-holomorphic representatives of Whitney disks between generators. The construction of $\mathcal{C}(D)$ is purely algebraic, with algebraic objects associated to each 'piece' of the knot diagram with no reference being made to a corresponding Heegaard diagram or Whitney disk.

But, as in the construction of Heegaard diagrams associated to knot diagrams for which Kauffman states correspond to generators (see Section 2.1), from a thickened up projection of any partial knot diagram one can produce a partial Heegaard diagram: simply the excised piece of the Heegaard diagram for the full knot. This partial Heegaard diagram, although not used in the constructions of [46, 49], does motivate the definitions of the maps in the DA-bimodules. This Heegaard diagram interpretation will be explained further in Section 4.2.2, but the very recent work of Ozsváth-Szabó in [48, Sec. 2.6] also gives the correspondence between the algebraic objects and appropriate partial Heegaard diagrams.

More formally, the complex $\mathcal{C}(D)$ is defined as follows.

Definition 4.2 *Let D be an oriented knot diagram for the oriented knot K, such that every maximum, minimum and crossing appears at a different y-coordinate in the projection of the knot diagram to the (x, y)-plane. Call such a knot diagram a* special knot diagram *for K. Mark the global minimum of a special knot diagram, such that any Kauffman state of this diagram has the edge containing the global minimum as the distinguished edge.*

As described in [49, Secs. 1,8], the complex $(\mathcal{C}(D), \partial)$ associated to a special knot diagram diagram D is a chain complex over $\mathcal{R}'$, generated by Kauffman states for the special knot diagram. This complex is bigraded, with two integer gradings Δ and A, such that the complex splits as

$$\mathcal{C}(D) = \bigoplus_{\delta,s} C_\delta(D,s).$$

Here, δ is the Δ-grading, and s the A-grading (or Alexander grading).

These graded components are then equipped with the actions:

$$U : \mathcal{C}_\delta(D,s) \to \mathcal{C}_{\delta-1}(D,s-1),$$
$$V : \mathcal{C}_\delta(D,s) \to \mathcal{C}_{\delta-1}(D,s+1),$$
$$\partial : \mathcal{C}_\delta(D,s) \to \mathcal{C}_{\delta-1}(D,s),$$

where U and V act on elements of the chain complex over $\mathcal{R}' \cong \mathbb{F}[U,V]/UV$ by multiplication. The differential ∂ will be defined later in Section 4.5.

Note, the bigrading of the chain complex $\mathcal{C}(D)$ is not the same as the (gr_U, gr_V)-graded complex $CFK_{\mathcal{R}'}(K)$. This is most clearly seen by the fact that multiplication by U and multiplication by V affect both gradings in this setting, whereas in $CFK_{\mathcal{R}'}(K)$ gr_V is unaffected by multiplication by U, and gr_U is unaffected by multiplication by V.

The Δ-grading of $\mathcal{C}(D)$ is identical to that of the Δ-grading introduced in Section 2.2, and can be related back to the familiar Maslov grading from [39], by the equation $\Delta = M - A$. Furthermore, the Δ and A gradings can be read off directly from the Kauffman states, in a similar method to Definition 2.2.

Definition 4.3 *Let C be the collection of crossings in a special knot diagram D. Then for x a Kauffman state of this special knot diagram, the integer valued gradings $\Delta(x)$ and $A(x)$ are defined as*

$$\Delta(x) = \sum_{c\in C} \Delta(c),$$
$$A(x) = \sum_{c\in C} A(c),$$

where $\Delta(c)$ and $A(c)$ are the local contributions at each crossing as displayed in Figure 4.1.

Remark 4.4 *Note, for a Kauffman state x, since the values $\Delta(x)$ and $A(x)$ are determined through local contributions, one can consider the same total of local contributions*

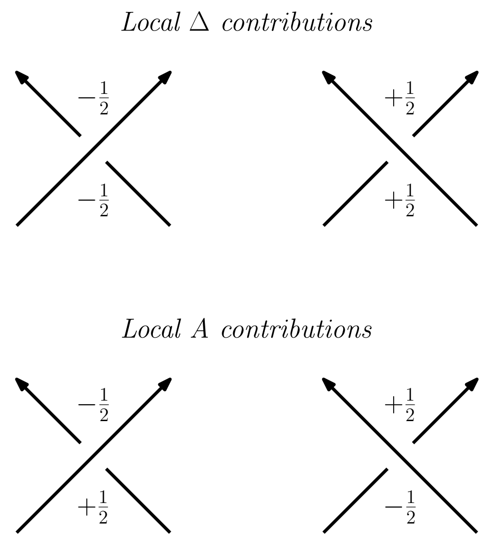


Figure 4.1: *The contributions to Δ and A at each crossing, following [49, Fig. 1]*

for a Kauffman state for a partial knot diagram. This is discussed in more detail when considering the specific Type D structures associated to three strand pretzel knots.

Although the Ozsváth-Szabó's invariant $\mathcal{C}(D)$ and Dai et al's invariant $CFK_{\mathcal{R}'}(K)$ have different gradings, the filtration provided is the same: i.e. in terms of U and V powers. An example is given for the trefoil $T_{2,3}$ in Figure 4.2, which is a simplification of Figure 1.5.

As remarked in [49, Sec. 1], and [46, Cor. 11.11], if one sets $U = V = 0$, the resulting complex – denoted $\widehat{\mathcal{C}}(D)$ – has a homology that has (by construction) an Euler characteristic that is equal to the (symmetric) Alexander polynomial $\Delta_K(q)$. Modifying the grading slightly back to the Maslov grading $d = \Delta - A$, one has:

$$\chi\left(H_*\left(\widehat{\mathcal{C}}(D)\right)\right) = \sum_{d\in\mathbb{Z}}(-1)^d rk\left(H_*\left(\widehat{\mathcal{C}}_d(D,s)\right)\right)q^s = \Delta_K(q).$$

In a similar way to classical knot Floer homology and Zemke's reformulation, one can take subcomplexes and quotient complexes as formulated in Section 1.2.3 and extract bigraded groups and associated homology theories that are also knot invariants. For example, setting $V = 0$ yields a complex $\mathcal{C}^-(D)$ that was originally proposed to be conjecturally equivalent to $CFK^-(K)$, see [46]. Once more, this equivalence was recently proven in [48, Thm. 1.1].

As will be elaborated upon later, the complex $\mathcal{C}(D)$ is the result of taking an appropriate tensor product of a Type D structure and an A_∞-module. These are modules associated

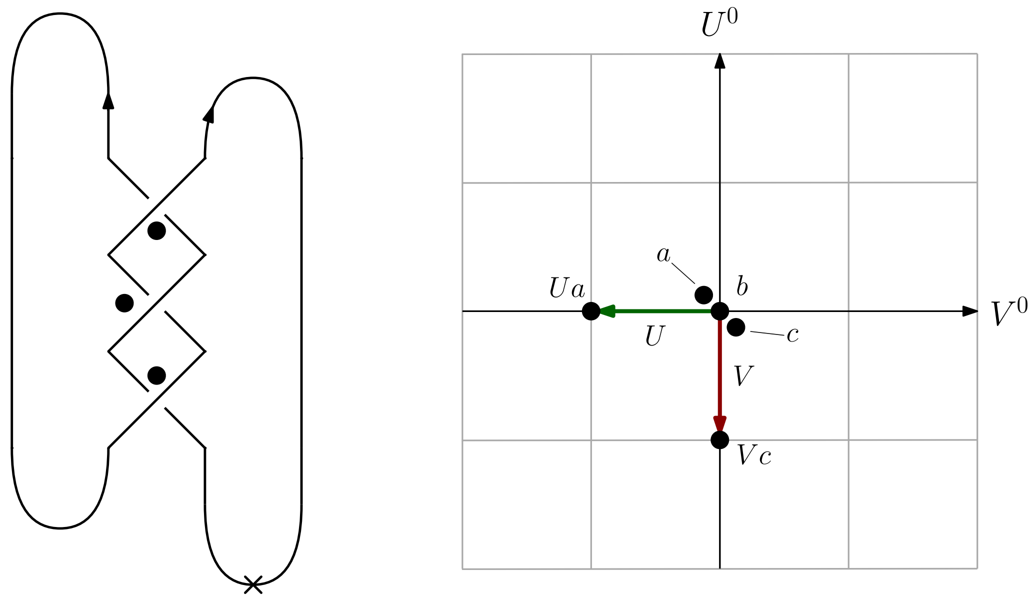


Figure 4.2: *A special knot diagram D for the trefoil $T_{2,3}$. The complex shown is then a common pictorial simplification of the complex $\mathcal{C}(D)$ – which is identical to the representation of $CFK_{\mathcal{R}'}(T_{2,3})$. Recall, Figure 1.5 gives the diagrammatic representation of $CFK_{\mathcal{R}}(T_{2,3})$. Here, only the differential is displayed, and one can yield the non-simplified diagram by setting $UV = 0$ in Figure 1.5. The state b is shown in the knot diagram, with a the Kauffman state with the leftmost region occupied at the bottom crossing, and c the Kauffman state with the leftmost region occupied at the top crossing. Following Definition 4.3, the (Δ, A)-bigrading of a is $(-1, 1)$, the bigrading of b is $(-1, 0)$, and the bigrading of c is $(-1, -1)$.*

to partial knot diagrams (respectively upper and lower knot diagrams), with a common differential graded algebra A associated to each object.

4.2 Differential graded algebras

For a special knot diagram as defined in Definition 4.2, except at finitely many values, a generic line $y = \ell$ will intersect the special knot diagram at $2m$ points, which can be labelled using the set $\{1, 2, \ldots, 2m\}$. In [49], building upon the definition in [46], Ozsváth-Szabó associate to every one of these level sets a differential graded algebra (DGA), which is a type of A_∞-algebra.

Definition 4.5 *[25, Def. 2.1] Over the ground ring $\mathbb{F}$, an A_∞-algebra $\mathcal{A}$ is a graded $\mathbb{F}$-module, equipped with $\mathbb{F}$-linear multiplication maps*

$$\mu_i : \mathcal{A}^{\otimes i} \to \mathcal{A}[2-i],$$

defined for all $i \geq 1$, such that for all i one has that

$$\sum_{i+j=n+1} \sum_{\ell=1}^{n-j+1} \mu_i(a_1 \otimes \cdots \otimes a_{\ell-1} \otimes \mu_j(a_\ell \otimes \cdots \otimes a_{\ell+j-1}) \otimes a_{\ell+j} \otimes \cdots \otimes a_n) = 0.$$

All tensor products are taken over the idempotent ring of the algebra, $\mathcal{I}(\mathcal{A})$.

More intuitively, the terms in this relation correspond to an equivalence class of trees with two vertices. Let the vertices of degree $j+1$ of the tree represent operations μ_j, where j incident edges are above the vertex, representing the j-inputs to μ_j. Trees are considered equivalent when the collapse of an edge between the two vertices yields the same graph. This is displayed in Figure 4.3.

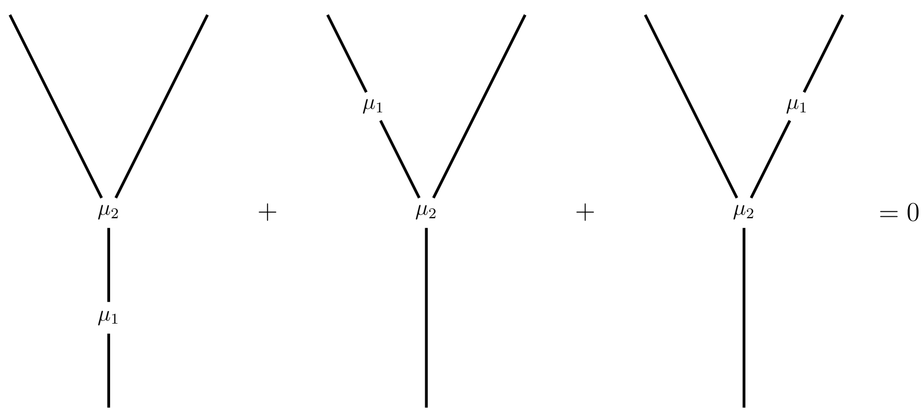


Figure 4.3: *The collapsing tree relation for $i = 2$. Each edge within the tree represents a tensor coordinate $\mathcal{A}$, for $\mathcal{A}$ an A_∞-algebra. The vertices correspond to maps μ_i, with the $i+1$ incident edges, i of which point upwards, representing the domain of the map. Collapsing the edge between the two vertices in any tree yields an identical tree with a single degree 3 vertex.*

A differential graded algebra A is then an A_∞-algebra such that the multiplication maps $\mu_i = 0$ for $i \geq 3$. The (possibly non-zero) maps μ_1 and μ_2 in a DGA can then be thought of as a differential and product in the graded algebra respectively. Differential graded algebras have been used in the calculation of bordered Heegaard Floer homology of 3-

manifolds, with examples including the torus algebra [25, Ch. 11] and the algebra defined for matched circles [25, Ch. 3].

In [24, 25], modules are defined over these algebras which correspond to partial Heegaard diagrams for a three-manifold, which are then pieced together algebraically in order to calculate the Heegaard Floer homology of the three manifold. This setting motivates the construction of $\mathcal{C}(D)$ through taking appropriate tensor products of A_∞-modules and Type D structures. The algebras in the construction of bordered Heegaard Floer homology track the interaction of domains in the partial Heegaard diagrams with the boundary, similar to the interpretation presented here in Section 4.2.2, and recently outlined in depth in [48].

4.2.1 The algebra $\mathcal{A}(n)$

Following the definitions in [49, Sec. 2], the algebra $\mathcal{A}(n)$ used in the construction of the knot invariant $\mathcal{C}(D)$ is defined as the extension of an algebra $\mathcal{B}(2n, n)$. As highlighted above, a special knot diagram cuts the line $y = \ell$ in $2n$ places at any point. One can then index these intersection points by the set $\{1, 2, \ldots, 2n\}$. One then defines I-states in this algebra $\mathcal{B}$ as follows.

Definition 4.6 *Let x be an n-element subset of $\{1, 2, \ldots, 2n-1\} \subset \{1, 2, \ldots, 2n\}$. Then I_x is an idempotent or I-state in $\mathcal{B}(2n, n)$ that can be represented by n occupied positions, where each position is to the right of some wall i, as in Figure 4.4.*

For every I-state, one has that $\mu_1(I_x) = 0$, and

$$\mu_2(I_x, I_y) = \begin{cases} I_x & \text{if } x = y, \\ 0 & \text{else.} \end{cases}$$

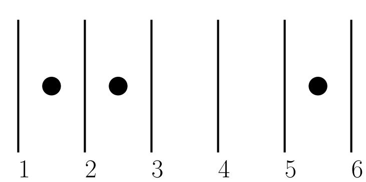


Figure 4.4: *A geometric interpretation of the idempotent, or I-state I_{125} in $\mathcal{B}(6, 3)$. Each region bounded by walls in this diagram is then assigned the label matching the wall to the left.*

This definition, restricting the possible occupied positions to the subset $\{1, 2, \ldots, 2n-1\}$, is in fact a truncated version of the idempotents presented in [49]. The possible truncations of the idempotents are presented in [27, Sec. 3.4]. As explained in [49, Prop. 8.2] and [46, Sec. 12], since the bigraded chain complex $\mathcal{C}(D)$ is generated by the Kauffman states of a special knot diagram, and by construction this has the distinguished edge as the global minimum, one can restrict all algebraic objects to an algebra with this truncated idempotent ring and yield the appropriate knot invariant.

This will be clarified in Section 4.2.2: one can interpret the idempotents as providing information about where in the special knot diagram there is an occupied state. The regions in Figure 4.4 must, excepting if they are incident to the global minimum, have a Kauffman state somewhere within the region. The idempotent I_x then has the interpretation that $i \in x$ as an n-element subset of $\{1, 2, \ldots, 2n-1\}$ if the region to the right of wall i does not have an marked point in this region in the subset of the knot diagram with $y \geq \ell$.

Motivated by [46, 49], pure algebra elements in $\mathcal{B}(2n, n)$ are defined by triples $[I_x, I_y, w]$, where I_x, I_y are I-states in $\mathcal{B}(2n, n)$, and $w \in \left(\frac{1}{2}\mathbb{Z}\right)^{2n}$ is a half-integral weight. Informally, the idempotents I_x and I_y will be referred to as incoming and outgoing idempotents respectively. Hence, one can represent an algebra element $b \in \mathcal{B}(2n, n)$ as $I_x \cdot b \cdot I_y$, to give information regarding the incoming and outgoing idempotents. The pure algebra elements in $\mathcal{B}(2n, n)$ are thus defined as follows.

Definition 4.7 1. *Idempotent elements: Let the triple $[I_x, I_x, \overrightarrow{0}]$ denote an idempotent element of $\mathcal{B}(2n, n)$. Here, I_x is an I-state as defined in Definition 4.6. Note, the weight of idempotent elements is 0 in every coordinate. Together, the idempotents make the ring of idempotents $\mathcal{I}(\mathcal{B})$, and one can define the unital element*

$$\mathbf{1} = \sum_{\substack{\text{all } n\text{-element subsets of} \\ \{1,2,\ldots,2n-1\}}} \left[I_x, I_x, \overrightarrow{0}\right].$$

2. *L_i: For x an n-element subset of $\{1, 2, \ldots, 2n-1\}$, such that $i \in x$, but $(i-1) \notin x$, define $y = (x \backslash \{i\}) \cup \{i-1\}$. Then, the element $I_x \cdot L_i \cdot I_y$ is defined as the triple $[I_x, I_y, \frac{1}{2}e_i]$, for e_i the standard basis element of $\mathbb{Z}^{2n}$ with 1 in the i^{th} coordinate.*

3. *R_i: Similarly, let x be an n-element subset of $\{1, 2, \ldots, 2n-1\}$ such that $i \notin x$, but $(i-1) \in x$. Then define $y = (x \backslash \{i-1\}) \cup \{i\}$. The element $I_x \cdot R_i \cdot I_y$ is then defined as the triple $[I_x, I_y, \frac{1}{2}e_i]$.*

4. *U_i: Let U_i^x denote the triple $[I_x, I_x, e_i]$ for some I-state I_x. Then, let U_i denote the formal sum*

$$U_i = \sum_{\substack{I_x \\ \{i,i-1\}\cap x \neq \emptyset}} U_i^x.$$

When specifying the specific idempotents associated with U_i, the notation used will often be $I_x \cdot U_i \cdot I_x = U_i^x$.

The elements L_i and R_i can be intuitively thought of as taking an incoming idempotent, and moving the marked position in region i left across wall i for L_i, or moving the marked position $i-1$ right across wall i for R_i. In a slight abuse of notation, in a similar way to the definition of U_i, the terms L_i and R_i may also denote the formal sum of all triples

$$L_i = \sum_{\substack{I_x \\ i \in x \\ (i-1)\notin x}} [I_x, I_y, \frac{1}{2}e_i],$$

$$R_i = \sum_{\substack{I_x \\ (i-1) \in x \\ i\notin x}} [I_x, I_y, \frac{1}{2}e_i].$$

Here, define I_y in the appropriate way for each of the terms, as shown in Definition 4.7.

Remark 4.8 *As described above, every pure algebra element in $\mathcal{B}(2n, n)$ has an associated incoming and outgoing idempotent. When I-states were defined in Definition 4.6, they were described as n-element subsets of $\{1, 2, \cdots, 2n-1\}$. Throughout this work, idempotent elements in $\mathcal{B}(2n, n)$ will often be denoted as simply I_x for some n-element subset x. However, as seen in Definition 4.7, they are more formally triples $\left[I_x, I_x, \overrightarrow{0}\right]$, with the same incoming and outgoing n-element subsets, and 0 weight in every component.*

To define a differential graded algebra $\mathcal{B}(2n, n)$, one must also carefully define the maps $\mu_1 : \mathcal{B}(2n, n) \to \mathcal{B}(2n, n)$ and $\mu_2 : \mathcal{B}(2n, n) \otimes \mathcal{B}(2n, n) \to \mathcal{B}(2n, n)$, recalling that all higher maps are 0 for a DGA. For now, the grading information will be omitted, since the full algebra $\mathcal{A}(n)$ will be equipped both with an Alexander multigrading, and a homological grading Δ, following [46, 49].

Definition 4.9 *Define $\mathcal{B}(2n, n)$ as the algebra generated over $\mathbb{F} \cong \mathbb{Z}/2$ by the elements above, namely:*

$$\mathcal{B}(2n, n) = \langle I_x, L_i, R_j, U_p \rangle \,,$$

taking all possible I-states I_x, and all possible L_i, R_j and U_p as defined in Definition 4.7.

Define $\mu_1 : \mathcal{B}(2n, n) \to \mathcal{B}(2n, n)$ *to be trivial, so every element in* $\mathcal{B}(2n, n)$ *lies in* $ker(\mu_1)$*. Then, for any two algebra elements* $A = [I_x, I_y, w_A]$ *and* $B = [I_r, I_z, w_b]$*, with weights* $w_A, w_B \in \frac{1}{2}\mathbb{Z}^{2n}$*, define the product* $\mu_2(A, B) := A \cdot B$ *as*

$$\mu_2(A, B) = \begin{cases} [I_x, I_z, w_A + w_B] & \textit{when } I_y = I_r, \\ 0 & \textit{else.} \end{cases}$$

Here, one is taking $w_A + w_B$ to be the component-wise sum of the two vectors. Note, this means that the outgoing idempotent of A must be equal to the incoming idempotent of B in order to have a non-zero product. Furthermore, use the convention that the product

$$[I_x, I_x, e_i] \cdot [I_x, I_x, e_i] = [I_x, I_x, 2e_i] = U_i^x \cdot U_i^x = (U_i^x)^2.$$

Before extending this algebra $\mathcal{B}(2n, n)$ to define the algebra $\mathcal{A}$, used in the construction of $\mathcal{C}(D)$, the definition in [49] takes the quotient of the above to define $\mathcal{B}_0$.

Definition 4.10 *Define* $\mathcal{B}_0$ *as the quotient algebra of* $\mathcal{B}(2n, n)$*, as follows.*

$$\mathcal{B}_0 = \mathcal{B}(2n, n)/\sim$$

where $\sim$ *denotes the relations:*

1. $L_{i+1} \cdot L_i = 0$ *for every* i*.*

2. $R_i \cdot R_{i+1} = 0$ *for every* i*.*

3. $U_i^x = [I_x, I_x, e_i] = 0$ *when* $\{i, i-1\} \cap x = \emptyset$*, for every* i*.*

   After taking this quotient, all non-zero elements in $\mathcal{B}_0$ *have the same weights as in* $\mathcal{B}(2n, n)$*, and the products* μ_2 *and differential* μ_1 *are defined as in Definition 4.9.*

In the literature, the idempotents associated to the algebra elements $L_{i+1} \cdot L_i$ and $R_i \cdot R_{i+1}$ are said to be 'far', see [46, Def. 3.5]. Using Figure 4.4, an algebra element is then equal to zero from one of the first two relations if the marked position by some wall moves across more than one wall.

However, when taken with the appropriate idempotents, the terms $L_i \cdot L_{i+1}$ and $R_{i+1} \cdot R_i$ are non-zero, as these would involve moving two marked positions across one wall each, see Figure 4.5.

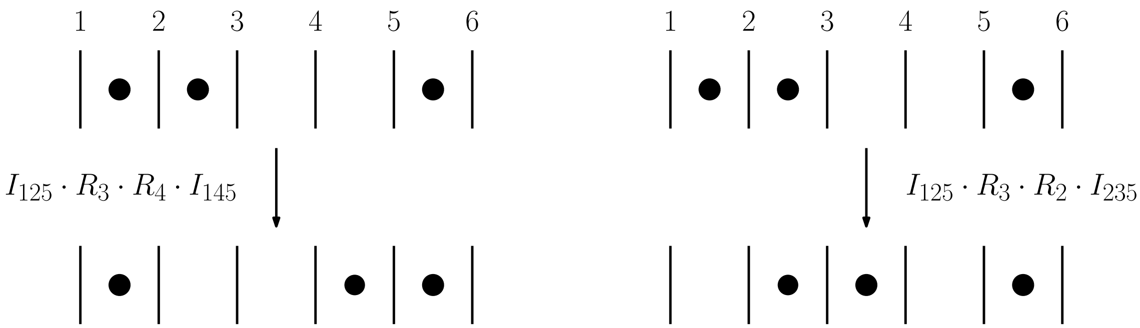


Figure 4.5: *A geometric interpretation of the two algebra elements* $[I_{125}, I_{145}, \frac{1}{2}(e_3 + e_4)]$ *and* $[I_{125}, I_{235}, \frac{1}{2}(e_2 + e_3)]$ *in* $\mathcal{B}(6,3)$*. The first algebra element is zero in the quotient* $\mathcal{B}_0$*, since a marked point moves across two walls, whereas the second algebra element is non-zero.*

As remarked upon, the algebra $\mathcal{B}(2n, n)$ is associated to every level set $y = \ell$ of a special knot diagram, since a special knot diagram intersects the line in $2n$ points for some natural number n. The part of this knot diagram above this level set – i.e. the intersection of the special knot diagram with $\{y \geq l\}$ – is called an *upper knot diagram.* An upper knot diagram provides a complete matching[1] on the set $\{1, 2, \ldots, 2n\}$, obtained through following the arc incident with position i at line $y = \ell$ through the upper knot diagram to some other position j on the line $y = \ell$. From the upper knot diagram at this level, one thus defines a term C_{ij}.

Definition 4.11 *Let* M *be the complete matching on* $\{1, 2, \ldots, 2n\}$ *arising from the upper knot diagram above the line* $y = \ell$*. For every pair* $\{p, q\} \in M$*, define elements*

$$C_{pq} = \sum_{I_x} [I_x, I_x, e_p + e_q] \,.$$

Note that although the number of strands may not change as one passes from the line $y = \ell$ to $y = \ell - 1$, the matching may change due to the change in the upper knot diagram. For example a crossing between strands i and $i + 1$ will swap i and $i + 1$ in the matching M to yield a new matching M'. With these matching elements, the definition of the algebra $\mathcal{A}(n)$ is as follows – see [49, Sec. 2.1], with grading conventions provided by [46, Sec. 2].

[1]A complete matching M on the set $\{1, 2, \cdots, 2k\}$ is a partition of the set into k subsets, each with two distinct elements.

Definition 4.12

$$\mathcal{A}(n) = \mathcal{B}_0 \cup \langle C_{pq} \rangle_{pq \in M} \,/ \sim$$

where $\sim$ denotes the following relations.

1. *$C_{pq} \cdot X = X \cdot C_{pq}$ for X any element not equal to C_{pq}.*
2. *$C_{pq} \cdot C_{pq} = 0$, for any matching element C_{pq}.*

Recall, $\mathcal{B}_0$ is already the quotient of the algebra $\mathcal{B}(2n, n)$ as defined in Definition 4.10.

The only non-zero differentials $\mu_1 : \mathcal{A}(n) \to \mathcal{A}(n)$ are given by

$$\mu_1(C_{pq}) = U_p \cdot U_q,$$

for every matching element C_{pq} with $\{pq\} \in M$.

The grading $\Delta(X) \in \mathbb{Z}$ for an algebra element $X \in \mathcal{A}(n)$ is then given by

$$\Delta(X) = \#\,(C_{pq} \text{ dividing } X) - \sum_{i=1}^{2n} w_i(X),$$

where $w_i(X)$ is the i^{th} coordinate of the weight $w(X)$. The weights $w(X)$ defined above provide a second $\frac{1}{2}\mathbb{Z}^{2n}$-grading, called the Alexander multi-grading.

The algebra $\mathcal{A}(n)$ then splits as a direct sum with these gradings. One can decompose

$$\mathcal{A}(n) = \bigoplus_{\substack{d \in \mathbb{Z} \\ \ell \in \left(\frac{1}{2}\mathbb{Z}\right)^{2n}}} \mathcal{A}_{d,\ell},$$

such that the maps μ_1 and μ_2 in $\mathcal{A}(n)$ act by

$$\mu_1 : \mathcal{A}_{d,\ell} \to \mathcal{A}_{d-1,\ell}, \qquad \mu_2 : \mathcal{A}_{d_1,\ell_1} \otimes \mathcal{A}_{d_2,\ell_2} \to \mathcal{A}_{d_1+d_2,\ell_1+\ell_2}.$$

Using the above definition, Ozsváth-Szabó proved in [46] the useful proposition that pure algebra elements are uniquely determined as follows.

Proposition 4.13 *[46, Prop. 3.9] A pure, non-matching element $[I_x, I_y, w(X)] = X \in \mathcal{A}(n)$ is uniquely characterised by the idempotents I_x, I_y and the weight $w(X)$.*

Remark 4.14 *The proof of this proposition presented in [46, Prop. 3.9] uses a formulation of the algebra $\mathcal{B}(2n, n)$ in terms of an identification of $\mathbb{F}[U_1, \cdots, U_{2m}]$-modules, presented in [46, Sec. 3.1]. Using this proposition, non-matching elements of the algebra $\mathcal{A}(n)$ can be thought of as determined by these triples.*

As a consequence, one can make the observation that, for certain idempotents, $I_x \cdot L_i \cdot R_i \cdot I_x = U_i^x$. *To see this, note that if* $L_i = [I_x, I_y, \frac{1}{2}e_i]$, *and* $R_i = [I_y, I_x, \frac{1}{2}e_i]$, *then since elements are defined uniquely by their idempotents and weight, one has that*

$$\mu_2(L_i, R_i) = [I_x, I_x, e_i] = U_i^x.$$

4.2.2 Interpretation of algebra elements on a Heegaard diagram

The algebra $\mathcal{A}(n)$ is defined purely algebraically: i.e. by triples as given in Definitions 4.9, 4.10 and 4.12, together with a complete matching on $\{1, 2, \dots, 2n\}$. As highlighted by Figure 4.4, idempotents $I_x = [I_x, I_x, \overrightarrow{0}]$ have an interpretation in terms of arcs intersecting the level set $\{y = \ell\}$ in a special knot diagram, but algebra elements R_i, L_j and U_p also have a similar interpretation.

A special knot diagram intersects the line $y = \ell$ in $2n$ points for some n, and using the construction in Theorem 2.3 one can yield a Heegaard diagram from a knot diagram by considering the 'thickened up' surface of the knot as a handlebody. Such a Heegaard diagram associated to a knot would intersect this level in $2n$-circles, with a local picture as displayed in Figure 4.6. Algebra elements then have an interpretation as the intersection of a domain in the partial Heegaard diagram with the boundary at the level $y = \ell$. This has recently been explained in more depth in [48, Sec. 13], which formalises how the algebra elements correspond to regions in upper and partial Heegaard diagrams.

Motivated by the idea that algebra elements correspond to regions bound by α and β curves in partial Heegaard diagrams arising from the Kauffman state construction of [36], one can associate the following regions in the local picture about $y = \ell$, with visual representation as demonstrated in Figure 4.6.

- The element L_i corresponds to the 'back' of the tube for strand i.
- The element R_i corresponds to the 'front' of the tube for strand i.
- The element U_i corresponds to the whole of the tube for strand i.
- For $pq \in M$ a matching, the element C_{pq} would correspond to the whole of the tubes for strands p and q, with the domain in the upper knot diagram connecting these tubes.

In this way, one can see that the view of U_i as the product of L_i and R_i corresponds to the fact that the sum of the domains on the back and front of tube i would be the whole tube.

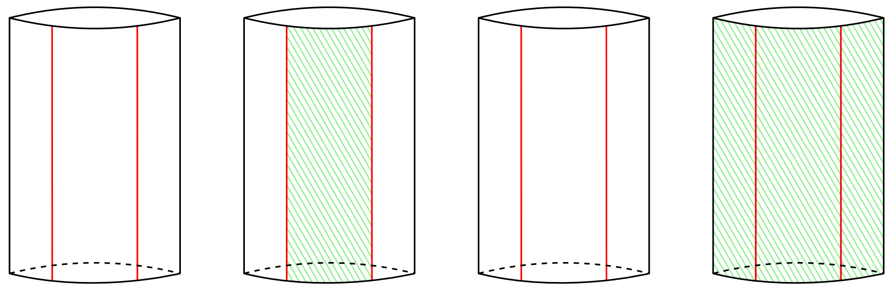


Figure 4.6: *The representation of the algebra element* $[I_{13}, I_{23}, \frac{1}{2}e_2{+}e_4] = I_{13}{\cdot}R_2{\cdot}U_4{\cdot}I_{23} \in \mathcal{A}(2)$ *in a partial Heegaard diagram. Note that* $\mathcal{A}(2)$ *in a special knot diagram is associated to four arcs intersecting the line* $y = \ell$ *for some* ℓ.

Furthermore, this gives some intuition as to the idempotent elements in the algebra. The outgoing idempotent of the element $I_{13} \cdot R_2U_4 \cdot I_{23}$ pictured in Figure 4.6 is the I-state I_{23}. Since only the front of the second tube (corresponding to arc 2 in a special knot diagram) is shaded, then in order to yield a valid domain in a Heegaard diagram, one would need to have an intersection point with a β curve somewhere along this α curve below $y = \ell$.

It is then relatively simple to check that appropriate β curves – for example corresponding to the β-curves at crossings between arcs 1 and 2 or 2 and 3 – require an intersection point on one of the α-curves of the tube corresponding to a point of a Kauffman state in this region in order to yield a valid domain according to appropriate restrictions on the corners of such domains (see [23, 53]).

Hence, when using this algebra, one can carry the intuition that an idempotent state I_x means that there is some decoration of a Kauffman state in region i for $i \in x$ below $y = \ell$. Equivalently, this would imply that there is no decoration of a Kauffman state in this region above $y = \ell$. Figure 4.14 may clarify this slightly by introducing an example of a DA-bimodule map between Kauffman state generators using a corresponding partial Heegaard diagram.

4.3 Type D structures

As described earlier in the chapter, one of the algebraic objects used in the construction of the invariant $\mathcal{C}(D)$ by Ozsváth-Szabó is a type of object called a Type D structure, defined over an A_∞-algebra A. The A_∞-algebra in question will be the algebra $\mathcal{A}(n)$ as introduced in Section 4.2.1. Such objects are associated to upper knot diagrams.

Definition 4.15 *For D a special knot diagram for a knot K, there are only finitely many levels ℓ such that the level set is at a crossing, maximum, or minimum. For all other choices of ℓ, an upper knot diagram is then the part of the special knot diagram above this chosen level, namely $D \cap \{y \geq \ell\}$.*

Note, this subset of the special knot diagram also provides a matching M on the set $\{1, 2, \ldots, 2n\}$, as discussed in the previous section.

In the construction of the knot invariant defined by Ozsváth-Szabó in [47, 49], the A_∞-algebra $\mathcal{A}(n)$ is a differential graded algebra. As a consequence, the only Type D structures considered in this thesis are those over differential graded algebras. Specialising to this case, the definition of a Type D structure is as follows, as presented in [25, Def. 2.18].

Definition 4.16 *Let M be a graded $\mathbb{F}$-module, and fix a differential graded algebra $\mathcal{A}$, also over $\mathbb{F}$. Let $\partial^1 : M \to (\mathcal{A} \otimes M)[1]$ be a map satisfying the condition*

$$0 = (\mu_2 \otimes Id_M) \circ (Id_{\mathcal{A}} \otimes \partial^1) \circ \partial^1 + (\mu_1 \otimes Id_M) \circ \partial^1 : M \to \mathcal{A} \otimes M.$$

Denote by ${}^{\mathcal{A}}M$ the pair (M, ∂^1). A module and map pair satisfying this compatibility condition is a Type D structure over the differential graded algebra $\mathcal{A}$.

This compatibility condition is more easily pictured using trees, as in the case of the A_∞-algebra relations. See Figure 4.7 for a visual representation of this. For the more general setting of a Type D structure defined over an A_∞-algebra that is not a differential graded algebra, the reader is referred to [24, Sec. 2.2.3].

Importantly, the tensor products within the Type D structure relations are taken over the ring of idempotents $\mathcal{I}(\mathcal{A})$ as defined earlier, and any element in the module M of the Type D structure thus has an associated idempotent. Hence, for the map ∂^1 to be non-zero, if $\partial^1(x) = a \otimes y$, where $a \in \mathcal{A}$ and $x, y \in M$, then a must have an (outgoing) idempotent matching that of y. For example, $\partial^1(I_f \cdot x) = I_f \cdot a \cdot I_g \otimes I_h \cdot y$ is only non-zero when $I_g = I_h \in \mathcal{I}(\mathcal{A})$.

A useful intuition when considering Type D structures is that the compatibility condition roughly corresponds to the $d^2 = 0$ relation for chain complex. In the tree on the left in Figure 4.7, one takes one differential in $\mathcal{A}$, and one in M; whereas on the right the

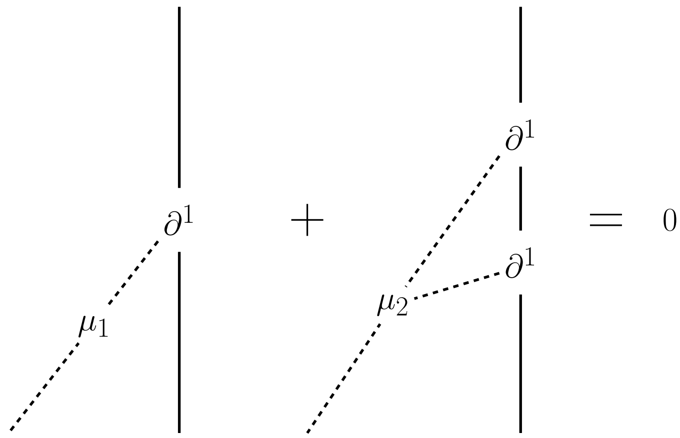


Figure 4.7: *The Type D relation for ${}^{\mathcal{A}}M$, where $\mathcal{A}$ is a DGA. The solid edges represent elements of M, and the dashed edges elements in $\mathcal{A}$.*

differential is taken twice in M.

4.3.1 Gradings and adaptation to a one-manifold

The algebra $\mathcal{A}(n)$ introduced by Ozsváth-Szabó in [49] is a graded algebra, and the Type D structures used in the construction of $\mathcal{C}(D)$ admit similar gradings: namely the half-integer valued Δ-grading, and the multi-grading $\mathcal{S}$.

This multi-grading is not quite the same as the Alexander multi-grading, defined before as the weight of an algebra element. Instead, the multi-grading $\mathcal{S}$ for a Type D structure is a quotient of $\frac{1}{2}\mathbb{Z}^{2n}$ that is determined by the upper-knot diagram. The Alexander multi-grading in the algebra $\mathcal{A}(n)$ takes values in $\frac{1}{2}\mathbb{Z}^{2n}$, so the weight w of an algebra element $a \in \mathcal{A}$ can be thought of as a half-integer valued function on the points $D \cap \{y = \ell\}$. This is denoted by $w(a)$.

An upper knot diagram can be thought of as a one-manifold W with boundary $\partial W = Y$. The zero-manifold Y is the intersection of the upper knot diagram with the level $\{y = \ell\}$. From the relative long exact sequence in cohomology, there is a map

$$H^0\left(Y; \frac{1}{2}\mathbb{Z}\right) \to H^1\left(W, \partial W; \frac{1}{2}\mathbb{Z}\right)$$

given by $y \mapsto d^0(y)$. This provides an action of $H^0(Y)$ on $H^1(W, \partial W)$.

Definition 4.17 *[49, Def. 2.6] A Type D structure ${}^{\mathcal{A}}X$ is called adapted to the one-manifold W if X is graded by $\mathcal{S} = H^1(W, \partial W)$ with the above action. Furthermore, one must have that:*

- *There is an additional* $\frac{1}{2}\mathbb{Z}$*-grading* Δ_X *such that when* $\partial^1(x) \to b \otimes y \in \mathcal{A} \otimes X$*, one has that*

$$\Delta(b) + \Delta_X(y) = \Delta_X(x).$$

- X *is a finite-dimensional vector space over* $\mathbb{F}$*.*

Informally, forcing that all Type D structures are adapted to one-manifolds allows one to consistently assign gradings to each piece — and moreover to each generator of Type D structures and DA bimodules.

The generators of the Type D structures used in the construction of $\mathcal{C}(D)$ are in correspondence with upper Kauffman states. Upper Kauffman states can be assigned half-integer valued gradings from the local contributions at each crossing, as seen in Figure 2.2, as was the case for full Kauffman states of a knot diagram.

Moreover, the other algebraic objects used in the construction of the invariant, such as DA-bimodules, are adapted to their corresponding one-manifolds, see [46, 49].

Since the Type D structures used within this thesis are constructed iteratively as tensor products — see Section 4.5 — all of the Type D structures considered are adapted to a one-manifold by construction. The full detail is presented in [46, Sec. 3.9] and [49, Def. 2.6].

Following [46, Sec. 2.4], a Type D structure over $\mathcal{A}(n)$, ${}^{\mathcal{A}(n)}X$ admits gradings $d \in \mathbb{Z}, s \in S$ such that X splits as the direct sum

$${}^{\mathcal{A}(n)}X = \bigoplus_{d \in Z, s \in S} X_{d,s}.$$

Furthermore, the map ∂^1 for the Type D structure acts as follows:

$$\partial^1 : X_{d,s} \longrightarrow \bigoplus_{\substack{d_0+d_1=d-1 \\ s_0+s_1=s}} \mathcal{A}_{d_0,s_0} \otimes X_{d_1,s_1}.$$

Equivalence of Type D structures

In the literature, Type D structures are sometimes referred to as Type D modules [24]. Moreover, following [24, Rmk. 2.2.28], Type D structures over a differential graded algebra $\mathcal{A}$ give rise to a differential graded category ${}^{\mathcal{A}}_{u}\mathbf{Mod}$. One can suitably define the module maps between Type D structures to be Type D structure homomorphisms if they obey the following relation — see [25, Def. 2.18]

Definition 4.18 *Let (M, ∂_M) and (N, ∂_N) be Type D structures over the same algebra $\mathcal{A}$. Then, if one defines the map $\phi : M \to \mathcal{A} \otimes N$ such that*

$$(\mu_2 \otimes \mathbf{1}_N) \circ (\mathbf{1}_{\mathcal{A}} \otimes \phi) \circ \partial_M + (\mu_2 \otimes \mathbf{1}_N) \circ (\mathbf{1}_{\mathcal{A}} \otimes \partial_N) \circ \phi + (\mu_1 \otimes \mathbf{1}_N) \circ \phi = 0,$$

then ϕ is a Type D structure homomorphism.

Note, that the algebra in question does not change here. From such homomorphisms between Type D structures, one can define a homotopy h between homomorphisms in a similar way to that of chain complexes. Two Type D structures are then thought of as equivalent if the composition of maps between the two type D structures are homotopic to the identity. It is homotopy classes of Type D structures that are then objects of the category above, and homomorphisms give the morphisms.

4.3.2 Visual representation of Type D structures

As an aid to the inductive proofs that will be much featured here, one can visualise Type D structures as directed weighted graphs (possibly with loops), an example of which can be seen in Figure 4.8.

The vertices of such a weighted, directed graph correspond to the elements of the Type D structure ${}^{\mathcal{A}}M = (M, \partial^1)$. Then, if there is a non-zero map from $x \in M$ such that $a \otimes y$ appears in the result $\partial^1(x)$, one would draw a directed edge from the vertex corresponding to x to the vertex corresponding to y. The weight on this edge would thus correspond to the algebra element $a \in \mathcal{A}$.

At no point has the restriction been made that $\partial^1(x)$ is a pure element of $\mathcal{A} \otimes M$. Sometimes, this may be the case, which will be denoted by $\partial^1(x) = a \otimes y$ for such elements $x, y \in M$, $a \in \mathcal{A}$. However, if one has that

$$\partial^1(x) = \sum (a_i \otimes y_i) \in \mathcal{A} \otimes M,$$

then each summand will be denoted by either $\partial^1(x) \to a_i \otimes y_i$, or $a_i \otimes y_i \in \partial^1(x)$. This is notationally easier, and also highlights the link between directed edges in the graph and non-zero maps in the Type D structure.

All of the Type D structures studied in the computation of invariants for three strand pretzel knots are *standard type D structures*, as defined by [49, Sec. 2.8].

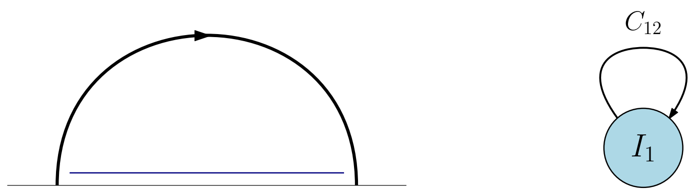


Figure 4.8: *Example of a directed graph corresponding to the Type D structure for the unique global maximum. Note, that such a directed graph can have loops.*

Definition 4.19 *For $\mathcal{A}(n)$ the DGA defined above, and ${}^{\mathcal{A}(n)}X$ a Type D structure, ${}^{\mathcal{A}(n)}X$ is said to be standard if it is adapted to a one-manifold with boundary, and for every $x \in X$, one has that*

$$\partial^1(x) = \left(\sum_{pq \in M} C_{pq}\right) \otimes x + \epsilon(x),$$

where M is the matching induced by the upper knot diagram on which the algebra depends, and $\epsilon(x)$ is a sum of elements $b \otimes y$, where b is an element of $\mathcal{B}_0 \subset \mathcal{A}(n)$, and $y \in X$.

A Type D structure over $\mathcal{A}(n)$ is then standard if in the associated graph every vertex has n self-loops, weighted by the n matching elements, and all of the other directed edges from that vertex are weighted by non-matching elements.

For simplicity, the graphs corresponding to Type D structures shown here will often omit the self-loops, since these would decorate every vertex, and by construction all of the Type D structures within the construction by Ozsváth-Szabó are standard. This is proven in [49, Prop. 8.3].

4.3.3 Type D structures of upper knot diagrams

The generators of the Type D structure for upper knot diagrams correspond to *upper Kauffman states*. The correspondence between Kauffman states for a special knot diagram and intersection points in the Heegaard diagram constructed from the projection [36, Sec. 2.2] gives a similar correspondence between upper Kauffman states and intersection points in a suitable partial Heegaard diagram associated to the upper knot diagram.

Definition 4.20 *For an upper knot diagram, regions are either closed, and so bounded by arcs of the upper knot diagram, or are not closed, so are bounded by arcs of the upper knot diagram and the horizontal level. The upper knot diagram intersects this horizontal level at $2n$ points, enclosing $2n-1$ regions.*

An upper Kauffman state $I_p \cdot X$ then corresponds to decorating a crossing in each of the closed regions and $n-1$ of the non-closed regions, as in the definition of a Kauffman state presented in Definition 2.1. The n remaining non-closed regions which are undecorated then determine an n-element subset of $\{1, 2, \cdots, 2n-1\}$, denoted p. The idempotent associated to the upper Kauffman state X is then denoted I_p.

In this way, if $i \in p$, then the decoration in this region must be lower in the knot diagram. It is worth noting however that not all upper Kauffman states extend to Kauffman states of the full knot diagram. However, it is true that all Kauffman states for a full knot diagram do restrict to upper Kauffman states at every level y. An example of an upper Kauffman state for an upper knot diagram of the three-strand trefoil is presented in Figure 4.9.

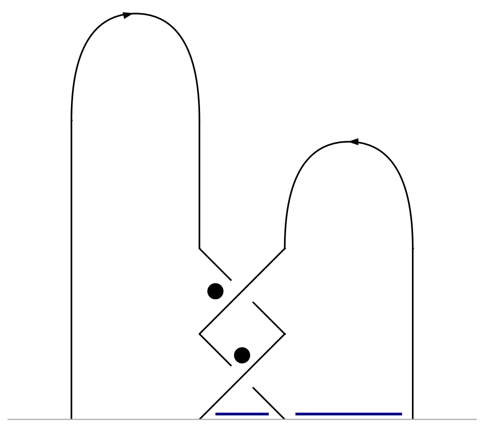


Figure 4.9: *An upper Kauffman state $I_{23} \cdot X$ for an upper knot diagram of the right-handed trefoil. Completing any upper Kauffman state to a full Kauffman state, the distinguished edge in the special knot diagram is by convention the global minimum. Note here, that the undecorated non-closed regions are to the right of the second and third intersection points with the horizontal level. Hence, the associated idempotent to this upper Kauffman state is I_{23}.*

The idempotents associated to each generator are important for dictating the possible position of lower Kauffman states. They can also restrict the possible maps within the Type D structure.

Lemma 4.21 *In a Type D structure ${}^{\mathcal{A}}M$, all non-zero maps $\partial^1 : {}^{\mathcal{A}}M \to \mathcal{A} \otimes {}^{\mathcal{A}}M$ are such that*

$$\partial^1(I_p \cdot X) \to [I_p, I_q, w_a] \otimes I_q \cdot Y = I_p \cdot a \cdot I_q \otimes I_q \cdot Y.$$

Note, the proof of this statement is obvious, as the tensor product $\mathcal{A} \otimes {}^{\mathcal{A}}M$ is taken over the ring of idempotents $\mathcal{I}(\mathcal{A})$. Hence, the outgoing idempotent of the algebra element, and the associated idempotent to the Type D generator must match.

Example 4.22 *As an example, assume one has a Type D structure with elements* $I_{134} \cdot X$ *and* $I_{135} \cdot Y$*. The only half-integer weight algebra element between these idempotents is* $I_{134} \cdot R_5 \cdot I_{135}$*. Then, if there is a map in the Type D structure between* X *and* Y*, one has* $\partial^1(X) \to I_{134} \cdot m_1 \cdot R_5 \cdot m_2 \cdot I_{135} \otimes Y$*, where* m_1 *and* m_2 *are integer weight elements in* $\mathcal{A}$*.*

Also, if there is no non-zero algebra element between the two idempotents, then there is no map between generators with these idempotents. As an example, since $R_4 \cdot R_5 = 0 \in \mathcal{A}$*, one has that there is no possible arrow between* $I_{123} \cdot X$ *and* $I_{125} \cdot Y$*.*

4.3.4 Simplification of Type D structures

One of the strengths of the computer implementation of the determination of the invariant $\mathcal{C}(D)$ for a special knot diagram D in [47] is the fact that the Type D structure can be simplified at every step (Morse event) to yield a (filtered) homotopy equivalent Type D structure with fewer generators.

Following the language of [49, Sec. 13.2], this simplification is the 'contraction of arrows' in the Type D structure. For ${}^{\mathcal{A}}X$ a Type D structure, the aim is to yield a Type D structure ${}^{\mathcal{A}}Y$ such that for every $y \in Y$, $\partial^1(y)$ has no terms with an algebra element of non-zero weight. Recall, in $\mathcal{A}$, the only non-zero weight elements are the idempotent elements, which sum together to form the unital element $\mathbf{1}$. In [49, Def. 3.2], Ozsváth-Szabó define the following.

Definition 4.23 *A Type D structure* ${}^{\mathcal{A}}X$ *is defined to be small if for every* $x, y \in X$ *with* $\Delta(x) = \Delta(y) + 1$*, one has that the* $\mathcal{A} \otimes y$ *coefficient of* $\partial^1(x)$ *is zero.*

Ozsváth-Szabó then prove in [49, Lem. 13.3] that any standard, Δ-graded, finitely generated Type D structure is homotopy equivalent to a small, finitely generated, standard, Δ-graded Type D structure over the same algebra.

The lemma, and associated proof, are very similar to the zig-zag lemma as presented in [63, Sec. 3.1] and the edge reduction algorithm presented in [21, Sec. 2.6]. More generally, [63, Thm. 5] proves the following, when $\mathcal{A}$ has some unital element $\mathbf{1}$.

Theorem 4.24 (Zig-Zag Lemma) *Let* $\mathcal{G}$ *be a set of generators for the Type D structure*

${}^{\mathcal{A}}X$. Then for any $x \in \mathcal{G}$, one can expand $\partial^1(x)$ as

$$\partial^1(x) = \sum_{y \in \mathcal{G}} c_{xy} \otimes y,$$

for algebra elements $c_{xy} \in \mathcal{A}$. An algebra element $e \in \mathcal{A}$ is invertible if there exists some element f such that $f \cdot e = e \cdot f = \mathbf{1}$. Denote the inverse of algebra element e by e^{-1}.

If $a, b \in \mathcal{G}$ are such that $c_{ab} \in \mathcal{A}$ is invertible, the Type D structure X' generated by $\mathcal{G}' = \mathcal{G} \backslash \{a, b\}$ is homotopy equivalent to X, with map $\partial^1_{X'}$ defined by

$$\partial^1_{X'}(x) = \sum_{y \in \mathcal{G}'} \left(c_{xy} + c_{xb} c_{ab}^{-1} c_{ay}\right) \otimes y.$$

Note, that the only invertible elements in $\mathcal{A}(n)$ are the idempotent elements, and so the simplification of Type D structures to 'small' Type D structures is essentially an implementation of the zig-zag lemma.

The zig-zag lemma is a well-known method for simplifying chain complexes by removing an acyclic pair. As stated in Theorem 4.24 presented above, there is a similar simplification of Type D structures. Moreover, [63, Sec. 3.1] continues to prove that similar simplifications exist for DA-bimodules and A_∞-modules.

However, a crucial method in this thesis is the determination of Type D structures through inductive proofs, and it is thus helpful to consider how a Type D structure changes upon extending the upper knot diagram. The correspondence between upper Kauffman states and generators of the Type D structure is useful in this, and so in the inductive proofs of Type D structures for three strand pretzel knots presented in Chapter 5, the Type D structures are not simplified in the intermediate stages.

4.4 DA-bimodules

The algebraic objects associated to the Morse events in a special knot diagram are DA-bimodules over the differential graded algebras. Such Morse events are crossings, maxima and minima, and by construction of the special knot diagram necessarily occur at finitely many distinct values y_i.

The specific examples of these DA-bimodules as defined by [49] will be given later in this section, but first the general definition of DA-bimodules over a DGA is presented following [46, Sec. 2.6] and [24]. As with Type D structures, such bimodules can be defined over a general A_∞-algebra, but since the algebra $\mathcal{A}(n)$ is a differential graded algebra, one can specialise and define a DA-bimodule as follows.

Definition 4.25 *[24, Def. 2.2.43] Let $\mathcal{A}$ and $\mathcal{B}$ be differential graded algebras over the rings $\mathbf{k}, \mathbf{j}$ respectively. A type DA-bimodule ${}^{\mathcal{A}}\mathcal{M}_{\mathcal{B}}$ is a graded $(\mathbf{k}, \mathbf{j})$-bimodule, with $(\mathbf{k}, \mathbf{j})$-linear maps*

$$\delta^1_{1+j} : \mathcal{M} \otimes \mathcal{B}^{\otimes j} \to \mathcal{A} \otimes \mathcal{M}.$$

The tensor products are taken over the rings of idempotents in each of the A_∞-algebras. These maps also satisfy similar compatibility conditions to A_∞-algebras and Type D structures.

To define these compatibility relations, first note that one can compose δ^1_{1+j} and δ^1_{1+k} maps by separating the algebra inputs. One can define the maps

$$\Delta_n : \mathcal{A}^n \to \sum_{i+j=n} \mathcal{A}^i \otimes \mathcal{A}^j.$$

Using this, one can define $\Delta = \sum_n \Delta_n$.

Then, define $\delta^1 = \sum_{j \geq 0} \delta^1_{1+j}$, the sum of those δ^1 maps taking any number of algebra inputs. Define the maps δ^k for $k > 1$ inductively, so

$$\delta^{i+1} = \left(\mathbf{1}_{\mathcal{A}^{\otimes i}} \otimes \delta^1\right) \circ \left(\delta^i \otimes \mathbf{1}_{\mathcal{B}^{\otimes}}\right) \circ \left(\mathbf{1}_{\mathcal{M}} \otimes \Delta\right).$$

Figure 4.10 displays this definition of δ^{i+1} from the composition of δ^i and δ^1 pictorially, which is perhaps easier to understand. Note, the base case for the induction is the δ^1 map defined above. For each $j \geq 0$, the maps must satisfy the following compatibility conditions.

$$\begin{aligned}
0 = & \left(\mu_1^{\mathcal{A}} \otimes \mathbf{1}_{\mathcal{M}}\right) \circ \delta^1_{1+j}(x \otimes a_1 \otimes \cdots \otimes a_j) \\
& + \sum_{k=1}^{j} \delta^1_{1+j}\left(x \otimes a_1 \otimes \cdots \otimes a_{k-1} \otimes \mu_1^{\mathcal{B}}(a_k) \otimes a_{k+1} \otimes \cdots \otimes a_j\right) \\
& + \sum_{k=1}^{j-1} \delta^1_j\left(x \otimes a_1 \otimes \cdots \otimes a_{k-1} \otimes \mu_2^{\mathcal{B}}(a_k \otimes a_{k+1}) \otimes a_{k+2} \otimes a_j\right) \\
& + \left(\mu_2^{\mathcal{A}} \otimes \mathbf{1}_{\mathcal{M}}\right) \circ \delta^2_{1+j} \circ \Delta_j.
\end{aligned}$$

Once more, this definition can be intuitively thought of as forcing that the sum of the possible ways to 'differentiate' twice is zero.

- In the first part of this sum, one differentiates once in $\mathcal{A}$, and once in $\mathcal{M}$.

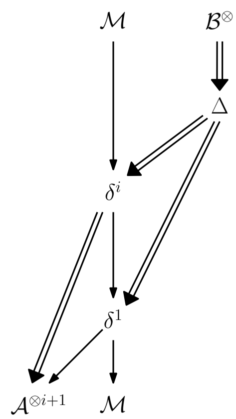


Figure 4.10: *Pictorial representation of the maps* $\delta^{i+1} : M \otimes \mathcal{B}^{\otimes} \to \mathcal{A}^{\otimes i+1} \otimes M$.

- In the second and third parts of this sum, one differentiates once in $\mathcal{B}$, and once in $\mathcal{M}$.
- In the fourth part of the sum, one differentiates twice in $\mathcal{M}$.

The DA-bimodule relation is displayed in Figure 4.11 for $\delta^1_{1+2} : M \otimes \mathcal{B} \otimes \mathcal{B} \to \mathcal{A} \otimes M$. Elements of a DA-bimodule then have an associated incoming and outgoing idempotent. Unlike with algebra elements, such as $I_{123} \cdot R_4 \cdot I_{124}$, whose incoming idempotent is displayed on the left, following the notation for a bimodule as ${}^{\mathcal{A}}\mathcal{M}_{\mathcal{B}}$, the incoming idempotent (associated to $\mathcal{B}$) is presented on the right, and the outgoing (associated to $\mathcal{A}$) on the left. An example might be $I_{12} \cdot X \cdot I_{134}$, where $I_{12} \in \mathcal{A}(2) = \mathcal{A}$, and $I_{134} \in \mathcal{A}(3) = \mathcal{B}$.

Remark 4.26 *As described in Definition 4.25, when defining the map* $\delta^1 = \sum_k \delta^1_k$ *in a* DA*-bimodule, all tensor products are taken in the ring of idempotents for the algebra in question.*

Consequently, if there is some map $\delta^1_{1+k} : {}^{\mathcal{A}}\mathcal{M}_{\mathcal{B}} \otimes \mathcal{B}^{\otimes k} \to \mathcal{A} \otimes {}^{\mathcal{A}}\mathcal{M}_{\mathcal{B}}$*, such that there are bimodule elements* $X, Y \in \mathcal{M}$*, and algebra elements* $b_i \in \mathcal{B}$ *and* $a \in \mathcal{A}$ *with*

$$\delta^1_{1+k}(X, b_1, b_2, \ldots, b_k) \to a \otimes Y,$$

then in order for this map to be non-zero one must have the following restrictions upon idempotents.

- *The element* $X \in {}^{\mathcal{A}}\mathcal{M}_{\mathcal{B}}$ *with associated idempotents* $I_x \cdot X \cdot I_{x_0}$*. Hence,* $I_x \in \mathcal{I}(\mathcal{A})$*, and* $I_{x_0} \in \mathcal{I}(\mathcal{B})$*.*
- *Idempotents* $I_{x_{i-1}} \cdot b_i \cdot I_{x_i}$ *for each* $b_i \in \mathcal{B}$ *in the sequence.*

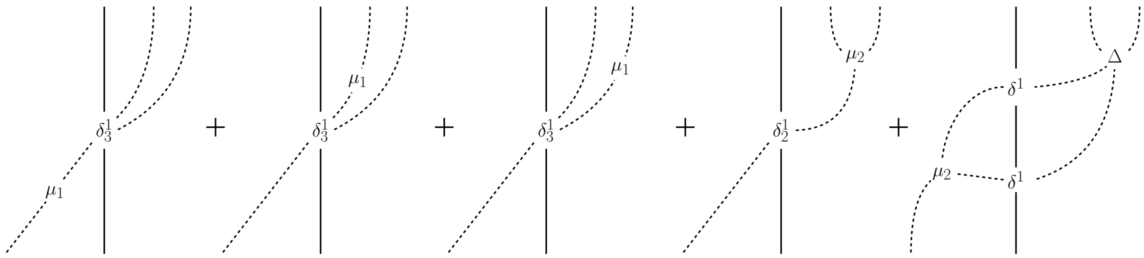


Figure 4.11: *Figure displaying the DA-bimodule compatibility condition of the map δ^1_{1+2}. If this expression sums to zero, and the same is true for all other values of j, then the bimodule ${}^{\mathcal{A}}\mathcal{M}_{\mathcal{B}}$ is said to be a DA-bimodule. Note, the last term in the sum is the sum of all possible ways to divide two algebraic inputs as input to $\delta^1 \circ \delta^1$.*

- $I_x \cdot a \cdot I_y$. *Note, that the idempotent I_x is the same as the outgoing idempotent of $X \in {}^{\mathcal{A}}\mathcal{M}_{\mathcal{B}}$.*
- $I_y \cdot Y \cdot I_{x_k}$, *where the incoming idempotent for the element Y is the same as that associated to the element b_k.*

4.4.1 DA-bimodules associated to Morse events

For DA-bimodules as defined by [49] in their construction of the invariant $\mathcal{C}(D)$, every Morse event has an associated bimodule ${}^{\mathcal{B}}\mathcal{M}_{\mathcal{A}}$. Here, $\mathcal{B}$ is the outgoing algebra $\mathcal{A}(n')$ associated to the bottom of the Morse event (the lower value of y in the special knot diagram), and $\mathcal{A}$ is the incoming algebra $\mathcal{A}(n)$ associated to the top of the Morse event.

Note, that if the event is a maximum or minimum, one would have $n' = n+1$ or $n' = n-1$ respectively. Whereas if the event is a crossing, one would have $n = n'$. However, in nearly all cases, the incoming and outgoing differential graded algebras associated to a DA-bimodule are different, since any crossing, maximum or minimum changes the upper knot diagram, and so the matching. The algebra $\mathcal{A}(n)$ has elements C_{pq} associated to arcs in the upper knot diagram matching arc p with arc q. More properly, one should annotate each algebra $\mathcal{A}_M(n)$ in order to demonstrate that there is a dependence upon the matching, however this is hopefully clear from context.

The only case in which the incoming and outgoing algebra of a DA-bimodule associated to a Morse event are equal is when there is a crossing between strands i and $i+1$, and there is an element $\{i, i+1\} \in M$, for M the associated matching of the incoming algebra.

Although the definitions for the bimodules associated to crossings and maxima are exactly as presented in [49], the definition of the bimodule associated to a minimum in [49, Sec. 7.2] has been adapted to the case of a three-strand pretzel knot. More specifically, Ozsváth-Szabó only present the explicit definition of the bimodule for a minimum between strands one and two. Since every special knot diagram admits an isotopy such that all minima always occur between these strands, at the expense of introducing additional crossings, this is sufficient for the construction. However, it is algebraically simpler not to introduce additional crossings that may complicate the determination of the Type D structure at any level of the upper knot diagram, and so a specialisation to the case of three strand pretzel knots is presented here in Section 4.6.2.

Exceptionally, the global minimum of the special knot diagram contains by convention the distinguished edge used in the determination of any Kauffman state. This minimum will have an A_∞-module associated to it, rather than a DA-bimodule. Moreover, as one can determine from the interpretation of idempotent states within an upper knot diagram as presented in Section 4.2.2, the idempotents associated with the global minimum differ from those of local minima.

In particular, idempotent elements I_x that are an associated idempotent to a generator of a Type D structure or DA-bimodule indicate that there is a decoration of a Kauffman state in the region above when $i \notin x$. This must necessarily be true for the incoming idempotents of all generators of a non-global minima between strands i and $i+1$, because the local diagram associated to a non-global minimum has no positions that may be marked by a Kauffman state[2]. This is explained more fully in terms of preferred idempotents in the construction of the specific bimodule.

However this property of the associated incoming idempotent is not true for the global minimum: since the distinguished arc is by construction the global minimum, and in the construction of [36] there can be no decoration placed in the region incident to the global minimum. As such, one would require — since the global minimum is between strands 1 and 2 — that 1 is in the incoming idempotent for any generator of the object associated to the global minimum, since this implies there is no Kauffman state in this region above this horizontal level.

As noted, the generators of a DA-bimodule have an associated incoming and outgoing idempotent, and the ring of idempotents for an algebra is the ring over which tensor

[2] Recall, Kauffman states mark one of the quadrants at each crossing in unoccupied regions.

products are defined. The generators of the DA-bimodules associated to crossings, maxima and minima are all in one-to-one correspondence with the valid partial Kauffman states for the Morse event in the special knot diagram, which have a pair of idempotents associated to them.

Definition 4.27 *Let D be a special knot diagram for an oriented knot K, such that there is only a single Morse event between $y_1 < y_2$. Then the associated bimodule ${}^{\mathcal{A}}\mathcal{M}_{\mathcal{B}}$ to this Morse event has generators that are in one-to-one correspondence with partial Kauffman states for this subset of the special knot diagram D: that is decorations of the subset of the knot diagram agreeing with the interpretation of the idempotents as described in Section 4.2.2.*

A partial Kauffman state $I_p \cdot X \cdot I_q \in {}^{\mathcal{A}}\mathcal{M}_{\mathcal{B}}$ has an associated incoming idempotent $I_q \in \mathcal{I}(\mathcal{B})$, and an outgoing idempotent $I_p \in \mathcal{I}(\mathcal{A})$. If $\mathcal{M}$ is the bimodule associated to any crossing that is not between strands i and $i+1$, then when the Kauffman state decoration at this crossing is placed in a region i incident with the line y_2 (the upper horizontal level), one must then have that $i \in I_q$ but $i \notin I_p$.

If M is the bimodule associated to a maximum or minimum, then the generators of the bimodule correspond to partial Kauffman states with no decorations, simply all possible valid assignments of incoming and outgoing idempotent.

Since DA-bimodules are associated to Morse events, and such Morse events can be thought of as one-manifolds with boundary, there is a corresponding notion of a DA-bimodule being adapted to the underlying one-manifold. This is similar to the adaptation of a Type D structure to a one-manifold as presented in Definition 4.17. Informally, being adapted to a one-manifold ensures consistency in gradings in the construction, particularly under taking tensor products of these objects, as outlined in Section 4.5.

Definition 4.28 *[49, Def. 2.6] From the construction of [49], to every Morse event there is an underlying one-manifold W, to which a DA-bimodule is associated. The boundary of W can be partitioned as $\partial W = Y_1 \sqcup Y_2$, where Y_1 is the finite collection of points at the 'top' boundary of W, and Y_2 the collection of points at the bottom.*

The Alexander multi-grading in each algebra $\mathcal{A}_1$ and $\mathcal{A}_2$ associated to the level sets the top and bottom of W respectively can be thought of as half-integer valued functions on Y_1 and Y_2 — i.e. taking values in $H^0(Y_i;\mathbb{Q})$. From the relative long exact sequence, there *is a corresponding action* $H^0(Y_1;\mathbb{Q}) \oplus H^0(Y_2;\mathbb{Q}) \to H^1(W,\partial W;\mathbb{Q})$ *given by* $(y_1,y_2) \mapsto -d^0(y_1)+d^0(y_2)$.

The DA-bimodule ${}^{\mathcal{A}_1}M_{\mathcal{A}_2}$ *is then defined to be adapted to* W *if:*

- M *is multi-graded by* $H^1(W,\partial W)$ *as described above, with additional* $\mathbb{Q}$*-grading* Δ_M*, compatible with the* Δ*-grading on the algebra. Namely, if there is a map* $\delta_{1+k}(m,a_1,\ldots,a_k) \to b \otimes y$*, then*

$$\Delta(b)+\Delta_M(y) = \Delta_M(m) - k + 2 + \sum_{i=1}^{k} \Delta(a_i).$$

- M *is a finite dimensional vector space over* $\mathbb{F}$.

This condition of being adapted to a one-manifold will always be satisfied for the DA-bimodules used in the construction of the invariant $\mathcal{C}(D)$ as described in [49]. For the DA-bimodules associated to crossings, this condition means that generators of the DA-bimodule have an associated $\left(\frac{1}{4}\mathbb{Z}\right)^{2n}$-valued grading, where $2n$ is the number of strands in the local diagram of the crossing.

4.4.2 DA-bimodules associated to crossings

As described in Definition 4.27, the generators of bimodules associated to crossings correspond to partial Kauffman states for each crossing. However, although there are only four marked positions at each crossing (corresponding to the four cardinal directions N, E, S, W), there can be more than four generators of the DA-bimodule, since there could be two states with the same cardinal direction but different idempotents.

For a positive crossing between strands i and $i+1$, the associated DA-bimodule is $\mathcal{P}^i$, and for a negative crossing between the same strands the DA-bimodule is $\mathcal{N}^i$. The generators of the DA-bimodules $\mathcal{P}^i$ and $\mathcal{N}^i$ are in one to one correspondence, and indeed the bimodules are said to be 'opposite', following [49, Def. 3.5].

Definition 4.29 *If* ${}^{\mathcal{A}}\mathcal{M}_{\mathcal{B}}$ *is a DA-bimodule, with* $\mathcal{A}$ *and* $\mathcal{B}$ *both examples of the DGA* $\mathcal{A}(n)$ *defined in Definition 4.12, then there is an opposite bimodule* ${}^{\mathcal{C}}\mathcal{N}_{\mathcal{D}}$*, with the same generating set as* $\mathcal{M}$.

For the differential graded algebra $\mathcal{A}(n)$*, following [46, Sec. 5.5], define the map* $o : \mathcal{A}(n) \to$

$\mathcal{A}(n)$ by

$$\begin{aligned} o(R_i) &= L_i & o(L_i) &= R_i \\ o(U_i) &= U_i & o(C_{ij}) &= C_{ij} \\ o(a \cdot b) &= o(b) \cdot o(a) & o(I_x) &= I_x. \end{aligned}$$

Then, the opposite bimodule ${}^{\mathcal{C}}\mathcal{N}_{\mathcal{D}}$ is such that $\mathcal{C} = \mathcal{A}$, $\mathcal{D} = \mathcal{B}$, and with maps $\delta^1_{i+1} : \mathcal{N} \otimes \mathcal{B}^{\otimes i} \to \mathcal{A} \otimes \mathcal{N}$ as follows.

- *If in $\mathcal{M}$ there was some map δ^1_1 such that $\delta^1_1(X) = b \otimes Y$, then in $\mathcal{N}$ there is a map $\delta^1_1(Y) = o(b) \otimes X$.*
- *More generally, if there is a map $\delta^1_{i+1}(X, a_1, \ldots, a_i) = b \otimes Y$ in $\mathcal{M}$, there is a corresponding map $\delta^1_{i+1}(Y, o(a_i), \ldots, o(a_1)) = o(b) \otimes X$ in $\mathcal{N}$.*

Since the bimodules $\mathcal{P}^i$ and $\mathcal{N}^i$ are opposites of each other, defining one fully and applying the definition of opposite bimodules over $\mathcal{A}(n)$ is sufficient to define the other.

Generators of the bimodules $\mathcal{P}^i$ and $\mathcal{N}^i$

Corresponding to the four cardinal directions, one can separate the generators of a bimodule corresponding to a crossing into four types. In [49, Sec. 3.2], Ozsváth-Szabó then define these types as follows.

Definition 4.30

$$N = \sum_{i \in I_x} I_x \cdot N \cdot I_x \qquad E = \sum_{\substack{i+1 \in I_x \\ i \notin I_x}} I_{x \setminus \{i+1\} \cup i} \cdot E \cdot I_x.$$

$$S = \sum_{i \notin I_x} I_x \cdot S \cdot I_x \qquad W = \sum_{\substack{i-1 \in I_x \\ i \notin I_x}} I_{x \setminus \{i-1\} \cup i} \cdot W \cdot I_x$$

When taking a tensor product of the DA-bimodule associated to a crossing with a Type D structure, as will be defined in Section 4.5, only one idempotent representative from a class will be picked out by the tensor product. This is because the generators of a Type D structure corresponding to an upper knot diagram have a single associated idempotent. However, when defining the maps δ^1 in a DA-bimodule for a crossing, algebra inputs to the map will be considered that may not be valid for all possible idempotents.

Following Definition 4.28, one can assign a multi-grading to all generators of $\mathcal{P}^i$ and $\mathcal{N}^i$.

Definition 4.31 *With the generators N, E, S and W for the DA-bimodule $\mathcal{P}^i$, as outlined in Definition 4.30, the $\left(\frac{1}{4}\mathbb{Z}\right)^{2n}$-valued grading, gr, of each generator is as follows:*

$$gr(N) = \frac{1}{4}(e_i + e_{i+1}) \qquad gr(E) = \frac{1}{4}(-e_i + e_{i+1})$$
$$gr(S) = \frac{1}{4}(-e_i - e_{i+1}) \qquad gr(W) = \frac{1}{4}(e_i - e_{i+1}).$$

For the same generators in $\mathcal{N}^i$, the corresponding gradings are (-1) times the grading for $\mathcal{P}^i$, as outlined by [46, Sec. 4.4].

Maps in the bimodule $\mathcal{P}^i$

The maps in a DA-bimodule are most easily described using a weighted, directed graph, similar to the weighted directed graph for Type D structures as defined in Section 4.3.2. In particular, the vertices of such a graph correspond to generators of the DA-bimodule, and the weights correspond to algebraic inputs and outputs.

More specifically, if there is a map $\delta^1_{1+k}(X, b_1, b_2, \ldots, b_k) \to a \otimes Y$ in the DA-bimodule ${}^{\mathcal{A}}\mathcal{M}_{\mathcal{B}}$, then from the vertex corresponding to X to the vertex corresponding to Y there is a directed edge with weight $a \otimes (b_1, b_2, \ldots, b_k)$. Edges corresponding to maps δ^1_1 without any algebraic input may also be highlighted by a dashed line.

Correspondingly, the graph depicted in Figure 4.12 defines the maps δ^1_1 and δ^1_2 between generators of different types in the bimodule $\mathcal{P}^i$, following [49, Sec. 3.2].

Between generators of the same type, one also has the following maps, where X denotes any cardinal generator.

- $\delta^1_2(X, a \cdot b) = a \cdot \delta^1_2(X, b)$ for algebra element a with weight outside of $span\{e_i, e_{i+1}\}$. Algebra elements with weight outside the crossing region are said to commute with the map δ^1_2.
- $\delta^1_2(N, L_i L_{i+1}) = L_i L_{i+1} \otimes N$, and $\delta^1_2(N, R_{i+1} R_i) = R_{i+1} R_i \otimes N$.
- $\delta^1_2(X, U_i \cdot a) = U_{i+1} \cdot \delta^1_2(X, a)$ when U_i and U_{i+1} are non-zero following the idempotent conditions described in Remark 4.26, and 0 otherwise.
- $\delta^1_2(X, U_{i+1} \cdot a) = U_i \cdot \delta^1_2(X, a)$ likewise following Remark 4.26.
- Similarly, with the idempotents as described in Remark 4.26, one has $\delta^1_2(X, U_i U_{i+1} \cdot a) = U_i U_{i+1} \cdot \delta^1_2(X, a)$.

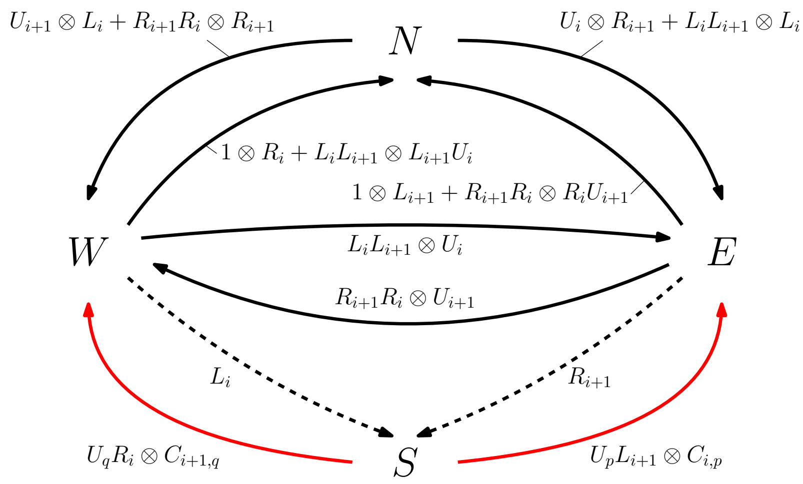


Figure 4.12: *The maps δ_1^1 and δ_2^1 in the DA-bimodule $\mathcal{P}^i$, following the definition presented in [49, Sec. 3.2]. The maps highlighted in red feature the matching elements $C_{i,p}$ and $C_{i+1,q}$. These are only non-zero when i and $i+1$ are not matched in the incoming algebra associated to $\mathcal{P}^i$.*

The last three points correspond to the crossing switching the role of i and $i+1$, which is hopefully clarified by the interpretation of DA-bimodules and their associated maps as regions in partial Heegaard diagrams, as described in Section 4.4.4.

The only remaining non-zero maps δ_k^1 in the DA-bimodule $\mathcal{P}^i$ are the maps $\delta_3^1(S,-,-)$. Those maps δ_3^1 used in the calculations within this thesis are:

$$\delta_3^1(S, R_i, R_{i+1}) = R_i \otimes E.$$

$$\delta_3^1(S, U_i, U_{i+1}) = L_{i+1}U_i \otimes E.$$

$$\delta_3^1(S, L_{i+1}, L_i) = L_{i+1} \otimes W.$$

$$\delta_3^1(S, U_{i+1}, U_i) = R_iU_{i+1} \otimes W.$$

$$\delta_3^1(S, L_{i+1}, U_i) = L_{i+1} \otimes N.$$

$$\delta_3^1(S, R_i, U_{i+1}) = R_i \otimes N.$$

The list of all possible maps $\delta_3^1(S,-,-)$ in $\mathcal{P}^i$ is slightly more extensive, and the full description can be found in [49, p. 21]. Within the calculations presented in Chapter 5, only the above maps are ever used when the tensor products between the Type D structures associated to three-strand pretzel knots and $\mathcal{P}^i$ are taken. With the complete list, one can verify in every calculation that the required idempotents and algebra elements for the other maps are not found in the subject Type D structures.

For the opposite bimodule $\mathcal{N}^i$, as described in Definition 4.29 the DA-bimodule maps can be completely recovered from the definition of the bimodule $\mathcal{P}^i$. Intuitively, using the directed graph representation of bimodule maps as presented in Figure 4.12, to find the maps for the opposite bimodule, reverse the direction of all arrows, swap L_j for R_j and R_j for L_j, and reverse the order of any terms in parentheses. For example, in $\mathcal{N}^i$, there is a non-zero map

$$\partial_3^1(N, U_{i+1}, L_i) = L_i \otimes S,$$

corresponding to the last bullet point in the list above.

The DA-bimodules $\mathcal{P}^i$ and $\mathcal{N}^i$ are as defined above, but the proof that the maps δ_1^1, δ_2^1 and δ_3^1 satisfy the structure relations presented in Definition 4.25 is omitted here. For the bimodule $\mathcal{P}^i$, this is proven in [49, Prop. 3.3], and the fact that $\mathcal{N}^i$ is a DA-bimodule follows from the fact that an opposite bimodule to a DA-bimodule is a DA-bimodule, see [46, Prop. 5.15].

4.4.3 DA-bimodules associated to maxima

In [49, Sec. 5.2], Ozsváth-Szabó define the DA-bimodule Ω^i corresponding to the Morse event of a maximum introduced to the left of strand i in the special knot diagram. As remarked in Section 4.4.1, the incoming and outgoing algebra for this bimodule are different: the incoming algebra is $\mathcal{A}(n)$ for some $n \in \mathbb{N}$, and the outgoing algebra is $\mathcal{A}(n+1)$. By necessity, since there is an arc between the new strand i and new strand $i+1$, the matching element $C_{i,i+1}$ must be a matching element in $\mathcal{A}(n+1)$.

A slight special case is that of the unique global maximum, ${}^{\mathcal{A}(1)}t\Omega^1$. Here, t denotes the fact that this is the 'terminal' maximum. Since there is no incoming algebra in this case (or, alternatively, the empty algebra), to the global maximum one associates a Type D structure with a single generator $C_{12} \in \mathcal{A}(1)$. This Type D structure is presented as a weighted directed graph in Figure 4.8.

The definition of the DA-bimodule ${}^{\mathcal{A}_2(n+1)}\Omega^i_{\mathcal{A}_1(n)}$ is exactly as is presented in [49, Sec. 5.2]. One can specialise the construction to when the maximum introduced gives either the left-most or the right-most strands, however the DA-bimodule is defined generally, and the truncation of the idempotents to the case of knots gives the appropriate simplification.

Motivated by the interpretation of idempotents in terms of possible positions of marked points in upper Kauffman states, the generators of the DA-bimodule Ω^i correspond to idempotent pairs, with outgoing idempotent in $\mathcal{I}(\mathcal{A}_2)$ and incoming idempotent in $\mathcal{I}(\mathcal{A}_1)$.

More specifically, the generators of the DA-bimodule ${}^{\mathcal{A}_2}\Omega^i_{\mathcal{A}_1}$ correspond to compatible idempotent pairs where the outgoing idempotent is said to be *allowed*.

Definition 4.32 *For y an n-element subset of $\{1, 2, \cdots, 2n-1\}$ for some n, the idempotent $I_y \in \mathcal{I}(\mathcal{A}_2)$ is defined to be an allowed idempotent if $i \in y$ and $|\{i-1, i+1\} \cap y| \leq 1$.*

Allowed idempotents are then separated into the following three types, based upon the intersection of y with the set $\{i-1, i+1\}$.

- *I_y is of type X if $y \cap \{i-1, i, i+1\} = \{i-1, i\}$.*
- *I_y is of type Y if $y \cap \{i-1, i, i+1\} = \{i, i+1\}$.*
- *I_y is of type Z if $y \cap \{i-1, i, i+1\} = \{i\}$.*

Following [49, Sec. 5.2], one can find a map from idempotents in $\mathcal{A}_1 = \mathcal{A}(n)$ to idempotents in $\mathcal{A}_2 = \mathcal{A}(n+1)$. In ${}^{\mathcal{A}_2}\Omega^i_{\mathcal{A}_1}$, this is a map $\phi_i : \{1, 2, \ldots, 2n\} \to \{1, 2, \ldots, 2n+2\}$, defined by

$$\phi_i(j) = \begin{cases} j & \text{if } j \leq i-1, \\ j+2 & \text{if } j \geq i. \end{cases}$$

Note that the map ϕ_i is not surjective, since the elements $\{i, i+1\} \cap \mathrm{im}(\phi_i) = \emptyset$. Using the map ϕ_i, one can construct a map from *allowed* idempotents in $\mathcal{A}_2 = \mathcal{A}(n+1)$ to idempotents in $\mathcal{A}_1$. Define this map ψ as

$$\psi(x) = \begin{cases} \phi_i^{-1}(x) & \text{if } i+1 \notin x, \\ \phi_i^{-1}(x) \cup \{i-1\} & \text{if } i+1 \in x. \end{cases}$$

The map ψ is then used in the definition of the generators of the DA-bimodule Ω^i: recall, every generator is a compatible idempotent pair, with an allowed outgoing idempotent. If this allowed idempotent is I_x (of any type), one has that the incoming idempotent of this generator is $I_{\psi(x)}$.

Definition 4.33

- *For every allowed idempotent $I_x \in \mathcal{I}(\mathcal{A}_2)$ of type X, define the generator $I_x \cdot X_x \cdot I_{\psi(x)}$.*
- *For every allowed idempotent of $I_y \in \mathcal{I}(\mathcal{A}_2)$ of type Y, define the generator $I_y \cdot Y_y \cdot I_{\psi(y)}$.*

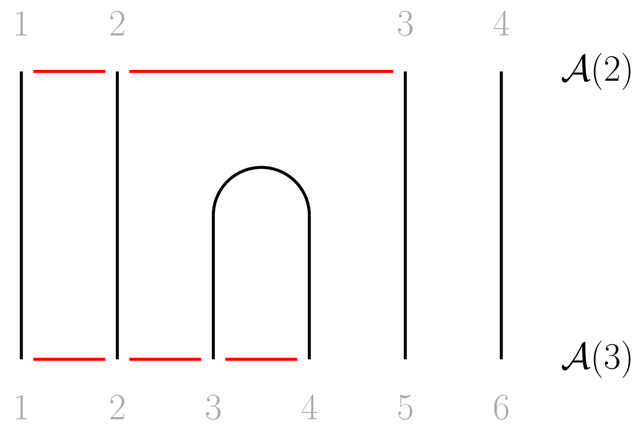


Figure 4.13: *An example generator for the bimodule* ${}^{\mathcal{A}(3)}\Omega^3_{\mathcal{A}(2)}$. *This is a generator* $I_{123} \cdot X_{123} \cdot I_{12}$ *of type* X, *with highlighted incoming and outgoing idempotents. The idempotent* $I_{123} \in \mathcal{I}(\mathcal{A})$ *is an allowed idempotent, since* $3 \in \{1,2,3\}$, *and* $|\{2,4\} \cap \{1,2,3\}| = 1$.

- *For every allowed idempotent* I_z *of type* Z, *define the generator* $I_z \cdot Z_z \cdot I_{\psi(z)}$.

Each allowed idempotent is an $(n+1)$ *element subset of* $\{1, 2, \dots, 2n+1\}$. *The incoming idempotent (rightmost) is then an* n*-element subset of* $\{1, 2, \dots, 2n-1\}$.

Truncating the idempotents to the case of knots, as explained previously, one sees that the bimodule Ω^1 thus has no idempotents of type X, and the bimodule Ω^{2n+1} has no idempotents of type Y, since the outgoing idempotents would be outside the permitted range. An example of a generator of type X for the bimodule ${}^{\mathcal{A}(3)}\Omega^2_{\mathcal{A}(2)}$ is presented in Figure 4.13.

In the DA-bimodule Ω^i, the maps $\delta^1_{1+k} : \Omega^i \otimes \mathcal{A}(n)^{\otimes k} \to \mathcal{A}(n+1) \otimes \Omega^i$ with $k > 0$ take as algebraic inputs sequences of algebra elements in $\mathcal{A}(n)$. Following [49, Lem. 5.2], one can then use the map ψ in order to find a correspondence between algebra elements in $\mathcal{A}(n)$ and $\mathcal{A}(n+1)$. Lemma 4.34 is quoted from [49, Lem. 5.2].

Lemma 4.34 *For* I_x *an allowed idempotent in* $\mathcal{A}_2 = \mathcal{A}(n+1)$, *and* I_y *an idempotent in* $\mathcal{A}_1 = \mathcal{A}(n)$, *such that* $\psi(x)$ *and* y *are not 'far apart', then there is an allowed idempotent state* I_z *in* $\mathcal{A}_2$ *such that* $\psi(z) = y$, *so that there is a map*

$$\Phi_x : I_{\psi(x)} \cdot \mathcal{A}_1 \cdot I_y \to I_x \cdot \mathcal{A}_2 \cdot I_z,$$

with the following properties:

- Φ_x *maps the portion of* $I_{\psi(x)} \cdot \mathcal{B}_1 \cdot I_{\psi(z)}$ *with weights in span*$\{e_1, e_2, \dots, e_{2n}\}$ *surjectively onto the portion of* $I_x \cdot \mathcal{B}_2 \cdot I_z$ *with weights in span* $\{\{e_1, \dots, e_{2n+2}\} \backslash \{e_i, e_{i+1}\}\}$.

- Φ_x *satisfies*

$$\Phi_x(U_j \cdot a) = U_{\phi_i(j)} \cdot \Phi_x(a) \qquad \text{and} \qquad \Phi_x(C_p \cdot a) = C_{\phi_i(p)} \cdot \Phi_x(a)$$

for any $j \in \{1, \ldots, 2n\}$ *and* $p \in M$ *the matching for* $\mathcal{A}_1$.

Moreover, the state z *is uniquely characterised by the existence of map* Φ_x.

Example 4.35 *As an example, consider the algebra element* $I_{23} \cdot L_2U_3 \cdot I_{13} \in \mathcal{A}(2)$. *In* ${}^{\mathcal{A}(3)}\Omega^3_{\mathcal{A}(2)}$, *the map* Φ_{235} *carries this element to* $I_{235} \cdot L_2U_5 \cdot I_{135}$. *Both* I_{235} *and* I_{135} *are allowed idempotents in* Ω^3, *with* $\phi_3(I_{235}) = I_{23}$ *and* $\phi_3(I_{135}) = I_{13}$.

Moreover, the map Φ_{235} *takes* $L_2U_3 \in \mathcal{A}(2)$, *an element of weight* $\frac{1}{2}e_2 + e_3$, *to* $L_2U_5 \in \mathcal{A}(3)$, *an element of weight* $\frac{1}{2}e_2 + e_5 = \frac{1}{2}e_{\phi_3(2)} + e_{\phi_3(3)}$.

Using this correspondence between algebra elements in the incoming algebra and outgoing algebras of Ω^i, one can then define the maps δ^1 for this bimodule.

Definition 4.36 *Let* X *be the sum of all generators of* Ω^i *of type* X. *Likewise, define* Y *and* Z *as the sum of all generators of the corresponding types, defined by Definition 4.33. The maps* $\delta^1_1 : \Omega^i \to \mathcal{A}(n+1) \otimes \Omega^i$ *are then defined as follows:*

$$\begin{aligned} \delta^1_1(X) &= C_{i,i+1} \otimes X + R_{i+1}R_i \otimes Y, \\ \delta^1_1(Y) &= C_{i,i+1} \otimes Y + L_iL_{i+1} \otimes X, \\ \delta^1_1(Z) &= C_{i,i+1} \otimes Z. \end{aligned}$$

The maps $\delta^1_2 : \Omega^i \otimes \mathcal{A}(n) \to \mathcal{A}(n+1) \otimes \Omega^i$ *are then defined by the* Φ_x. *Namely, for generator* Q_x *corresponding to the allowed idempotent* I_x, *let* z *be the allowed idempotent defined by the map* Φ_x. *Then*

$$\delta^1_2(Q_x, a) = \Phi_x(a) \otimes Q_z.$$

The maps $\delta^1_j = 0$ *for* $j \geq 3$.

An important part of this definition is that for the algebra element $a \in \mathcal{A}(n)$, the map $\Phi_x(a)$ defines the allowed idempotent I_z, as remarked in Lemma 4.34.

The maps δ^1_1 and δ^1_2 as defined then satisfy the DA-bimodule structure relations, outlined in Definition 4.25. This fact is proven in [46, Thm 8.3].

4.4.4 Interpretation of DA-bimodules in partial Heegaard diagrams

As described above, for example in Figure 4.6, the algebra elements in $\mathcal{A}(n)$ have a representation in a subset of the Heegaard diagram derived from a special knot diagram.

With the same interpretation of the algebra elements, and motivated by the [46, Sec. 4.4], one can view the maps $\delta^1_{1+k} : \mathcal{M} \otimes \mathcal{B}^{\otimes k} \to \mathcal{A} \otimes \mathcal{M}$ in terms of domains in the partial Heegaard diagrams corresponding to Morse events. Specifically, the algebraic inputs (elements of $\mathcal{B}^{\otimes k}$) represent the sum of domains corresponding to the algebra elements as outlined in Section 4.2.2 at the top of the Morse event. Likewise, an element outgoing algebra $a \in \mathcal{A}$ has a corresponding domain exiting at the bottom of the Morse event.

Using the association of the differential graded algebra $\mathcal{B}$ to the top of some Morse event, and $\mathcal{A}$ to the bottom of the same Morse event, the algebra elements represent sums of domains intersecting the upper and lower boundary of the corresponding partial Heegaard diagram. The corners of the domains — as introduced in Remark 1.6 — correspond to the intersection points in a partial Heegaard diagram that are in bijection with the partial Kauffman states for this Morse event, see Section 4.4.1.

Furthermore, for the map to be non-zero, the sum of domains must satisfy the same conditions as corners of the domains in a full Heegaard diagram. If there is a map between two different generators of the bimodule, then the corresponding intersection points in the partial Heegaard diagram should be acute or obtuse corners of the domain. Whereas if the map is from a generator to itself, then the corners in the Heegaard diagram are degenerate, as in the sense of [23].

Example 4.37 *As an example, consider the domain pictured in Figure 4.14. This is the partial Heegaard diagram associated to a positive crossing between strands one and two, namely associated to the DA-bimodule* $\mathcal{P}^1$.

In the definition of the maps δ^1 *in* $\mathcal{P}^1$*, one has that there is a non-zero map* $\delta^1_3(S, R_1, U_2) \to R_1 \otimes N$*. Here, the solid dot on the intersection between red* α *and blue* β *curves is in correspondence with partial Kauffman state* S*, and the open dot is in correspondence with the partial Kauffman state* N*. Together, the two algebra elements* R_1 *and* U_2 *form a domain with an obtuse corner at* N*, and an acute corner at* S*. This domain intersects the bottom of the partial Heegaard diagram at the front half of the first tube, corresponding to the algebra element* R_1*.*

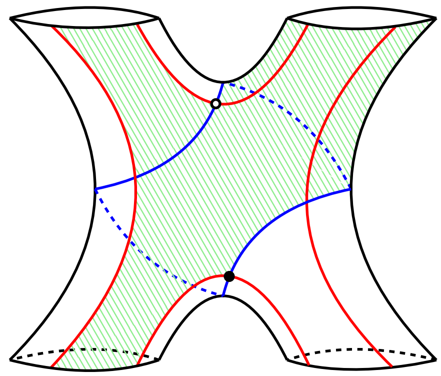


Figure 4.14: *A domain on a partial Heegaard diagram corresponding to $\mathcal{P}^1$. The domain shown corresponds to the map $\delta_3^1(S, R_1, U_2) \to R_1 \otimes N$. The U_2 input corresponds to the full tube incoming on the right, and R_1 the front half of the tube in both incoming and outgoing algebras.*

This correspondence between DA-bimodules and partial Heegaard diagrams has recently been further described by [48]. For example [48, Sec. 2.6] describes partial Heegaard diagrams for Morse events. Moreover, in this paper, Ozsváth-Szabó discuss the appropriate gluing of partial Heegaard diagrams, which corresponds to taking the box-tensor product of the algebraic objects defined over A_∞-algebras.

4.5 Tensor products of algebraic objects over A_∞-algebras

In order to utilise the strength of a cut and paste construction, it must be possible to construct $\mathcal{C}(D)$ from smaller algebraic objects. Moreover, from the physical interpretation of upper knot diagrams and partial knot diagrams, one should intuitively be able to construct a Type D structure from a DA-bimodule and a Type D structure, since by attaching a Morse event to an upper knot diagram one yields another upper knot diagram.

Algebraically, this operation is the box-tensor product $\boxtimes$ between a DA-bimodule and a Type D structure to yield another Type D structure.

As defined in [24, Sec. 2.3.2], given a bimodule ${}^{\mathcal{A}}\mathcal{M}_{\mathcal{B}}$ with map δ and Type D structure ${}^{\mathcal{B}}\mathcal{N}$ with map ∂, one defines the Type D structure ${}^{\mathcal{A}}X = {}^{\mathcal{A}}\mathcal{M}_{\mathcal{B}} \boxtimes {}^{\mathcal{B}}\mathcal{N}$. Generators of ${}^{\mathcal{A}}X$ correspond to elements of $\mathcal{M} \otimes \mathcal{N}$, with this tensor product taken over $\mathcal{I}(\mathcal{B})$. The map $\partial_X : X \to \mathcal{A} \otimes X$ is then defined by

$$\partial_X(m \otimes n) = \big(\delta_1^1(m) \otimes n\big) + \sum_{j \geq 1} (\delta_{1+j}^1 \circ \partial^j)(m \otimes n).$$

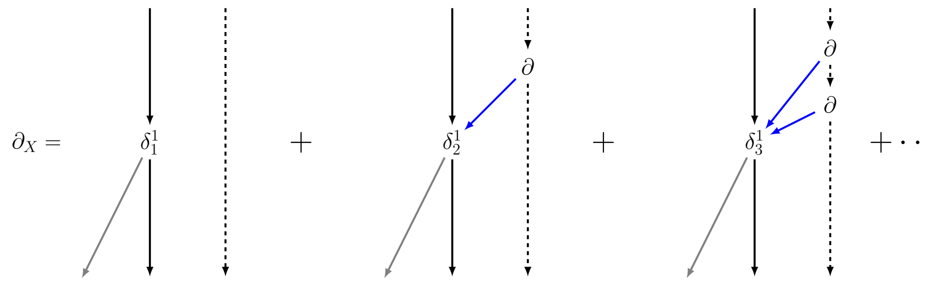


Figure 4.15: *Representation of how to take the tensor product of a Type D structure and a DA-bimodule, yielding another Type D structure.*

The tensor product forming a new Type D structure is demonstrated pictorially in Figure 4.15. Note, this is a finite sum (and so the product is well defined) if there is an $N \in \mathbb{N}$ such that the DA-bimodule map has $\delta^1_{1+j} = 0$ for $j > N$. In the construction of the algebraic invariant by Ozsváth-Szabó , this tensor product will be well defined for this reason.

The fact that the pair (X, ∂_X) defines a Type D structure is a specialisation of the result in [24, Prop. 2.3.10], which states that the tensor products of bimodules of certain types also result in bimodules. Tensor products of bimodules are only well defined when certain conditions on boundedness are satisfied, see [24, Sec. 2.2]. However, as proven in [49, Prop. 2.8], the Type D structures and DA-bimodules used in the construction of $\mathcal{C}(D)$ are bounded.

Remark 4.38 *By construction, the box-tensor product can only be taken when the outgoing algebra of the Type D structure matches the incoming algebra of the DA-bimodule. Since the tensor products in the definition of the map ∂_X are taken over the ring of idempotents for this common algebra, if there are no generators in the Type D structure $\mathcal{N}$ with a certain idempotent, then there are no generators of X with this as an incoming idempotent to the $\mathcal{M}$-tensor coordinate.*

This is a useful property, since a DA-bimodule may have many possible generators with different idempotents, for example in the case of the DA-bimodule associated to the maximum: Ω^i. The generators of this bimodule are all compatible pairs of idempotents with an allowed outgoing idempotent. However, under taking the tensor product with a Type D structure, only those generators in Ω^i with incoming idempotents matching the outgoing *idempotents of the Type D structure will appear in the tensor product.*

This motivates why three strand pretzel knots are so amenable to the cut and paste determination of $\mathcal{C}(D)$. Placing each set of crossings together in the special knot diagram, the possible upper Kauffman states that generate the Type D structure restrict the possible idempotents that give non-zero tensor products after the next crossing is added. Hence, the Type D structures at any point have relatively few (and well structured) generators, making inductive methods for the determination of Type D structures easier to utilise.

4.5.1 The tensor product of two bimodules

As described by [24, Sec. 2.3.2], under certain restrictions, it is possible to take the box-tensor product of two bimodules to yield another. In particular, the tensor product of two DA-bimodules is another DA-bimodule. The composition of two partial knot diagrams is also a partial knot diagram, and has an associated DA-bimodule in this construction.

Let $\left({}^{\mathcal{A}}\mathcal{M}_{\mathcal{B}}, \delta^1_M\right)$ and $\left({}^{\mathcal{B}}\mathcal{N}_{\mathcal{C}}, \delta^1_N\right)$ be bimodules over A_∞-algebras A, B and C. One can then define the bimodule ${}^{\mathcal{A}}\mathcal{X}_{\mathcal{C}}$, which is generated by the elements of $\mathcal{M} \otimes_{\mathcal{I}(\mathcal{B})} \mathcal{N}$. Note, this tensor product is taken over $\mathcal{I}(\mathcal{B})$ the ring of idempotents for the A_∞-algebra $\mathcal{B}$. As in the case of a tensor product between a Type D structure and a DA-bimodule, the outgoing algebra of the rightmost DA-bimodule must match the incoming algebra of the leftmost DA-bimodule. This once more places restrictions upon the generators of the ${}^{\mathcal{A}}\mathcal{X}_{\mathcal{C}}$, due to the enforced compatibility of idempotents.

The map δ^1_X for the bimodule ${}^{\mathcal{A}}\mathcal{X}_{\mathcal{C}}$ is defined similarly to the map shown in Figure 4.15. For any DA-bimodule, say ${}^{\mathcal{F}}\mathcal{Y}_{\mathcal{G}}$ with map δ^1, recall that one can define the map $\delta^k : \mathcal{Y} \otimes \mathcal{G}^{\otimes} \to \mathcal{F}^{\otimes k} \otimes \mathcal{Y}$ iteratively, as demonstrated in Figure 4.10. Using this, the map $\delta^1_X : \mathcal{X} \otimes \mathcal{C}^{\otimes} \to \mathcal{A} \otimes \mathcal{X}$ is defined as

$$\delta^1_X = \sum_{n \geq 0} (\delta^1_{M,1+n} \otimes Id_{\mathcal{N}}) \circ (Id_{\mathcal{M}} \otimes \delta^n_N).$$

Once more, this is more intuitively displayed pictorially, as in Figure 4.16. Note that in the left-most tensor product there is only a single bimodule map, since the map δ^1_X has only a single algebra output.

Observe, if one specifies that in the bimodule ${}^{\mathcal{B}}\mathcal{N}_{\mathcal{C}}$ the maps $\delta^1_{N,1+k} = 0$ for all $k > 0$, then $\mathcal{N}$ would be a Type D structure over $\mathcal{B}$, and the map displayed in Figure 4.16 would be precisely as displayed in Figure 4.15.

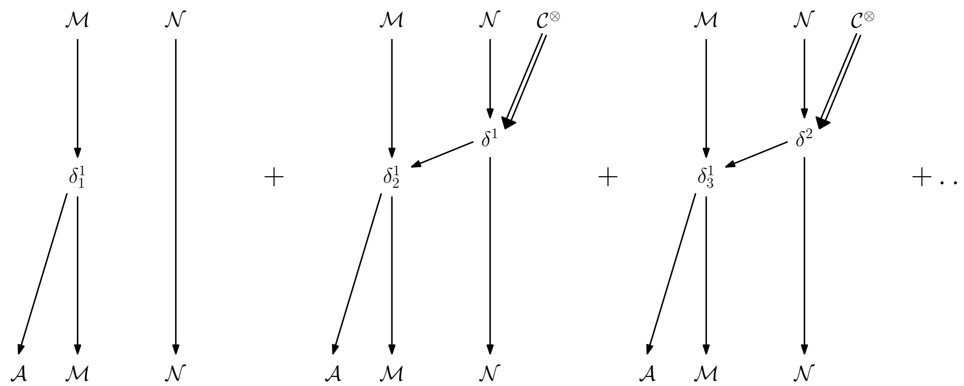


Figure 4.16: *Pictorial representation of the definition of the DA-bimodule map* $\delta^1_{\mathcal{M}\boxtimes\mathcal{N}}$ *for the bimodule* ${}^{\mathcal{A}}\mathcal{M}_{\mathcal{B}} \boxtimes {}^{\mathcal{B}}\mathcal{N}_{\mathcal{C}}$*, as displayed in [24, Fig. 4].*

The fact that this map $\delta^1_X : \mathcal{X} \otimes \mathcal{C}^{\otimes} \to \mathcal{A} \otimes \mathcal{X}$ satisfies the structure relations of a DA-bimodule as outlined in Definition 4.25 is proved in [24, Prop. 2.3.10]. The proof is not presented here, but is a relatively simple consequence of the fact that the maps in both $\mathcal{M}$ and $\mathcal{N}$ satisfy the structure relations.

Remark 4.39 *Determining the DA-bimodule associated to k half-twists between neighbouring strands would make the determination of the Type D structure at any point in an n-strand pretzel knot simpler, and possible extend the amenability of this construction to pretzel knots with more than three strands. This is beyond the scope of the current thesis, but could provide a useful direction for further work.*

4.5.2 Yielding a chain complex via box-tensor product

As remarked above, if a DA-bimodule has a trivial incoming algebra, it has the structure of a Type D structure. Likewise, if the output algebra of the DA-bimodule is empty, the DA-bimodule has the structure of an A_∞-module, see Definition 4.40. Hence, one can specialise the definition of a tensor product between two DA-bimodules presented above to define the tensor product of an A_∞-module and other algebraic objects.

Following [46, Sec. 2.5], let $({}^{\mathcal{A}}X, \partial)$ be a Type D structure, and $(M_{\mathcal{A}}, m_{1+i})$ an A_∞-module. Recall, from Definition 4.40, that each $m_{1+j} : M \otimes \mathcal{A}^{\otimes j} \to M$ takes j algebraic inputs from $\mathcal{A}$. The box tensor product $M \boxtimes X$ is then defined as the module generated by the tensor product $M \otimes_{\mathcal{I}(\mathcal{A})} X$, with map

$$\partial_{M\boxtimes X}(t \otimes x) = \sum_{j=0}^{\infty} (m_{j+1} \otimes Id_x) \circ (t \otimes \partial^j(x)).$$

The resulting pair $(M \boxtimes X, \partial_{M\boxtimes X})$ forms a chain complex when either X is a bounded Type D structure or M is a bounded A_∞-module. This is demonstrated in [25, Lem. 2.30], and by construction, the invariant $\mathcal{C}(D)$ of [49] is thus a well-defined chain complex.

This notion of boundedness is important in making sure that tensor products are well defined. Roughly speaking, this notion of boundedness is when the sum displayed above is finite. For example, a Type D structure ${}^{\mathcal{A}}X$ is a bounded type D structure if for each $x \in X$, there is some n such that for $i > n$, $\partial^i(x) = 0$. Boundedness for A_∞-modules is defined similarly, see [24, Def. 2.2.18] and [24, Def. 2.2.23].

By virtue of being adapted to one-manifolds associated to special knot diagrams, the objects used in the construction of $\mathcal{C}(D)$ are bounded, as proved by [49, Prop. 2.8].

4.6 A_∞-modules

As remarked in the special case of the module assigned to the global maximum in Section 4.4.3, the Type D structure ${}^{\mathcal{A}(1)}t\Omega^1$ can be thought of as a DA-bimodule with empty incoming algebra. Hence, the compatibility conditions outlined in Definition 4.25 and Figure 4.11 simplify to the relations necessary for a module to be a Type D structure as presented in Definition 4.16.

A_∞-modules are in some sense dual to Type D structures, being equivalent to DA-bimodules with empty outgoing algebra. As presented in [25, Def. 2.5], when $\mathcal{A}$ is a differential graded algebra, define an A_∞-module as follows.

Definition 4.40 *An A_∞-module $\mathcal{M}_A$ over differential graded algebra $\mathcal{A}$ is a graded $\mathbb{F}$-module $\mathcal{M}$, with operations*

$$m_{1+i} : \mathcal{M} \otimes \mathcal{A}^{\otimes i} \to \mathcal{M}[1-i],$$

for all $i \geq 0$, such that the following compatibility condition holds.

$$\begin{aligned} 0 = & \sum_{i+j=n+1} m_i \left(m_j(x \otimes a_1 \otimes \dots a_{j-1}) \otimes a_j \otimes \dots a_{n-1}\right) \\ & + \sum_{l=1}^{n-1} m_n(x \otimes a_1 \otimes \dots \otimes a_{l-1} \otimes \mu_1(a_l) \otimes \dots \otimes a_{n-1}) \\ & + \sum_{l=1}^{n-2} m_{n-1}(x \otimes a_1 \otimes \dots \otimes a_{l-1} \otimes \mu_2(a_l \otimes a_{l+1}) \otimes \dots a_{n-1}). \end{aligned}$$

An inspection of these relations, and those presented in Definition 4.25, reveals that the A_∞-module relations are a specialisation to the case where the algebra output of every map δ^1_{1+j} is trivial. Furthermore, following [24, 25], one can define A_∞-modules over a general A_∞-algebra, rather than the special case of a differential graded algebra.

Type D structures are associated to upper knot diagrams, and similarly A_∞-algebras are associated to lower knot diagrams in the construction of [49]. Then, as described in [1, Sec. 4.3], the generators of these A_∞-module are in correspondence with the partial Kauffman states of this lower diagram.

Remark 4.41 *The duality between Type D structures and A_∞-modules can be formalised in terms of the cobar resolution of a differential graded algebra. Informally, if there is an A_∞-module $M_{\mathcal{A}}$, such that for some $x \in M$ with map $m_{1+i}(x, a_1, \ldots, a_i) = y$, then there is a Type D structure with the same generators, such that $\delta^i(x) = a_1^* \otimes \cdots \otimes a_i^* \otimes y$. These algebra elements a_j^* are in the dual algebra to $\mathcal{A}$, denoted $\mathcal{A}'$. This duality is more explicitly described in [24, Rmk. 2.2.35] and [63, Def. 6].*

4.6.1 The terminal minimum as an A_∞-module

In [49, Sec. 7], Ozsváth-Szabó associate a DA-bimodule ${}^{\mathcal{A}(n)}\mho^1_{\mathcal{A}(n+1)}$ to the Morse event of a minimum between strands one and two. The case of a generic minimum $\mho^i$ between strands i and $i+1$ is defined inductively in [49, Sec. 7.5], so that

$$\mho^i = \mho^{i-1} \boxtimes \mathcal{P}^i \boxtimes \mathcal{P}^{i-1}.$$

Defining the DA-bimodule in this way, one need only introduce the specific maps and generators for the DA-bimodule $\mho^1$, and add crossings to the special knot diagram to yield an isotopic special knot diagram with all non-global minima between strands one and two. The isotopy in question is given in Figure 4.17. Note that the expense of this procedure is adding more crossings, and so a greater number of Kauffman states as generators. This makes inductive calculations more complicated.

The global minimum is treated differently, as this is by construction the distinguished edge of the decorated projection associated to Kauffman states of a special knot diagram. Associate to the global minimum the bimodule $t\mho$.

Let $\mathcal{R}' = \mathbb{F}[U, V]/(UV)$ be the same ring as defined previously. The overall chain complex $\mathcal{C}(D)$ for a special knot diagram D is a chain complex over $\mathcal{R}'$, and so a module over $\mathcal{R}'$.

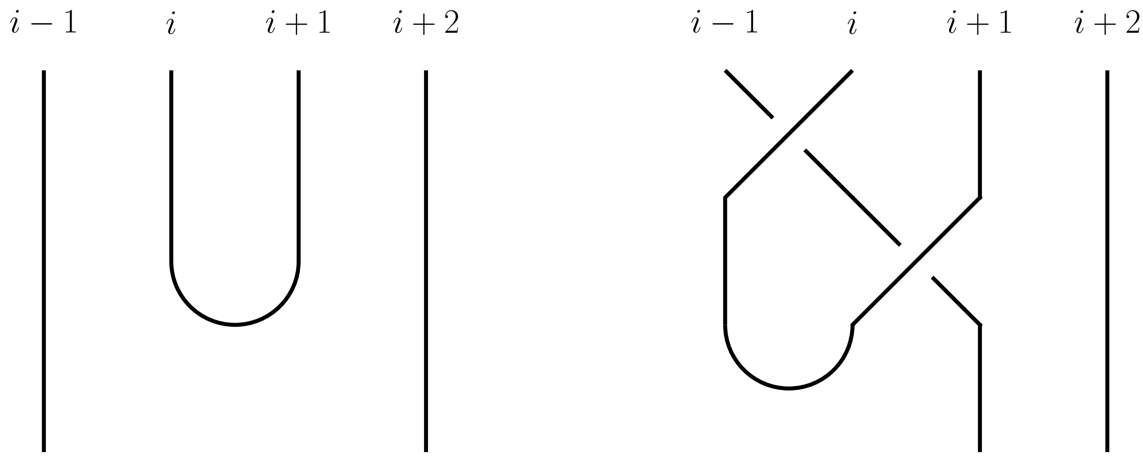


Figure 4.17: *Diagram showing the isotopy required in the inductive definition of the DA-bimodule $\mho^i$ associated to a minimum between strands i and $i+1$. Although all special knot diagrams admit isotopies to special knot diagrams with all minima between the first and second strands, this comes at the expense of adding additional crossings.*

The Kauffman states for a special knot diagram correspond to generators of $\mathcal{C}(D)$ over this ring $\mathcal{R}'$.

This ring $\mathcal{R}'$ can be given the structure of a differential graded algebra, and moreover a type of bimodule called an (left,right) AA-bimodule. The full definition of an AA-bimodule will not be presented here, although it can be found in [24, Def. 2.2.38]. Furthermore, one can yield another (left,right) AA-bimodule under taking the box-tensor product with a DA-bimodule. Namely, there is a well-defined operation (under certain boundedness conditions, which these bimodules satisfy) such that ${}_{\mathcal{R}'}\mathcal{R}'_{\mathcal{R}'} \boxtimes {}^{\mathcal{R}'}X_{\mathcal{A}(1)}$ is a (left,right) AA-bimodule.

Definition 4.42 *Let $\mathcal{R}'$ be defined as above. Define the operation $\mu_2 : \mathcal{R}' \otimes \mathcal{R}' \to \mathcal{R}'$ by $P \otimes Q \mapsto P \cdot Q$, where $\cdot$ denotes multiplication of polynomials in $\mathbb{F}[U,V]$, followed by setting $UV = 0$. Equip $\mathcal{R}'$ with the additional operation $\mu_1 : \mathcal{R}' \to \mathcal{R}'$ such that $P \mapsto 0$ for all $P \in \mathcal{R}'$. Together, μ_1 and μ_2 give $\mathcal{R}'$ the structure of a differential graded algebra.*

Define the bimodule ${}_{\mathcal{R}'}\mathcal{R}'_{\mathcal{R}'}$ to be generated by the elements of $\mathcal{R}'$, together with actions $m_{1,1,0} : \mathcal{R}' \otimes \mathcal{R}' \otimes \mathcal{R}' \to \mathcal{R}'$, $m_{0,1,1} : \mathcal{R}' \otimes \mathcal{R}' \otimes \mathcal{R}' \to \mathcal{R}'$ and $m_{0,1,0} : \mathcal{R}' \otimes \mathcal{R}' \otimes \mathcal{R}' \to \mathcal{R}'$

defined by:

$$m_{1,1,0}(r \otimes s \otimes u) = \mu_2(r, s)$$
$$m_{0,1,1}(r \otimes s \otimes u) = \mu_2(s, u)$$
$$m_{0,1,0}(r \otimes s \otimes u) = \mu_1(s).$$

Let all other relations $m_{i,1,k} : \mathcal{R}'^{\otimes i} \otimes \mathcal{R}' \otimes \mathcal{R}'^{\otimes k}$ *be zero. Then,* ${}_{\mathcal{R}'}\mathcal{R}'_{\mathcal{R}'}$ *has the structure of a (left,right) AA-module as defined by [24, Def. 2.2.38].*

Using Definition 4.42 and viewing $\mathcal{R}'$ as an AA-bimodule, define as follows the AA-bimodule Y, motivated by [49, Sec. 8.2].

$$_{\mathcal{R}'}Y_{\mathcal{A}(1)} = {}_{\mathcal{R}'}\mathcal{R}'_{\mathcal{R}'} \boxtimes {}^{\mathcal{R}'}t\mho_{\mathcal{A}(1)}.$$

The definition presented here is slightly different than that presented in [49], since from the beginning of the construction within this thesis, it has been assumed that the idempotents of the differential graded algebras have been truncated to the case of knots. This is left until just before the addition of the global minimum in Ozsváth-Szabó's construction, see [49, Prop. 8.2] and [46, Rmk. 11.2].

The effect of this is that the original definition of $t\mho$ by Ozsváth-Szabó has an output algebra for $t\mho$ that is $\mathcal{S} = \mathbb{F}[u, v]/(uv)$. However, as remarked in the proof of [49, Prop. 8.2], restricting to the idempotents considered here can be done for all the bimodules used in the construction, and the effect of this is that the output algebra of $t\mho$ is then $\mathcal{R}' \subset \mathcal{S}$, where $u^2 = U, v^2 = V$.

Definition 4.43 *Let* ${}^{\mathcal{R}'}t\mho_{\mathcal{A}(1)}$ *be generated by a single element* $Q_1 \cdot I_1$*. Then, define the maps*

$$\delta^1_2(Q_1, U_1^k) = U^k \otimes Q_1$$
$$\delta^1_2(Q_1, U_2^k) = V^k \otimes Q_1$$
$$\delta^1_2(Q_1, 1) = 1 \otimes Q_1$$
$$\delta^1_2(Q_1, U_1^k U_2^l) = 0 \otimes Q_1$$
$$\delta^1_2(Q_1, C_{12}) = 0 \otimes Q_1.$$

Let all other maps δ^1_j *be zero.*

Using the AA-bimodule ${}_{\mathcal{R}'}\mathcal{R}'_{\mathcal{R}'}$ *as defined above, define the AA-bimodule* ${}_{\mathcal{R}'}\mathcal{R}'_{\mathcal{R}'} \boxtimes {}^{\mathcal{R}'}t\mho_{\mathcal{A}(1)}$*.*

This has a single generator – corresponding to $1 \otimes Q_1$. The left action of $\mathcal{R}'$ corresponds to multiplication of this generator by polynomials in U or V in $\mathcal{R}'$.

Consequently, one can think of this as an A_∞-module $Y'_{\mathcal{A}(1)}$ generated by $1 \otimes Q_1$, $U^k \otimes Q_1$ and $V^j \otimes Q_1$, for all $k, j \in \mathbb{N}$. The non-zero maps $m_{1+n} : Y' \otimes \mathcal{A}(1)^{\otimes n}$ are then defined as follows:

$$\begin{aligned} m_2(X, 1) &= X && \text{for any generator } X \\ m_2(U^j \otimes Q_1, U_1^k) &= U^{j+k} \otimes Q_1 && \text{for any } j, k \in \mathbb{Z}_{\geq 0} \\ m_2(V^j \otimes Q_1, U_2^k) &= V^{j+k} \otimes Q_1 && \text{for any } j, k \in \mathbb{Z}_{\geq 0}. \end{aligned}$$

Definition 4.43 is strongly motivated by the correspondence between partial Heegaard diagrams for the global minimum and A_∞-modules. Using the construction of a Heegaard diagram associated to a special knot diagram, as in Section 2.1, there is a meridional curve on the partial Heegaard diagram corresponding to the distinguished edge in the decorated projection of the lower knot diagram, on either side of which are the z and w basepoint.

In the construction of the invariant $\mathcal{C}(D)$ for three strand pretzel knots, the global minimum will be oriented right to left, which corresponds with placing the z-basepoint on the right of the meridian, and the w-basepoint opposite. If the global minimum is oriented left to right, the role of U and V are switched, see [49, Sec. 8.2].

4.6.2 The other minima

Pretzel knots, and in particular three strand pretzel knots, admit knot diagrams such that all minima are at lower y-coordinates than the crossings and maxima, see Figure 5.1. In particular, in order to yield the special knot diagram associated to a three strand pretzel knot with an upper knot diagram as demonstrated in Figure 5.1, one would take the tensor product with the A_∞-module

$$Y'_{\mathcal{A}(1)} \boxtimes {}^{\mathcal{A}(1)}\mho^2_{\mathcal{A}(2)} \boxtimes {}^{\mathcal{A}(2)}\mho^2_{\mathcal{A}(3)}.$$

From the construction of [49, Sec. 7.5], the bimodule $\mho^2$ is inductively defined as $\mho^1 \boxtimes \mathcal{P}^2 \boxtimes \mathcal{P}^1$.

Using the fact that all idempotents used in the construction are truncated as described on page 61, one can define the the bimodule ${}^{\mathcal{A}(1)}\mho^1_{\mathcal{A}(2)}$ as follows.

Definition 4.44 *Let* ${}^{\mathcal{A}(1)}\mho^1_{\mathcal{A}(2)}$ *be generated by a single generator* $I_1 \cdot G_{23} \cdot I_{23}$. *In* $\mathcal{A}(2)$, *denote the matching elements by* $C_{1,\alpha+2}$ *and* $C_{2,\beta+2}$, *where* $\alpha \neq \beta$, *and each take one of the values* 1 *or* 2. *As a shorthand, let* $C_{1,\alpha+2}$ *be denoted by* C_1, *and* $C_{2,\beta+2}$ *by* C_2.

By virtue of being a knot, one has that C_{12} *is not a matching element in* $\mathcal{A}(2)$. *Then, the maps in the bimodule are defined as follows.*

$$\delta^1_{1+1+m}(G_{23}, U_2^m, C_1^{\otimes m}) = U_\alpha^m \otimes G_{23},$$

$$\delta^1_3(G_{23}, C_1, C_2) = C_{\alpha,\beta} \otimes G_{23},$$

$$\begin{aligned}\delta^1_*(G_{23}, U_2^m L_2, C_1^m, \ldots \\ \ldots, U_1^{a_1+1}, C_2^{\otimes a_1}, U_2^{b_1+1}, C_1^{\otimes b_1}, U_1^{a_2+1}, C_2^{\otimes a_2}, \ldots, \\ \ldots, U_1^{a_k+1}, C_2^{\otimes a_k}, U_2^n R_2, C_1^{\otimes n}) = U_\alpha^A U_\beta^B \otimes G_{23}.\end{aligned}$$

Here, $A = m + n + \sum b_i$, *and* $B = \sum a_i$, *with* $a_i, b_i, n, m \in \mathbb{Z}_{\geq 0}$.

This is not exactly the same as the definition presented in [49, Sec. 7.2], which is described in terms of walks on a directed graph with fixed start and endpoints, although this is similar to the presentation of this bimodule in [49, Sec. 13.1].

In order to provide an algebra input to the A_∞-module Y' that gives some non-zero module element output, one would require that at most one of A or B to be non-zero, since $UV = 0$ in $\mathcal{R}'$. In particular, this restricts the algebra inputs yielding non-zero maps in the A_∞-module $Y' \boxtimes \mho^1$.

The DA-bimodule $\widetilde{\mho}^2$

As mentioned, the downside of defining the DA-bimodule $\mho^2$ inductively for minima is that there is a cost to increasing the number of generators in the corresponding partial knot diagram.

Specifically, consider Figure 4.18. In the leftmost diagram, there is only a single possible Kauffman state. Let the corresponding bimodule be $\widetilde{\mho}^2$, and denote the generator by $I_1 \cdot Q \cdot I_{13}$. For the rightmost diagram, the corresponding DA-bimodule is

$${}^{\mathcal{A}(1)}\mho^2_{\mathcal{A}(2)} \cong {}^{\mathcal{A}(1)}\mho^1_{\mathcal{A}(2)} \boxtimes \mathcal{P}^2 \boxtimes \mathcal{P}^1.$$

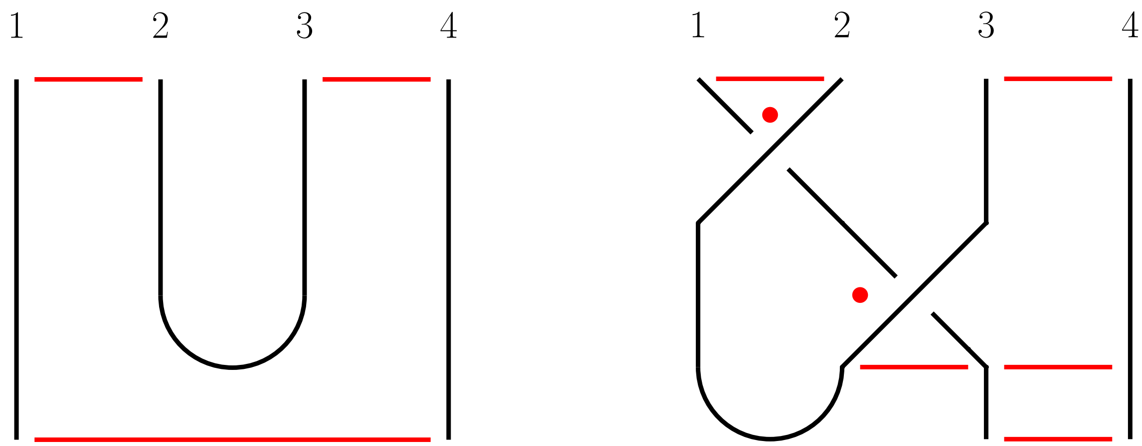


Figure 4.18: *Diagram of two partial knot diagrams, corresponding to the DA-bimodules $\widetilde{\mho}^2$ (left), and $\mho^2 \cong \mho^1 \boxtimes \mathcal{P}^2 \boxtimes \mathcal{P}^1$ (right), as defined by Ozsváth-Szabó .*

This DA-bimodule (the tensor product of three DA-bimodules) has three generators:

$$I_1 \cdot G \otimes W \otimes N \cdot I_{13}$$

$$I_1 \cdot G \otimes W \otimes E \cdot I_{23}$$

$$I_1 \cdot G \otimes N \otimes S \cdot I_{23},$$

where G is the generator of $\mho^1$ as defined in Definition 4.44. Of these three, only $G \otimes W \otimes N$ has idempotents that match the single generator of the bimodule $\widetilde{\mho}^2$.

Definition 4.45 *Let ${}^{\mathcal{A}(1)}\widetilde{\mho}^2_{\mathcal{A}(2)}$ be a bimodule generated by the single generator $I_1 \cdot Q \cdot I_{13}$. Let the map $\delta^1 : \widetilde{\mho}^2 \otimes \mathcal{A}(2)^{\otimes} \to \mathcal{A}(1) \otimes \widetilde{\mho}^2$ be defined by:*

$$\delta^1_3(Q, C_{2*}, C_{3*}) = C_{12} \otimes Q$$

$$\delta^1_4(Q, L_3, U_2, R_3) = 1 \otimes Q$$

$$\delta^1_2(Q, U_1^k) = U_1^k \otimes Q \qquad \textit{for } k \geq 0$$

$$\delta^1_2(Q, U_4^\ell) = U_2^\ell \otimes Q \qquad \textit{for } l \geq 0$$

$$\delta^1_{1+1+n}(Q, U_3^n, C_{2p}^{\otimes n}) = U_q^n \otimes Q \qquad \textit{for } n \geq 1,$$

where $q = 1$ when $p = 1$, and $q = 2$ when $p = 4$. Note, C_{23} cannot be an element of $\mathcal{A}(2)$, as the corresponding diagram would yield a link of two components. Define all other maps δ^1_k to be zero.

Lemma 4.46 *The bimodule ${}^{\mathcal{A}(1)}\widetilde{\mho}^2_{\mathcal{A}(2)}$ with the associated maps has the structure of a DA-bimodule.*

Proof Clearly, $\widetilde{\mho}^2$ has a single generator to which incoming and outgoing idempotents are associated. In order to prove that this has the structure of a DA-bimodule, it remains to verify that the structure relations in Definition 4.25 are satisfied.

Since δ^1_1 is trivial, start by considering δ^1_2. Examining Figure 4.11, as there is only a single algebra input to δ^1_2, it remains to verify that only the first two terms sum together to give 0.

For $\delta^1_2(Q, U_1)$, note that since $\mu_1(U_1) = 0$ in both $\mathcal{A}(1)$ and $\mathcal{A}(2)$, one has that

$$(\mu_1 \otimes \mathbf{1}) \circ \delta^1_2(Q, U_1) = 0$$
$$\delta^1_2 \circ (\mathbf{1} \otimes \mu_1)\,(Q, U_1) = 0.$$

The same is true for $\delta^1_2(Q, U_4)$, since $\mu_1(U_4) = 0$ in both the incoming and outgoing differential graded algebras.

For $\delta^1_3(Q, C_{2*}, C_{3*})$, note that the fourth and fifth terms in the sum displayed in Figure 4.11 will be zero, since $\delta^1_1 = 0$ and $\delta^1_2(Q, C_{2*} \cdot C_{3*}) = 0$. Likewise, no non-zero δ^1_3 map takes C_{3*} or $\mu_1(C_{3*})$ as an input, hence the second and third terms are zero. Finally, the first term is zero, since $\mu_1(C_{12}) = U_1U_2 \in \mathcal{A}(1)$. However, $I_1 \cdot U_1U_2 \cdot I_1$ has identical weight and idempotents to $I_1 \cdot L_1R_1R_2L_2 \cdot I_1$, which is zero in $\mathcal{A}(1)$. By Proposition 4.13, $U_1U_2 = 0$ in $\mathcal{A}(1)$, and so all terms in the sum are zero.

For the term $\delta^1_4(Q, L_3, U_2, R_3)$, note that since $\mu_1(1) = 0 \in \mathcal{A}(1)$, and all three algebra inputs are in the kernel of μ_1 in $\mathcal{A}(2)$, the first three terms in the sum of Figure 4.11 are zero. Similarly, since there are no non-zero δ^1_k terms with similar inputs for $k < 4$, the other terms in the sum are also zero. So the structure relation is satisfied for this set of inputs.

Similar logic applies for $\delta^1_{1+1+n}(Q, U_3^n, C_{2p}^{\otimes n})$. The first term is necessarily zero, since $\mu_1(U_q) = 0 \in \mathcal{A}(1)$. Likewise, the last two terms in the sum are zero, since δ^1 is zero if only U_3 or C_{2p} inputs are supplied. No non-zero arrow has U_2U_p as an input, so the second and third terms in the sum are zero. Hence, $\widetilde{\mho}^2$ is a DA-bimodule. ■

Using this definition, one can then define the A_∞-module $Y' \boxtimes \widetilde{\mho}^2_{\mathcal{A}(2)}$. This is a simple result of applying the tensor product as defined in Section 4.5.1, specialising to the case of an empty outgoing algebra for the leftmost DA-bimodule.

Lemma 4.47 *The A_∞-module $Y' \boxtimes \widetilde{\mho}^2_{\mathcal{A}(2)}$ has generators:*

$$1 \otimes G \otimes Q \cdot I_{13},$$
$$U^\ell \otimes G \otimes Q \cdot I_{13},$$
$$V^k \otimes G \otimes Q \cdot I_{13},$$

with maps $m_{1+i} : (Y' \boxtimes \widetilde{\mho}^2) \otimes \mathcal{A}(2)^{\otimes i} \to Y' \boxtimes \widetilde{\mho}^2$ defined as follows.

$$\begin{aligned}
m_2(U^\ell \otimes G \otimes Q, U_1^k) &= U^{k+\ell} \otimes G \otimes Q, && k, \ell \in \mathbb{Z}_{\geq 0}\\
m_2(V^s \otimes G \otimes Q, U_4^t) &= V^{t+s} \otimes G \otimes Q, && t, s \in \mathbb{Z}_{\geq 0}\\
m_4(U^\ell \otimes G \otimes Q, L_3, U_2, R_3) &= U^\ell \otimes G \otimes Q, && \ell \in \mathbb{Z}_{\geq 0}\\
m_4(V^s \otimes G \otimes Q, L_3, U_2, R_3) &= V^s \otimes G \otimes Q, && s \in \mathbb{Z}_{\geq 0}\\
m_{1+1+n}(U^p \otimes G \otimes Q, U_3^n, C_{12}^{\otimes n}) &= U^{p+n} \otimes G \otimes Q, && p \in \mathbb{Z}_{\geq 0}, n \in \mathbb{N}\\
m_{1+1+m}(V^q \otimes G \otimes Q, U_3^n, C_{24}^{\otimes m}) &= V^{q+m} \otimes G \otimes Q, && q \in \mathbb{Z}_{\geq 0}, m \in \mathbb{N}
\end{aligned}$$

All other maps in the A_∞-module are trivial. Note that only one of the last two relations may be non-zero, as the matching element in $\mathcal{A}(2)$ is either C_{12} or C_{24}.

The dual algebra and canonical bimodule

Having defined the bimodule $\widetilde{\mho}^2$, and the A_∞-module $Y' \boxtimes \widetilde{\mho}^2_{\mathcal{A}(2)}$ as described in Lemma 4.47, the aim is to prove an equivalence between this and $Y' \boxtimes \mho^2_{\mathcal{A}(2)}$, where $\mho^2$ is defined inductively following [49].

In [46, 49], a common method for demonstrating the equivalence of two DA-bimodules is to prove that the two yield identical Type D structures (over some algebra) after a tensor product is taken with an invertible bimodule.

Before explaining some of these terms, the dual algebra to $\mathcal{A}(n)$, denoted $\mathcal{A}'(n)$ needs to be introduced. This is defined in [49, Sec. 2.2], and after the usual truncation, can be described as follows.

Definition 4.48 *Adapting the definition of an I-state presented in Definition 4.6, let an I'-state be an $(n-1)$-element subset of $\{1, 2, \cdots, 2n-1\}$. Define the algebra $\mathcal{B}'$ in the same way as $\mathcal{B}$ was defined in Definition 4.9. Namely, elements in the algebra consist of triples $[I_x, I_y, w_i]$, where I_x and I_y are I'-states, and w_i is a weight in $\left(\frac{1}{2}\mathbb{Z}\right)^{2n}$.*

Then, augment $\mathcal{B}'$ with the elements E_i for $i \in \{1, \cdots, 2n\}$. More formally, let E_i denote the formal sum

$$E_i = \sum_{I_x} [I_x, I_x, e_i],$$

where I_x are the I'-states as defined above. Hence, define $\mathcal{A}'(n)$ as

$$\mathcal{A}'(n) = \mathcal{B}' \cup \langle E_i \rangle / \sim$$

where $\sim$ denotes the relations:

1. $L_{i+1} \cdot L_i = 0$ *for all* i.
2. $R_i \cdot R_{i+1} = 0$ *for every* i.
3. $U_i^x = [I_x, I_x, e_i] = 0$ *when* $\{i, i-1\} \cap x = \emptyset$, *for every* i.
4. $E_i \cdot b = b \cdot E_i$ *for any* $b \in \mathcal{B}$.
5. $E_i^2 = 0$ *for all* i.
6. *When* $\{i, j\} \notin M$, *where* M *is the associated matching as in the definition of* $\mathcal{A}(n)$, *then*

$$[\![E_i, E_j]\!] = E_i \cdot E_j + E_j \cdot E_i = 0.$$

Otherwise, this element is non-zero.

The only non-trivial map μ_1 is $\mu_1(E_i) = U_i$ for every i. Furthermore, define the integer valued grading $\Delta(a)$ by

$$\Delta(a) = \#\,(E_j \text{ dividing } a) - \sum_i w_i(a).$$

The duality of this algebra with the algebra $\mathcal{A}(n)$ is not discussed here, however is presented in more detail in [49, Sec. 2.4].

As described by [46, Sec. 2.6], when $\mathcal{B}$ and $\mathcal{C}$ are differential graded algebras, a (left,left) DD-bimodule ${}^{\mathcal{B},\mathcal{C}}X$ is a module X that is a Type D structure over $\mathcal{B} \otimes \mathcal{C}$. Using this notion, the canonical bimodule ${}^{\mathcal{A}(n),\mathcal{A}'(n)}\mathcal{K}$ is defined in [49, Sec. 2.3] as follows.

Definition 4.49 *Let the canonical bimodule ${}^{\mathcal{A}(n),\mathcal{A}'(n)}\mathcal{K}$ be generated by idempotent pairs $(I_x \otimes I_y) \cdot K_x$, where I_x is an n-element subset of $\{1, 2, \ldots, 2n-1\}$, and I_y is the comple-*

mentary $(n-1)$-element subset[3]. *Define the element $A \in \mathcal{A}(n) \otimes \mathcal{A}'(n)$ by*

$$A = \sum_i (L_i \otimes R_i + R_i \otimes L_i) + \sum_i U_i \otimes E_i + \sum_{\{i,j\} \in M} C_{ij} \otimes [\![E_i, E_j]\!] \in \mathcal{A}(n) \otimes \mathcal{A}'(n).$$

The map $\partial : \mathcal{K} \to \mathcal{A}(n) \otimes \mathcal{A}'(n) \otimes \mathcal{K}$ is defined by $\partial(v) = A \otimes v$, where this tensor product is taken over $\mathcal{I}(\mathcal{A}(n)) \otimes \mathcal{I}(\mathcal{A}'(n))$.

The fact that this map ∂, and the bimodule ${}^{\mathcal{A}(n),\mathcal{A}'(n)}\mathcal{K}$ satisfy the Type D relations is proven in [49, Lem. 2.2]. From [24, Def. 2.3.9], one can suitably define the tensor product of a DA-bimodule and DD-bimodule. If ${}^{\mathcal{A}}\mathcal{M}_{\mathcal{B}}$ is a DA-bimodule, and ${}^{\mathcal{B},\mathcal{C}}\mathcal{N}$ is a DD-bimodule, then there is a DD-bimodule ${}^{\mathcal{A},\mathcal{C}}X$ whose generators correspond to elements of $\mathcal{M} \otimes \mathcal{N}$.

The maps in the tensor product are defined similarly to the tensor product of a DA-bimodule and Type D structure, as in Section 4.5, excepting that the resulting algebra element in $\mathcal{C}$ is the product of the algebra elements in $\mathcal{C}$ from the map ∂^n_N. This is presented in more detail in [24]

Remark 4.50 *A useful result is that the canonical bimodule is invertible (see [49, Thm. 2.3]). A corollary of this is that if there are two DA-bimodules ${}^{\mathcal{A}}\mathcal{P}_{\mathcal{B}}$ and ${}^{\mathcal{A}}\mathcal{Q}_{\mathcal{B}}$, then if the tensor product ${}^{\mathcal{A}}\mathcal{P}_{\mathcal{B}} \boxtimes \mathcal{K}$ is equivalent to ${}^{\mathcal{A}}\mathcal{Q}_{\mathcal{B}} \boxtimes \mathcal{K}$, then the DA-bimodules are equivalent. This is used to great effect in the construction of the invariant $\mathcal{C}(D)$, for example in the proof of [49, Thm. 4.1], which states that the bimodules for positive and negative crossings satisfy the relation corresponding to the second Reidemeister move: i.e. $\mathcal{P}^i \boxtimes \mathcal{N}^i \cong Id \cong \mathcal{N}^i \boxtimes \mathcal{P}^i$.*

The DD-bimodule for a minimum

Unlike the definition of the DA-bimodule $\mho^2$, which is defined inductively, Ozsváth-Szabó define the DD-bimodule associated to a minimum between strands c and $c+1$ explicitly. Adapting their definition in [49, Sec 7.1] to the truncated algebras in question, the DD-bimodule $\mho_2$ is defined as follows.

Definition 4.51 *Define the Type DD-bimodule ${}^{\mathcal{A}(1),\mathcal{A}'(2)}\mho_2$ to be the bimodule as generated by P_2, corresponding to the idempotent pair $I_1 \otimes I_2 \in \mathcal{A}(1) \otimes \mathcal{A}'(2)$, namely*

$$(I_1 \otimes I_2) \cdot P_2.$$

[3] Hence, the two collections partition the set $\{1, 2, \ldots, 2n-1\}$. If y is not a complementary set to x, define $I_x \otimes I_y \cdot K_x$ to be zero.

Define the element $A \in \mathcal{A}(1) \otimes \mathcal{A}'(2)$ as

$$\begin{aligned} A = {} & U_1 \otimes E_1 + U_2 \otimes E_4 + 1 \otimes E_2 U_3 \\ & + U_\alpha \otimes [\![E_{\phi(\alpha)}, E_2]\!] E_3 \\ & + C_{\alpha,\beta} \otimes [\![E_{\phi(\alpha)}, E_2]\!][\![E_3, E_{\phi(\beta)}]\!]. \end{aligned}$$

Here, ϕ is the map $\phi_2 : \{1,2\} \to \{1,2,3,4\}$, as defined on page 87. The above uses the fact that 2 and 3 cannot be matching in $\mathcal{A}'(2)$, hence there are non-zero elements $[\![E_{\phi(\alpha)}, E_2]\!]$ and $[\![E_3, E_{\phi(\beta)}]\!]$, corresponding to the matching elements $C_{2,\alpha}$ and $C_{3,\beta}$ in $\mathcal{A}(2)$. Necessarily, $C_{\alpha,\beta} = C_{12} \in \mathcal{A}(1)$.

Then, the map $\partial^1 : \mho_2 \to \mathcal{A}(1) \otimes \mathcal{A}'(2) \otimes \mho_2$ is defined by $\partial^1(P_2) = A \otimes P_2$, similar to the map in the canonical bimodule $\mathcal{K}$.

By [49, Lem. 7.1], $\mho_2$ has the structure of a Type DD-bimodule, so this will not be proved here. However, as a Type DD-bimodule, one can define the tensor product $Y'_{\mathcal{A}(1)} \boxtimes {}^{\mathcal{A}(1),\mathcal{A}'(2)}\mho_2$. This is a simple consequence of the tensor products defined in [24, Def. 2.3.9] and Definition 4.43

Lemma 4.52 *The box-tensor product $Y' \boxtimes \mho_2$ has the structure of a Type D structure over the differential graded algebra $\mathcal{A}'(2)$. This module has generators $U^\ell \otimes G \otimes P_2$ and $V^k \otimes G \otimes P_2$, with $\ell, k \in \mathbb{Z}_{\geq 0}$. The maps ∂ in the Type D structure are defined as follows.*

$$\begin{aligned} \partial(U^\ell \otimes G \otimes P_2) &= (E_2 U_3) \otimes U^\ell \otimes G \otimes P_2 \\ &\qquad + (E_1 + [\![E_1, E_2]\!] E_3) \otimes U^{\ell+1} \otimes G \otimes P_2. \\ \partial(V^k \otimes G \otimes P_2) &= (E_2 U_3) \otimes V^k \otimes G \otimes P_2 \\ &\qquad + (E_4) \otimes V^{k+1} \otimes G \otimes P_2. \end{aligned}$$

Here, the matching elements are assumed to be C_{12} and C_{34}, but are easily adapted to the other case.

This is a Type D structure as a consequence of the fact that Y' is an A_∞-module, and $\mho_2$ a DD-bimodule. Recall, in Lemma 4.47 an A_∞-module $Y' \boxtimes \widetilde{\mho}^2$ was defined. Using the canonical bimodule $\mathcal{K}$, one can define a Type D structure over $\mathcal{A}'(2)$,

$$Y' \boxtimes \widetilde{\mho}^2_{\mathcal{A}(2)} \boxtimes {}^{\mathcal{A}(2),\mathcal{A}'(2)}\mathcal{K}.$$

Proposition 4.53 *The Type D structures $Y' \boxtimes \widetilde{\mho}^2_{\mathcal{A}(2)} \boxtimes {}^{\mathcal{A}(2),\mathcal{A}'(2)}\mathcal{K}$ and $Y' \boxtimes \mho_2$ over the differential graded algebra $\mathcal{A}'(2)$ are equivalent.*

Proof Firstly, the generators of $\left(Y' \boxtimes \widetilde{\mho}^2\right) \boxtimes \mathcal{K}$ are:

$$1 \otimes G \otimes Q \cdot I_{13} \otimes (I_{13} \otimes I_2) \cdot K_{13} = I_2 \cdot 1 \otimes G \otimes Q \otimes K_{13}$$
$$U^\ell \otimes G \otimes Q \cdot I_{13} \otimes (I_{13} \otimes I_2) \cdot K_{13} = I_2 \cdot U^\ell \otimes G \otimes Q \otimes K_{13}$$
$$V^k \otimes G \otimes Q \cdot I_{13} \otimes (I_{13} \otimes I_2) \cdot K_{13} = I_2 \cdot V^k \otimes G \otimes Q \otimes K_{13}.$$

Calculating the tensor product of $\left(Y' \boxtimes \widetilde{\mho}^2_{\mathcal{A}(2)}\right)$ and ${}^{\mathcal{A}(2),\mathcal{A}'(2)}\mathcal{K}$ essentially reduces to finding inputs for the maps $m_{1+i}\left((Y' \boxtimes \widetilde{\mho}^2) \otimes \mathcal{A}(2)^{\otimes i}\right)$ that are in the $\mathcal{A}(2)$ coordinate of the term $A \in \mathcal{A}(2) \otimes \mathcal{A}'(2)$ as defined in Definition 4.49. After finding the required inputs, the algebra coefficient in $\mathcal{A}'(2) \otimes \left(Y' \boxtimes \widetilde{\mho}^2_{\mathcal{A}(2)}\right)$ is the product of the algebra coefficients in the $\mathcal{A}'(2)$ in the coordinate of A.

- From the term $m_2(U^\ell \otimes G \otimes Q, U_1) = U^{\ell+1} \otimes G \otimes Q$, one pairs this with $U_1 \otimes E_1$ in A, to yield
$$\partial(U^\ell \otimes G \otimes Q \otimes K_{13}) \ni E_1 \otimes \left(U^{\ell+1} \otimes G \otimes Q \otimes K_{13}\right).$$
- Similarly, the term $m_2(V^k \otimes G \otimes Q, U_4) = V^{k+1} \otimes G \otimes Q$ is paired with the term $U_4 \otimes E_4$ in A to yield
$$\partial(V^k \otimes G \otimes Q \otimes K_{13}) \ni E_4 \otimes \left(V^{k+1} \otimes G \otimes Q \otimes K_{13}\right).$$
- For any generator X in $Y' \boxtimes \widetilde{\mho}^2$, the map $m_4(X, L_3, U_2, R_3) = X$ is paired with the terms $L_3 \otimes R_3$, $U_2 \otimes E_2$ and $R_3 \otimes L_3$ to yield
$$\partial(X \otimes K_{13}) \ni E_2 U_3 \otimes (X \otimes K_{13}).$$
This follows from the fact that in $\mathcal{A}'(2)$, the product of the rightmost tensor coordinates is $R_3 \cdot E_2 \cdot L_3 = E_2 \cdot R_3 \cdot L_3 = E_2 U_3$, since E_2 is central.
- Assuming the matching elements to be C_{12} and C_{34} (the other case is easily adapted), one has that the map $m_3(U^\ell \otimes G \otimes Q, U_3, C_{12})$ pairs with the terms $U_3 \otimes E_3$ and $C_{12} \otimes [\![E_1, E_2]\!]$ in A. This yields
$$\partial(U^\ell \otimes G \otimes Q \otimes K_{13}) \ni E_3[\![E_1, E_2]\!] \otimes \left(U^{\ell+1} \otimes G \otimes Q \otimes K_{13}\right).$$

There is a simple one-to-one correspondence between the generators of the two Type D structures, given by $G \otimes Q \otimes K_{13} \leftrightarrow G \otimes P_2$, with matching idempotents. Using this and the result of Lemma 4.52 gives an equivalence between the two modules, as the action of ∂ is the same on each. ■

Using Remark 4.50, which outlines that the canonical bimodule can be used to show that two DA-bimodules are equivalent, one has the following simple corollary.

Corollary 4.54 *The A_∞-modules $Y' \boxtimes \widetilde{\mho}^2$ and $Y' \boxtimes \mho^2$ are equivalent.*

Proof When the DA-bimodule $\mho^2$ was inductively defined, Ozsváth-Szabó proved in [49, Thm. 7.10] that there is an equivalence between the two Type DD-bimodules $\mho^2 \boxtimes \mathcal{K}$ and $\mho_2$. Including the notation denoting the algebras, this is the equivalence

$$^{\mathcal{A}(1)}\mho^2_{\mathcal{A}(2)} \boxtimes {}^{\mathcal{A}(2),\mathcal{A}'(2)}\mathcal{K} \cong {}^{\mathcal{A}(1),\mathcal{A}'(2)}\mho_2.$$

Hence, $Y' \boxtimes \mho_2 \cong Y' \boxtimes \mho^2 \boxtimes \mathcal{K}$. Applying the result of Proposition 4.53, one has that

$$Y' \boxtimes \widetilde{\mho}^2 \boxtimes \mathcal{K} \cong Y' \boxtimes \mho^2 \boxtimes \mathcal{K}.$$

From the invertibility of the canonical bimodule $\mathcal{K}$, as proven in [49, Thm. 2.3], one yields the result. See [49, Lem. 2.13] for example. ■

One can go further than this, and demonstrate that the two DA-bimodules $^{\mathcal{A}(1)}\mho^2_{\mathcal{A}(2)}$ and $^{\mathcal{A}(1)}\widetilde{\mho}^2_{\mathcal{A}(2)}$ are equivalent, by showing that the DD-bimodules $\mho^2 \boxtimes \mathcal{K}$ and $\widetilde{\mho}^2 \boxtimes \mathcal{K}$ are equal. However, it is sufficient when examining the minima to only prove equivalence of the A_∞-modules.

Remark 4.55 *Examining the motivation behind the definition of $\widetilde{\mho}^2$, one might question why there is no term $m_3(Q, U_2, C_{34}) = V \otimes Q$. From the DA-bimodule $\mho^2$, the only term with the matching idempotents to Q is*

$$I_1 \otimes G \otimes W \otimes N \cdot I_{13}.$$

One sees that $\delta^1_2(I_{13} \cdot N \cdot I_{13}, U_2) = I_{13} \cdot U_1 \otimes N$ in $\mathcal{P}^1$, yet $\delta^1_2(I_{23} \cdot W \cdot I_{13}, U_1) = I_{23} \cdot U_1 \otimes W = 0$ in $\mathcal{P}^2$.

Furthermore, of the three generators presented for $\mho^2$ on page 101, in the associated Heegaard diagram for $\mho^1 \boxtimes \mathcal{P}^2 \boxtimes \mathcal{P}^1$, there is a rectangular domain between the generators $G \otimes W \otimes E$ and $G \otimes N \otimes S$. With the recently proven equivalence between $\mathcal{C}(D)$ and classical knot Floer homology, such a domain would provide the term $G \otimes N \otimes S \ni \widehat{\partial}(G \otimes W \otimes E)$, and so both terms would be trivial in homology.

Using a very similar technique, one can define the DA-bimodule $^{\mathcal{A}(2)}\widetilde{\mho}^2_{\mathcal{A}(3)}$, and then the A_∞-module $Y' \boxtimes \widetilde{\mho}^2 \boxtimes \widetilde{\mho}^2_{\mathcal{A}(3)}$, corresponding to the lower Heegaard diagram of the global minimum, and minima between strands 2 and 3 and between strands 4 and 5. Both of these are defined here, but the equivalence with $Y' \boxtimes \widetilde{\mho}^2 \boxtimes \mho^2$ is omitted, since the calculation is a simple adaptation of the above.

Definition 4.56 *Corresponding to the appropriate partial Heegaard diagram, define the bimodule* ${}^{\mathcal{A}(2)}\widetilde{\mho}^2_{\mathcal{A}(3)}$ *to have a single generator,* $I_{13} \cdot H \cdot I_{135}$*. In the following definition, the matching elements in* $\mathcal{A}(3)$ *are assumed to be* C_{14}*,* C_{26} *and* C_{35}*, corresponding to the case of three-strand pretzel knots considered within this thesis. Definitions with other matching elements are very similar.*

Let the non-zero maps $\delta^1_{1+k} : \widetilde{\mho}^2 \otimes \mathcal{A}(3)^{\otimes k} \to \mathcal{A}(2) \otimes \widetilde{\mho}^2$ *be defined as follows.*

$$\begin{aligned}
\delta^1_2(H, U_1^k) &= U_1^k \otimes H \\
\delta^1_2(H, U_6^k) &= U_4^k \otimes H \\
\delta^1_2(H, U_5^\ell) &= U_3^\ell \otimes H \\
\delta^1_{1+1+\ell}(H, U_3^\ell, C_{26}^{\otimes \ell}) &= U_4^\ell \otimes H \\
\delta^1_4(H, L_5, U_4, R_5) &= 1 \otimes H \\
\delta^1_4(H, L_3, U_2, R_3) &= 1 \otimes H \\
\delta^1_3(H, C_{26}, C_{35}) &= C_{34} \otimes H \\
\delta^1_2(H, C_{14}) &= C_{12} \otimes H.
\end{aligned}$$

Once more, in [49, Sec. 7.1], Ozsváth-Szabó define the DD-bimodule ${}^{\mathcal{A}(2),\mathcal{A}'(3)}\mho_2$. However, the forced compatibility with the idempotent of the single generator of the A^∞-module $Y' \boxtimes \widetilde{\mho}^2_{\mathcal{A}(2)}$ means that there is only a single generator of this bimodule: $(I_{13} \otimes I_{24}) \cdot P_{24}$. Furthermore, the map $\partial(P_2) = A \otimes P_2$, where

$$\begin{aligned}
A = {} & U_1 \otimes E_1 + U_2 \otimes E_4 + U_3 \otimes E_5 + U_4 \otimes E_6 + 1 \otimes E_2 U_3 \\
& + U_4 \otimes [\![E_2, E_6]\!] E_3 + C_{34} + [\![E_2, E_6]\!][\![E_3, E_5]\!] + C_{12} \otimes [\![E_1, E_4]\!].
\end{aligned}$$

Using this, and the canonical bimodule ${}^{\mathcal{A}(3),\mathcal{A}'(3)}\mathcal{K}$, one can prove the equivalence between $Y' \boxtimes \widetilde{\mho}^2_{\mathcal{A}(2)} \boxtimes {}^{\mathcal{A}(2),\mathcal{A}'(3)}\mho_2$ and $Y' \boxtimes \widetilde{\mho}^2_{\mathcal{A}(2)} \boxtimes {}^{\mathcal{A}(2)}\widetilde{\mho}^2_{\mathcal{A}(3)} \boxtimes \mathcal{K}$ as Type D structures exactly as in Proposition 4.53.

Then, invertibility of the canonical bimodule yields an equivalence between $Y' \boxtimes \widetilde{\mho}^2 \boxtimes \widetilde{\mho}^2_{\mathcal{A}(3)}$ and $Y' \boxtimes \widetilde{\mho}^2 \boxtimes \mho^2_{\mathcal{R}(3)}$. The former is defined here, utilising Definition 4.56 and Section 4.5.1.

Definition 4.57 *Define the A_∞-module corresponding to the three minima, without additional introduced crossings as in the inductive definition, as $Y' \boxtimes \widetilde{\mho}^2 \boxtimes \widetilde{\mho}^2$. Let this be generated by the elements:*

$$\begin{gathered}(1 \otimes G \otimes Q \otimes H \cdot I_{135})\,, \\ \left(U^\ell \otimes G \otimes Q \otimes H \cdot I_{135}\right), \\ \left(V^j \otimes G \otimes Q \otimes H \cdot I_{135}\right).\end{gathered}$$

The maps in the A_∞-module $m_{1+j} : Y' \boxtimes \widetilde{\mho}^2 \boxtimes \widetilde{\mho}^2 \otimes \mathcal{A}(3)^{\otimes j} \to Y' \boxtimes \widetilde{\mho}^2 \boxtimes \widetilde{\mho}^2$ are defined as follows.

$$\begin{aligned}
m_2(U^\ell \otimes G \otimes Q \otimes H, U_1^k) &= U^{k+\ell} \otimes G \otimes Q \otimes H, && k, \ell \in \mathbb{Z}_{\geq 0} \\
m_2(V^s \otimes G \otimes Q \otimes H, U_6^t) &= V^{t+s} \otimes G \otimes Q \otimes H, && t, s \in \mathbb{Z}_{\geq 0} \\
m_4(U^\ell \otimes G \otimes Q \otimes H, L_3, U_2, R_3) &= U^\ell \otimes G \otimes Q \otimes H, && \ell \in \mathbb{Z}_{\geq 0} \\
m_4(V^s \otimes G \otimes Q \otimes H, L_3, U_2, R_3) &= V^s \otimes G \otimes Q \otimes H, && s \in \mathbb{Z}_{\geq 0} \\
m_4(U^\ell \otimes G \otimes Q \otimes H, L_5, U_4, R_5) &= U^\ell \otimes G \otimes Q \otimes H, && \ell \in \mathbb{Z}_{\geq 0} \\
m_4(V^s \otimes G \otimes Q \otimes H, L_5, U_4, R_5) &= V^s \otimes G \otimes Q \otimes H, && s \in \mathbb{Z}_{\geq 0} \\
m_{1+1+\ell}(V^r \otimes G \otimes Q \otimes H, U_3^\ell, C_{26}^{\otimes \ell}) &= V^{r+\ell} \otimes G \otimes Q \otimes H, && r \in \mathbb{Z}_{\geq 0}, \ell \in \mathbb{N} \\
m_{1+1+n}(U^p \otimes G \otimes Q \otimes H, U_5^n, C_{14}^{\otimes n}) &= U^{p+n} \otimes G \otimes Q \otimes H, && p \in \mathbb{Z}_{\geq 0}, n \in \mathbb{N}
\end{aligned}$$

Chapter 5

Inductive Arguments

As mentioned above, three strand pretzel knots are particularly amenable to study using the cut and paste argument of Ozsváth-Szabó, since the Type D structure at any point can be determined using inductive methods. In this chapter, the calculation of the Type D structure for the upper knot diagram of $P(2c+1, -2b-1, 2a)$ will be determined up until the three final minima. See Figure 5.1 for an example.

In what follows, much use will be made of the pictorial representation of Type D structures by directed graphs, see Section 4.3.2. Excepting the case of the Type D structures associated to Ω^1 and $\Omega^2 \boxtimes \Omega^1$, the self-arrows with weight given by $\sum_{pq \in \text{Matching}} C_{pq}$ will be omitted.

5.1 Initial maxima

In any special knot diagram, the program developed by Ozsváth-Szabó in [47, 49] starts with a Type D structure for the global maximum of the diagram. This is represented by the directed graph in Figure 4.8. The Type D structure ${}^{\mathcal{A}(1)}\Omega^1$ has a single generator corresponding to the idempotent I_1, and as a standard Type D structure has only a single self-arrow, given by the matching element C_{12}. Recall from Definition 4.11 that each matching element C_{pq} is the sum $\sum_{I_x} I_x \cdot C_{pq} \cdot I_x$, so forms a non-zero tensor product with all elements in the image of a map from a Type D structure.

More formally, ${}^{\mathcal{A}(1)}\Omega^1 = \langle I_1 \cdot P \rangle$, with the map ∂^1 given by

$$\partial^1(I_1 \cdot P) = C_{12} \otimes I_1 \cdot P \in \mathcal{A}(1) \otimes {}^{\mathcal{A}(1)}\Omega^1.$$

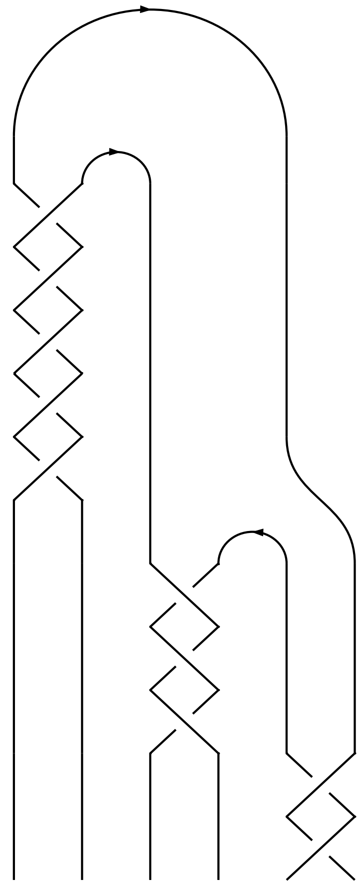


Figure 5.1: *An example of the (oriented) upper knot diagram for a three strand pretzel knot $P(2c+1, -2b-1, 2a)$. In this case, $c = 2$, $b = 1$ and $a = 1$. Note that every Morse event occurs at a different height.*

Then, consider the tensor product with the DA-bimodule corresponding to the next maximum ${}^{\mathcal{A}(2)}\Omega^2_{\mathcal{A}(1)}$. This is defined in [49, Sec. 5.2] and Section 4.4.3, but from the view of DA-bimodules as being generated by partial Kauffman states (see Definition 4.27), there are only two admissible generators, both corresponding to idempotents:

$$I_{12} \cdot X \cdot I_1,$$

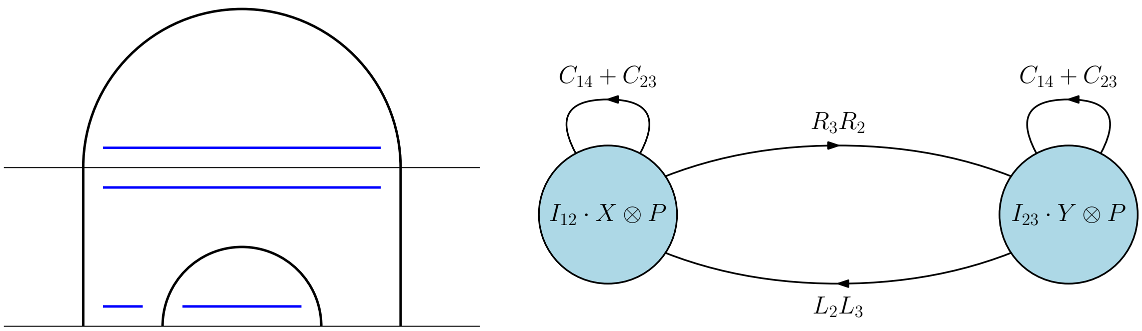
$$I_{23} \cdot Y \cdot I_1.$$

One can then take the tensor product of each with the generator $I_1 \cdot P \in {}^{\mathcal{A}(1)}\Omega^1$, following the procedure outlined in Section 4.5. A pictorial representation of this is given in

Figure 5.2a. Note, that in Ω^2, the maps δ^1_{1+k} are given by

$$\begin{aligned}
\delta^1_2(X, C_{12}) &= C_{14} \otimes X, \\
\delta^1_2(Y, C_{12}) &= C_{14} \otimes Y, \\
\delta^1_1(X) &= C_{23} \otimes X + R_3 R_2 \otimes Y, \\
\delta^1_1(Y) &= C_{23} \otimes Y + L_2 L_3 \otimes X.
\end{aligned}$$

This is exactly as presented in Section 4.4.3, with generators as defined in Definition 4.33. Then, taking the appropriate box-tensor product of the DA-bimodule and Type D structure yields the Type D structure ${}^{\mathcal{A}(2)}\Omega^2 \boxtimes \Omega^1$, as shown in Figure 5.2b.



(a) *Pictorial representation of the generator $I_{12} \cdot X \otimes P$.*

(b) *Type D structure for ${}^{\mathcal{A}(2)}\Omega^2 \boxtimes \Omega^1$*

Figure 5.2: *The Type D structure for the tensor product of the global maximum and the second maximum. Note, this is a standard Type D structure.*

5.2 First set of crossings

As can be seen in Figure 5.1, the next part of the special knot diagram for the pretzel knot $P(2c+1, -2b-1, 2a)$ are $2c+1$ positive crossings between strands 1 and 2. Determining the Type D structure after these crossings corresponds to taking the tensor product

$$\left({}^{\mathcal{A}(2)}\mathcal{P}^1_{\mathcal{A}(2)}\right)^{\boxtimes(2c+1)} \boxtimes {}^{\mathcal{A}(2)}\Omega^2 \boxtimes \Omega^1.$$

One can determine this Type D structure using induction on the number of crossings. Hence, one must first consider the base case of two maxima and a single positive crossing.

Inductive statement

Let P_{2k+1} be the statement that the Type D structure

$$\left({}^{\mathcal{A}(2)}\mathcal{P}^1_{\mathcal{A}(2)}\right)^{\boxtimes 2k+1} \boxtimes {}^{\mathcal{A}(2)}\Omega^2 \boxtimes \Omega^1$$

is as depicted by the weighted, directed graph in Figure 5.3, with the understanding that all Type D structures are standard, as defined by Definition 4.19. By convention, the self-arrows corresponding to matching elements are suppressed.

Moreover, it is worth noting that the algebras $\mathcal{A}(2)$ in the above tensor product are not all equal. Given that a positive crossing between the first and second strand switches the role of 1 and 2 in the algebra, one has that one of the copies of $\mathcal{A}(2)$ has matching elements $\{C_{13}, C_{24}\}$, while the other has matching elements $\{C_{14}, C_{23}\}$. However, one can see from a simple diagram that if one adds an odd number of positive crossings, the output algebra has matching elements $\{C_{13}, C_{24}\}$.

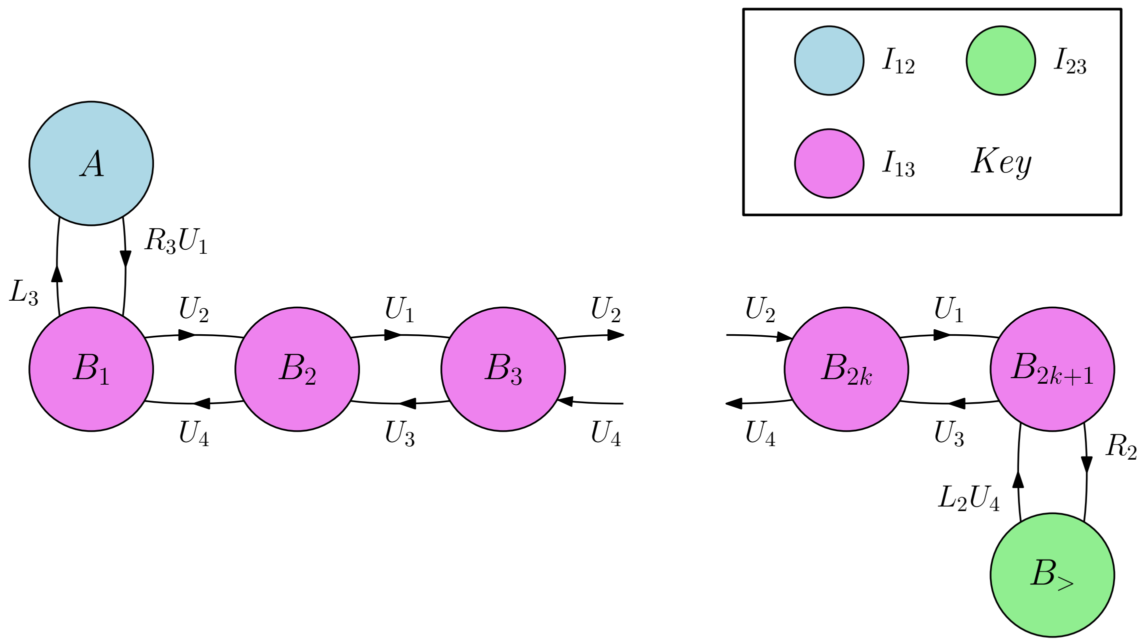


Figure 5.3: *Weighted graph describing the inductive structure of a set of $2k+1$ positive crossings $\mathcal{P}^1$ attached to two maxima. A denotes the Kauffman state with only North N states, B_k denotes that the E state occurs k crossings from the top, and $B_>$ denotes that one has only S states.*

Base case: 1 crossing, $k = 0$

The possible positions for Kauffman states in $\mathcal{P}^1$ are dictated by the idempotents associated to the Type D structure above, see Figure 5.2b. Using the definition of the cardinal generators N, S, E and W, one sees that the tensor product

$$\mathcal{P}^1 \boxtimes \Omega^2 \boxtimes \Omega^1$$

has only three possible generators:

$$\begin{aligned} A &:= I_{12} \cdot N \cdot I_{12} \otimes (X \otimes P) \\ B_1 &:= I_{13} \cdot E \cdot I_{23} \otimes (Y \otimes P) \\ B_> &:= I_{23} \cdot S \cdot I_{23} \otimes (Y \otimes P). \end{aligned}$$

Remark 5.1 *This notation indicates the possible position of the decorations in the Kauffman state. As described earlier in Section 2.2, and inspired by Eftekhary [5], one can separate Kauffman states for three strand pretzel knots by the position of the marked point in the region enclosed by the global maximum. Using the local grading information at each crossing, following Figure 4.1 one can also easily determine the grading of these elements of the Type D structure.*

In this case, and in the case of Figure 5.3, A denotes that one only has N generators on the strand, B_k denotes that the marked point in the region adjacent to the strand is the k^{th} from the top, and $B_>$ denotes that the generators on this strand are all S.

In order for a valid tensor product to be taken with the generator $I_{12} \cdot X \otimes P$, the incoming tensor product of a generator in $\mathcal{P}^1$ must be I_{12}. There is thus only one choice of generator in order to produce a valid term in $\mathcal{P}^1 \boxtimes \Omega^2 \boxtimes \Omega^1$, namely $I_{12} \cdot N \cdot I_{12}$.

This requirement is identical under any number of iterations of tensor product of $\mathcal{P}^1$, and this results in such states being A states. It is behaviour like this that makes inductive arguments possible in determining the Type D structure.

Let the state with idempotent I_{12} in $\Omega^2 \boxtimes \Omega^1$ be X, and let the other be Y. Proceeding with the calculation of the map d in the Type D structure $\mathcal{P}^1 \boxtimes \Omega^2 \boxtimes \Omega^1$, one has the following.

$$\begin{aligned}
d(N \otimes X) &= \delta_2^1(N, R_3R_2) \otimes Y, \\
&= R_3 \cdot \delta_2^1(N, R_2) \otimes Y, \\
&= R_3U_1 \otimes E \otimes Y, \\
&= R_3U_1 \otimes B_1. \\
d(E \otimes Y) &= \delta_1^1(E) \otimes Y + \delta_2^1(E, L_2L_3) \otimes X, \\
&= R_2 \otimes S \otimes Y + L_3 \cdot \delta_2^1(E, L_2) \otimes X, \\
&= R_2 \otimes B_> + L_3 \otimes N \otimes X, \\
&= R_2 \otimes B_> + L_3 \otimes A. \\
d(S \otimes Y) &= \delta_2^1(S, C_{14}) \otimes Y, \\
&= U_4L_2 \otimes E \otimes Y, \\
&= L_2U_4 \otimes B_1.
\end{aligned}$$

This fits the structure given in Figure 5.3 with $k = 0$. In the above calculation, use has been made of the fact that when tensoring by the bimodule $\mathcal{P}^i$, elements with weight outside $span\{e_i, e_{i+1}\}$ commute with the map δ. Moreover, one has that $L_2U_4 = U_4L_2$, since Proposition 4.13 states that elements are uniquely determined by their weights and idempotents, and these elements have equal idempotents and weights in $\frac{1}{2}\mathbb{Z}^4$. Hence, the base case is true.

Increasing the number of crossings

It is a relatively straightforward calculation to show that this holds for $k = 1$, i.e. the special knot diagram with three positive crossings. As remarked upon before, if in the Type D structure X one has an element $I_{1\ell} \cdot x$, then the only possible element in $\mathcal{P}^1$ with compatible idempotent is $I_{1\ell} \cdot N \cdot I_{1\ell}$. Hence, the only possible generator with x in the X tensor-coordinate is $I_{1\ell} \cdot N \otimes x$.

Abusing the notation slightly, tensoring with $\mathcal{P}^1$ once, yields a Type D structure with four generators

$$I_{12} \cdot A := N \otimes A$$

$$I_{13} \cdot B_1 := N \otimes B_1$$

$$I_{13} \cdot B_2 := E \otimes B_>$$

$$I_{23} \cdot B_> := S \otimes B_>.$$

Then, one has the following differentials.

$$\begin{aligned}
d(A) &= \delta_2^1(N, R_3U_1) \otimes B_1, \\
&= R_3 \cdot \delta_2^1(N, U_1) \otimes B_1, \\
&= R_3U_2 \otimes N \otimes B_1 = R_3U_2 \otimes B_1. \\
d(B_1) &= \delta_2^1(N, L_3) \otimes A + \delta_2^1(N, R_2) \otimes B_>, \\
&= L_3 \otimes N \otimes A + U_1 \otimes E \otimes B_>, \\
&= L_3 \otimes A + U_1 \otimes B_2. \\
d(B_2) &= \delta_1^1(E) \otimes B_> + \delta_2^1(E, L_2U_4) \otimes B_1, \\
&= R_2 \otimes S \otimes B_> + U_4 \otimes N \otimes B_1, \\
&= R_2 \otimes B_> + U_4 \otimes B_1. \\
d(B_>) &= \delta_2^1(S, C_{13}) \otimes B_>, \\
&= U_3L_2 \otimes E \otimes B_>, \\
&= L_2U_3 \otimes B_2.
\end{aligned}$$

This once more utilises the observation that elements with weights outside the crossing region commute with the map δ. For example, $\delta_2^1(N, L_3) = L_3 \cdot \delta_2^1(N, 1) = L_3 \cdot 1 \otimes N$. Moreover, tensoring by $\mathcal{P}^1$ swaps the role of 1 and 2, so one has that $\delta_2^1(N, U_1) = U_2 \otimes N$.

Tensoring once more by the DA-bimodule $\mathcal{P}^1$ swaps the role of 1 and 2, and the impact of this is that

$$d(S \otimes B_>) = \delta_2^1(S, C_{14}) = U_4L_2 \otimes E = L_2U_4 \otimes E.$$

Hence, the Type D structure $\left(\mathcal{P}^1\right)^{\boxtimes 3} \boxtimes \Omega^2 \boxtimes \Omega^1$ also conforms to the form described in Figure 5.3.

Using the inductive assumption

Assume that the Type D structure for $\left({}^{\mathcal{A}(2)}\mathcal{P}^1_{\mathcal{A}(2)}\right)^{\boxtimes 2k+1} \boxtimes {}^{\mathcal{A}(2)}\Omega^2 \boxtimes \Omega^1$ is as displayed in Figure 5.3 when $k = n$: i.e. that the statement P_{2n+1} is true. Since B_r and A have idempotents I_{13} and I_{12} respectively, the only compatible generator in $\mathcal{P}^1$ is N.

Tensoring once by $\mathcal{P}$, one has the following calculation for the differential, where $1 \leq r \leq n-1$ in B_{2r+1}, and $1 \leq r \leq n$ for B_{2r}.

$$\begin{aligned}
d(A) &= \delta_2^1(N, R_3U_1) \otimes B_1, \\
&= R_3U_2 \otimes N \otimes B_1. \\
d(B_1) &= \delta_2^1(N, L_3) \otimes A + \delta_2^1(N, U_2) \otimes B_2, \\
&= L_3 \otimes N \otimes A + U_1 \otimes N \otimes B_2, \\
&= L_3 \otimes A + U_1 \otimes B_2. \\
d(B_{2r}) &= \delta_2^1(N, U_4) \otimes B_{2r-1} + \delta_2^1(N, U_1) \otimes B_{2r+1}, \\
&= U_4 \otimes B_{2r-1} + U_2 \otimes B_{2r+1}. \\
d(B_{2r+1}) &= \delta_2^1(N, U_3) \otimes B_{2r} + \delta_2^1(N, U_2) \otimes B_{2r+2}, \\
&= U_3 \otimes B_{2r} + U_1 \otimes B_{2r+2}. \\
d(B_{2n+1}) &= \delta_2^1(N, U_3) \otimes B_{2n} + \delta_2^1(N, R_2) \otimes B_>, \\
&= U_3 \otimes B_{2n} + U_1 \otimes E \otimes B_>, \\
&= U_3 \otimes B_{2n} + U_1 \otimes B_{2n+2}. \\
d(B_{2n+2}) &= \delta_1^1(E) \otimes B_> + \delta_2^1(E, L_2U_4) \otimes B_{2n+1}, \\
&= R_2 \otimes S \otimes B_> + U_4 \otimes N \otimes B_{2n+1}, \\
&= R_2 \otimes B_> + U_4 \otimes B_{2n+1}. \\
d(B_>) &= \delta_2^1(S, C_{13}) \otimes B_>, \\
&= U_3L_2 \otimes E \otimes B_>, \\
&= L_2U_3 \otimes B_{2n+2}.
\end{aligned}$$

The graph describing the Type D structure for $\left(\mathcal{P}^1\right)^{2n+2} \boxtimes \Omega^2 \boxtimes \Omega^1$ is shown in Figure 5.4. The loops on each vertex have once more been suppressed, but one has that $\delta_2^1(X, C_{1p} + C_{2q}) = (C_{1q} + C_{2p}) \otimes X$ for any $X \in \mathcal{P}^1$: i.e. the roles of 1 and 2 are swapped.

This is the intermediate step in the calculation of the Type D structure $\left(\mathcal{P}^1\right)^{2(n+1)+1} \boxtimes \Omega^2 \boxtimes \Omega^1$, as one needs to tensor by $\mathcal{P}$ once more. Luckily, the calculation is practically

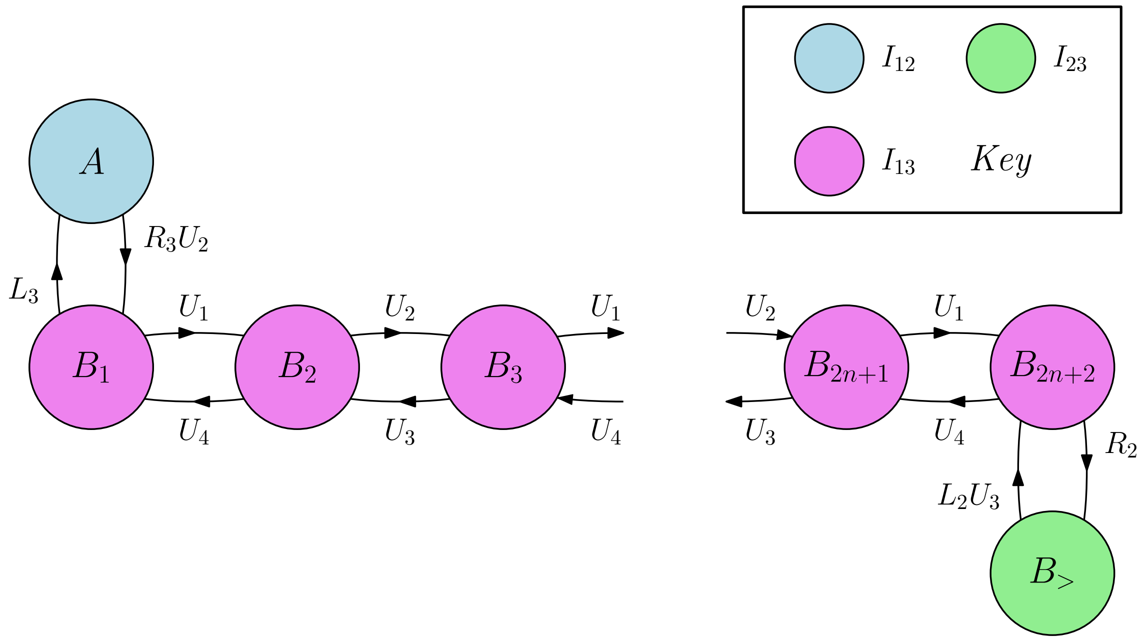


Figure 5.4: *Weighted, directed graph showing the intermediate step in the inductive proof, i.e. the Type D structure with an even number of positive crossings.*

identical to the above, but is included here for the sake of completeness.

$$
\begin{aligned}
d(A) &= \delta_2^1(N, R_3U_2) \otimes B_1, \\
&= R_3U_1 \otimes N \otimes B_1. \\
d(B_1) &= \delta_2^1(N, L_3) \otimes A + \delta_2^1(N, U_1) \otimes B_2, \\
&= L_3 \otimes N \otimes A + U_2 \otimes N \otimes B_2, \\
&= L_3 \otimes A + U_2 \otimes B_2. \\
d(B_{2r}) &= \delta_2^1(N, U_4) \otimes B_{2r-1} + \delta_2^1(N, U_2) \otimes B_{2r+1}, \\
&= U_4 \otimes B_{2r-1} + U_1 \otimes B_{2r+1}. \\
d(B_{2r+1}) &= \delta_2^1(N, U_3) \otimes B_{2r} + \delta_2^1(N, U_1) \otimes B_{2r+2}, \\
&= U_3 \otimes B_{2r} + U_2 \otimes B_{2r+2}. \\
d(B_{2n+2}) &= \delta_2^1(N, U_4) \otimes B_{2n+1} + \delta_2^1(N, R_2) \otimes B_>, \\
&= U_4 \otimes B_{2n+1} + U_1 \otimes E \otimes B_>, \\
&= U_4 \otimes B_{2n+1} + U_1 \otimes B_{2n+3}. \\
d(B_{2n+3}) &= \delta_1^1(E) \otimes B_> + \delta_2^1(E, L_2U_3) \otimes B_{2n+2}, \\
&= R_2 \otimes S \otimes B_> + U_3 \otimes N \otimes B_{2n+2}, \\
&= R_2 \otimes B_> + U_3 \otimes B_{2n+2}.
\end{aligned}
$$

$$\begin{aligned} d(B_>) &= \delta^1_2(S, C_{14}) \otimes B_>, \\ &= U_4L_2 \otimes E \otimes B_>, \\ &= L_2U_4 \otimes B_{2n+3}. \end{aligned}$$

This matches the form in Figure 5.3, as is required for the inductive proof. So, $P_{2n+1} \Rightarrow P_{2(n+1)+1}$, and the type D structure $\left(\mathcal{P}^1\right)^{2(n+1)+1} \boxtimes \Omega^2 \boxtimes \Omega^1$ is thus determined by induction. ■

Remark on the Type D relation.

As one can see from Figure 5.3, it does not appear at first that the Type D relation as defined in [25, Def. 2.18] and represented in Figure 4.7 is satisfied. This can be thought of as analogous to the relation $d^2 = 0$ in a chain complex. In this case, note that $d(d(A))$ contains the term $R_3U_1U_2 \otimes B_2$.

However, as one can see from the relation

$$0 = (\mu_2 \otimes Id_M) \circ (Id_{\mathcal{A}} \otimes \partial^1) \circ \partial^1 + (\mu_1 \otimes Id_M) \circ \partial^1,$$

for this to be a Type D structure the above term must be equal to zero. This is clear after including the information of the idempotents. The term B_2 has associated idempotent $I_{13} \cdot B_2$, and note that the element $I_{13} \cdot U_1U_2 \cdot I_{13}$ has the same weight and associated idempotents as $I_{13} \cdot R_2L_2L_1R_1 \cdot I_{13}$. From Proposition 4.13, this information uniquely determines an element in $\mathcal{A}(2)$, yet $L_2L_1 = 0$, and so $I_{13} \cdot U_1U_2 \cdot I_{13} = 0$.

Traversing two arrows and returning to the same position in the graph is cancelled using the differential $\mu_1 : \mathcal{A} \to \mathcal{A}$; for example one has that $d(d(B_2))$ contains the term $(U_1U_3 + U_4U_2) \otimes B_2$. However, in $\mathcal{A}(2)$, $\mu_1(C_{13} + C_{24}) = U_1U_3 + U_4U_2$, hence this term in $\mu_1 \circ d$ provides the cancellation. It is relatively straightforward to check this relation at all vertices in the directed graph, but this should necessarily be a valid Type D structure as the tensor product of a DA-bimodule and a Type D structure, see [24, Sec. 2.3].

5.3 The next maximum Ω^4

As one can see from Figure 5.1, the form one takes for the special knot diagram of a three strand pretzel knot requires the addition of the bimodule associated to a maximum to give the new fourth and fifth strands. The generators of this bimodule then correspond to the permissible idempotents in this situation.

Since this DA-bimodule is tensored with a Type D structure only featuring the idempotents I_{12}, I_{13} and I_{23}, one only needs to consider generators with incoming idempotents matching these.

Following [49, Sec. 5], one has a map

$$\phi_4 : \{1, 2, 3, 4\} \to \{1, 2, 3, 4, 5, 6\}$$

given by

$$\phi_4(j) = \begin{cases} j & \text{if } j \leq 3 \\ j + 2 & \text{if } j = 4. \end{cases}$$

This corresponds to the new integer assignments of strands after the tensor product with the maximum. Moreover, Ozsváth-Szabó define *allowed idempotent states* in $\mathcal{A}(3)$. If y is a three-element subset of $\{1, 2, \ldots, 6\}$, then y is an allowed idempotent state if $|y \cap \{3, 5\}| \leq 1$. Thinking of this visually, this means that in the regions on either side of the new maximum, at most one of them is occupied. See Figure 5.5 for examples of allowed idempotent states in this case.

Ozsváth-Szabó then divide the generators of the bimodule Ω^4 into classes X, Y and Z, depending upon the intersection of the preferred idempotent with the set $\{3, 4, 5\}$, see [49, Sec. 5].

$$X_{134} := I_{134} \cdot X_{134} \cdot I_{13}$$

$$X_{234} := I_{234} \cdot X_{234} \cdot I_{23}$$

$$Y_{145} := I_{145} \cdot Y_{145} \cdot I_{13}$$

$$Y_{245} := I_{245} \cdot Y_{245} \cdot I_{23}$$

$$Z_{124} := I_{124} \cdot Z_{124} \cdot I_{12}$$

The arrows δ^1_1 in the DA-bimodule split in each of the X, Y, Z cases. In particular, one

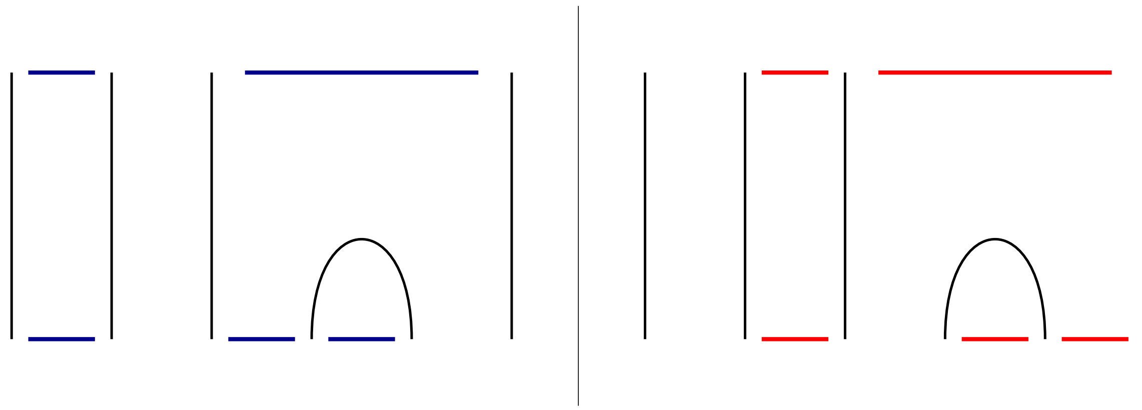


Figure 5.5: *The lower (outgoing) idempotents in this figure are examples of compatible, allowed idempotents in the case of the new maximum. Moreover, these two correspond to generators of the DA-bimodule Ω^4, namely X_{134} and Y_{245}.*

has the following.

$$\delta^1_1(X_{134}) = C_{45} \otimes X_{134} + R_5 R_4 \otimes Y_{145}$$
$$\delta^1_1(X_{234}) = C_{45} \otimes X_{234} + R_5 R_4 \otimes Y_{245}$$
$$\delta^1_1(Y_{145}) = C_{45} \otimes Y_{145} + L_4 L_5 \otimes X_{134}$$
$$\delta^1_1(Y_{245}) = C_{45} \otimes Y_{245} + L_4 L_5 \otimes X_{134}$$
$$\delta^1_1(Z_{124}) = C_{45} \otimes Z_{124}.$$

The remaining arrows δ^1_2 in Ω^4 are defined in [49, Lemma 5.2], which roughly gives a map Φ_x that translates elements from $\mathcal{A}(2)$ to algebra elements in $\mathcal{A}(3)$. Since the full detail is presented in the cited lemma, I will only state that this provides a correspondence between the weights as one would expect from the map ϕ defined above. So, Φ_{145} maps $I_{13} \cdot U_4 \cdot I_{13} \mapsto I_{145} \cdot U_6 \cdot I_{145}$.

There is some subtlety here, since one expects U_4 to map to U_6, but this is only when this is permitted by the idempotents. For example, with the map Φ_{134}, one has that $I_{13} \cdot U_4 \cdot I_{13} \mapsto I_{134} \cdot U_6 \cdot I_{134}$. However, from Definition 4.12, one has that $I_{134} \cdot U_6 \cdot I_{134} = 0$ in $\mathcal{A}(3)$.

The full DA-bimodule used in this case is presented in Figure 5.6, following from simple applications of the definitions in [49, Sec. 5] and [46, Sec. 8]. In particular, note the self-arrows, which are described in the caption. They contain terms from both δ^1_1 arrows, and from δ^1_2.

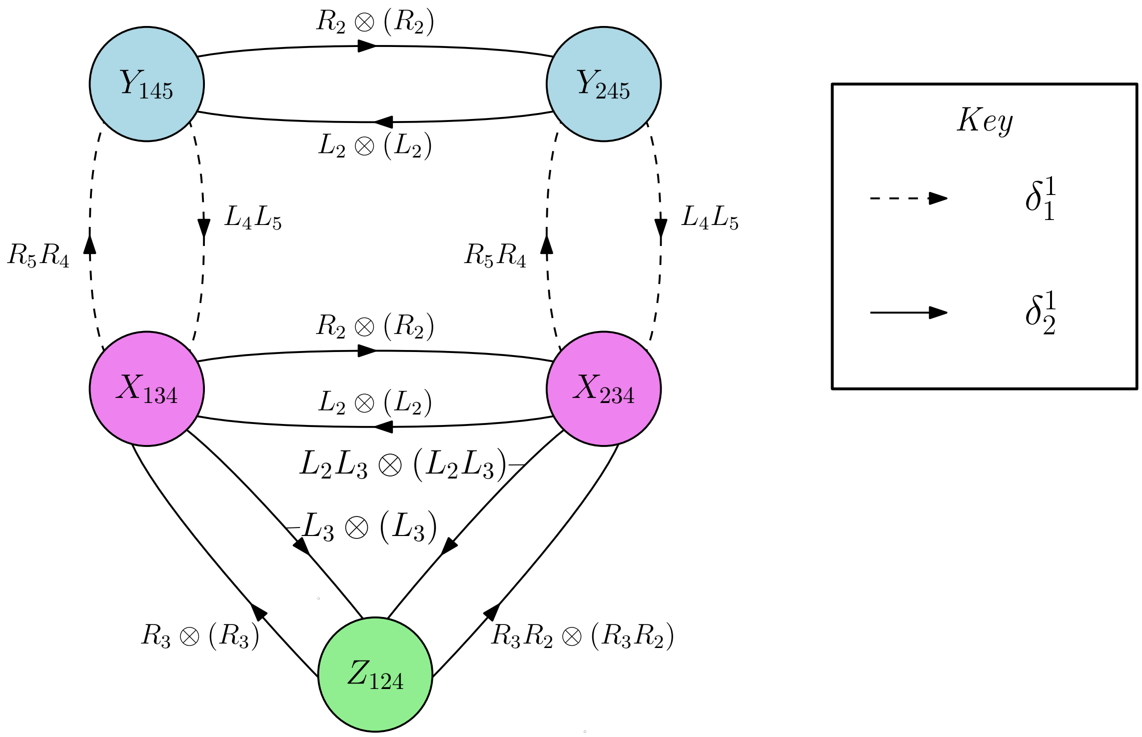


Figure 5.6: *This is a diagramattic representation of the DA-bimodule* Ω^4*. The elements with different outgoing idempotents are distinguished by colour. Moreover, the dashed arrows denote* δ^1_1 *maps, and the solid arrows* δ^1_2 *maps, with the bracketted coefficient being the algebraic input. There are also self arrows, which have coefficient* $C_{45} + C_{13} \otimes (C_{13}) + C_{24} \otimes (C_{26}) + U_{\phi_4(p)} \otimes (U_p)$.

5.3.1 Tensoring with the current Type D structure

As can be seen from the idempotents associated to each generator of the DA-bimodule Ω^4, one has the following set of generators of the Type D structure $\Omega^4 \boxtimes (\mathcal{P}^1)^{2c+1} \boxtimes \Omega^2 \boxtimes \Omega^1$.

$$A := Z_{124} \otimes A$$

$$B_k := X_{134} \otimes B_k$$

$$C_k := Y_{145} \otimes B_k$$

$$B_> := X_{234} \otimes B_>$$

$$C_> := Y_{245} \otimes B_>.$$

The states $B_>$ are upper Kauffman states which will not complete to full Kauffman states for the knot $P(2c + 1, -2b - 1, 2a)$ with diagram as shown in Figure 5.1 after the three minima are added. This can be seen from the fact that the idempotent I_{234} indicates the presence of a marked point in the region adjacent to the first and second set of crossings. Since by construction of the special knot diagram all positive crossings between the first and second strands have already been placed, this indicates that the marked point in this region must be placed adjacent to the second set of (negative) twists. But, since $3 \in \{234\}$, this indicates that all of the marked points on this set of twists must be North N states, which would contradict the position of the remaining marked point. However, it is still important to preserve this state in the calculations, because it may be used in later tensor products.

Proceeding with the calculation of the tensor product, one has the following. Since by construction the Type D structure is standard, if G is an element of the Type D structure, one has that $d(G) \ni (C_{13} + C_{26} + C_{45}) \otimes G$. For the sake of brevity, these terms have been omitted.

$$
\begin{aligned}
d(A) &= \delta_1^1(Z_{124}) \otimes A + \delta_2^1(Z_{124}, R_3U_1) \otimes B_1,\\
&= R_3U_1 \otimes X_{134} \otimes B_1,\\
&= R_3U_1 \otimes B_1.\\
d(B_1) &= \delta_1^1(X_{134}) \otimes B_1 + \delta_2^1(X_{134}, L_3) \otimes A + \delta_2^1(X_{134}, U_2) \otimes B_2,\\
&= R_5R_4 \otimes Y_{145} \otimes B_1 + L_3 \otimes Z_{124} \otimes A + U_2 \otimes X_{134} \otimes B_2,\\
&= R_5R_4 \otimes C_1 + L_3 \otimes A + U_2 \otimes B_2.\\
d(C_1) &= \delta_1^1(Y_{145}) \otimes B_1 + \delta_2^1(Y_{145}, L_3) \otimes A + \delta_2^1(Y_{145}, U_2) \otimes B_2,\\
&= L_4L_5 \otimes X_{134} \otimes B_1 + 0 \otimes A + U_2 \otimes Y_{145} \otimes B_2,\\
&= L_4L_5 \otimes B_1 + U_2 \otimes C_2.\\
d(B_{2r}) &= \delta_1^1(X_{134}) \otimes B_{2r} + \delta_2^1(X_{134}, U_4) \otimes B_{2r-1} + \delta_2^1(X_{134}, U_1) \otimes B_{2r+1}, \quad 1 \leq r \leq c\\
&= R_5R_4 \otimes Y_{145} \otimes B_{2r} + U_6 \otimes X_{134} \otimes B_{2r-1} + U_1 \otimes X_{134} \otimes B_{2r+1},\\
&= R_5R_4 \otimes C_{2r} + 0 \otimes B_{2r-1} + U_1 \otimes B_{2r+1},\\
&= R_5R_4 \otimes C_{2r} + U_1 \otimes B_{2r+1}.\\
d(C_{2r}) &= \delta_1^1(Y_{145}) \otimes B_{2r} + \delta_2^1(Y_{145}, U_4) \otimes B_{2r-1} + \delta_2^1(Y_{145}, U_1) \otimes B_{2r+1}, \qquad 1 \leq r \leq c\\
&= L_4L_5 \otimes X_{134} \otimes B_{2r} + U_6 \otimes Y_{145} \otimes B_{2r-1} + U_1 \otimes Y_{145} \otimes B_{2r+1},\\
&= L_4L_5 \otimes B_{2r} + U_6 \otimes C_{2r-1} + U_1 \otimes C_{2r+1}.\\
d(B_{2r+1}) &= \delta_1^1(X_{134}) \otimes B_{2r+1} + \delta_2^1(X_{134}, U_3) \otimes B_{2r} + \delta_2^1(X_{134}, U_2) \otimes B_{2r+1}, \quad 1 \leq r \leq c
\end{aligned}
$$

$$
\begin{aligned}
&= R_5R_4 \otimes Y_{145} \otimes B_{2r+1} + U_3 \otimes X_{134} \otimes B_{2r} + U_2 \otimes X_{134} \otimes B_{2r+2}, \\
&= R_5R_4 \otimes C_{2r+1} + U_3 \otimes B_{2r} + U_2 \otimes B_{2r+2}. \\
d(C_{2r+1}) &= \delta^1_1(Y_{145}) \otimes B_{2r+1} + \delta^1_2(Y_{145}, U_3) \otimes B_{2r} + \delta^1_2(Y_{145}, U_2) \otimes B_{2r+2}, \qquad 1 \leq r \leq c \\
&= L_4L_5 \otimes X_{134} \otimes B_{2r+1} + U_3 \otimes Y_{145} \otimes B_{2r} + U_2 \otimes Y_{145} \otimes B_{2r+2}, \\
&= L_4L_5 \otimes B_{2r+1} + 0 \otimes C_{2r} + U_2 \otimes C_{2r+2}, \\
&= L_4L_5 \otimes B_{2r+1} + U_2 \otimes C_{2r+2}. \\
d(B_{2c+1}) &= \delta^1_1(X_{134}) \otimes B_{2c+1} + \delta^1_2(X_{134}, U_3) \otimes B_{2c} + \delta^1_2(X_{134}, R_2) \otimes B_{>}, \\
&= R_5R_4 \otimes Y_{145} \otimes B_{2c+1} + U_3 \otimes X_{134} \otimes B_{2c} + R_2 \otimes X_{234} \otimes B_{>}, \\
&= R_5R_4 \otimes C_{2c+1} \otimes U_3 \otimes B_{2c} + R_2 \otimes B_{>}. \\
d(C_{2c+1}) &= \delta^1_1(Y_{145}) \otimes B_{2c+1} + \delta^1_2(Y_{145}, U_3) \otimes B_{2c} + \delta^1_2(Y_{145}, R2) \otimes B_{>}, \\
&= L_4L_5 \otimes X_{134} \otimes B_{2c+1} + U_3 \otimes Y_{145} \otimes B_{2c} + R_2 \otimes Y_{245} \otimes B_{>}, \\
&= L_4L_5 \otimes B_{2c+1} + 0 \otimes C_{2c} + R_2 \otimes C_{>}, \\
&= L_4L_5 \otimes B_{2c+1} + R_2 \otimes C_{>}. \\
d(B_{>}) &= \delta^1_1(X_{234}) \otimes B_{>} + \delta^1_2(X_{234}, L_2U_4) \otimes B_{2c+1}, \\
&= R_5R_4 \otimes Y_{245} \otimes B_{>} + L_2U_6 \otimes X_{134} \otimes B_{2c+1}, \\
&= R_5R_4 \otimes C_{>} + 0 \otimes B_{2c+1}, \\
&= R_5R_4 \otimes C_{>}. \\
d(C_{>}) &= \delta^1_1(Y_{245}) \otimes B_{>} + \delta^1_2(Y_{245}, L_2U_4) \otimes B_{2c+1}, \\
&= L_4L_5 \otimes X_{234} \otimes B_{>} + L_2U_6 \otimes Y_{145} \otimes B_{2c+1}, \\
&= L_4L_5 \otimes B_{>} + L_2U_6 \otimes C_{2c+1}.
\end{aligned}
$$

The above is represented in Figure 5.7, again as a weighted directed graph. Once more, the self-arrows are omitted, but correspond to the matching elements. Moreover, as mentioned previously, not all of the arrows between the B_k and B_{k+1} states in the Type D structure $\left(\mathcal{P}^1\right)^{2c+1} \boxtimes \Omega^2 \boxtimes \Omega^1$ are preserved under tensor product with Ω^4, due to the applications of the relations within the algebra to the idempotents.

The generators of the Type D structure have been labelled in such a way that they indicate the Kauffman state to which they extend in the full special knot diagram. As remarked before, not all of the pictured upper Kauffman states in Figure 5.7 extend to full Kauffman states.

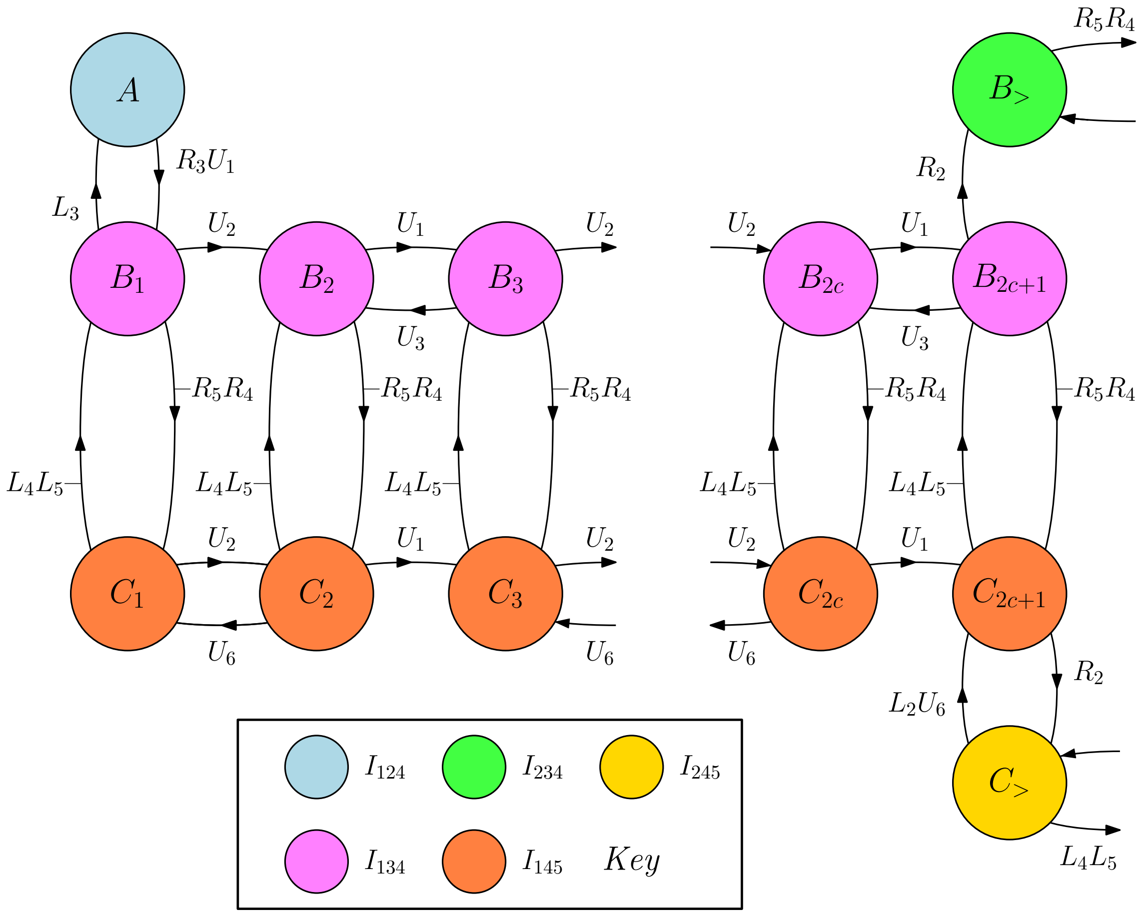


Figure 5.7: *Weighted, directed graph describing the Type D structure* $\Omega^4 \boxtimes \left(\mathcal{P}^1\right)^{2c+1} \boxtimes \Omega^2 \boxtimes \Omega^1$. *Self-arrows corresponding to the matching elements are once more omitted, as this is necessarily a standard Type D structure. The horizontal arrows incident to* $B_>$ *and* $C_>$ *are between these two states.*

Moreover, the same Type D relations are satisfied, despite it appearing at first that they are not. Taking as an example the 'double differential' starting at C_1, one sees that $d(d(C_1)) \ni U_2U_1 \otimes C_3$. But, the fact that the associated idempotent to this algebra element is $I_{145} \cdot U_2U_1 \cdot I_{145}$ determines that this must be zero due to having the same weight and idempotents as an algebra element containing the term $L_2L_1 = 0$.

5.4 Second set of crossings

Remark 5.2 (Remark on the following inductive proofs:) *In the following inductive processes, the DA-bimodules corresponding to the crossings will be box-tensored to the Type D structure one at a time. If* Q_r *is the inductive statement, the proof will at-*

tempt to show that Q_r implies Q_{r+2}. This could also be attempted by first determining the DA-bimodule for the box-tensor product of the two crossings, i.e. for negative crossings determining $\mathcal{N} \boxtimes \mathcal{N}$. As shown in [24], this is a DA-bimodule, which one can then box-tensor with a Type D structure to yield another Type D structure.

However, the benefit of doing this one bimodule at a time is that one need not specialise to the parity case considered. Here, the special knot diagram constructed will be for the knot $P(2c+1, -2b-1, 2a)$, so the next step is to take an odd number of negative crossings, then an even number of positive crossings. If the intermediate Type D structures are determined, one could then examine other cases. For example, although in [41] the $\widehat{HFK}(P(odd, -odd, odd))$ is determined, by fully defining the intermediate Type D structure one could use this to then determine the bordered invariant $\mathcal{C}(D)$ for this family.

Before proceeding with the calculation of the Type D structure after the second set of crossings, note that the DA-bimodule now utilised in the tensor product is $\mathcal{N}^3$, describing a negative crossing between strands 3 and 4. Although this is given a formal description in [46, Sec. 5.5], recall from Definition 4.29 this bimodule is the opposite of $\mathcal{P}^3$. Intuitively, one yields the description of $\mathcal{N}^3$ from $\mathcal{P}^3$ by reversing the direction of all arrows, and swapping L for R.

For ease of notation, define the module ${}^{\mathcal{A}(3)}X = {}^{\mathcal{A}(3)}\Omega^4 \boxtimes \left(\mathcal{P}^1\right)^{2c+1} \boxtimes \Omega^2 \boxtimes \Omega^1$ as the Type D structure defined through induction in Subsection 5.3.1. The aim is now to determine the Type D structure for $\left(\mathcal{N}^3\right)^{2b+1} \boxtimes X$ using similar methods.

The upper Kauffman states that correspond to generators of this module can again be split depending on the position of the marked points in each region. Figure 5.8 depicts the possible different positions of the marked points after adding additional crossings between strands 3 and 4. There are also other upper Kauffman states not depicted in this figure, such as $I_{234} \cdot B_{>}$. This has all S generators on the first set of crossings, and all N generators on the second set of crossings, and will not extend to a full Kauffman state of the three-strand pretzel knot $P\,(2c+1, -2b-1, 2a)$.

Before making the inductive statement, it is informative to determine the Type D structure $\mathcal{N}^3 \boxtimes X$, where X is defined as above. This has $(6c+10)$ generators, as can be seen from an examination of the possible upper Kauffman states. Using the description given in Figure 5.8, one has the following collection of upper Kauffman states corresponding to

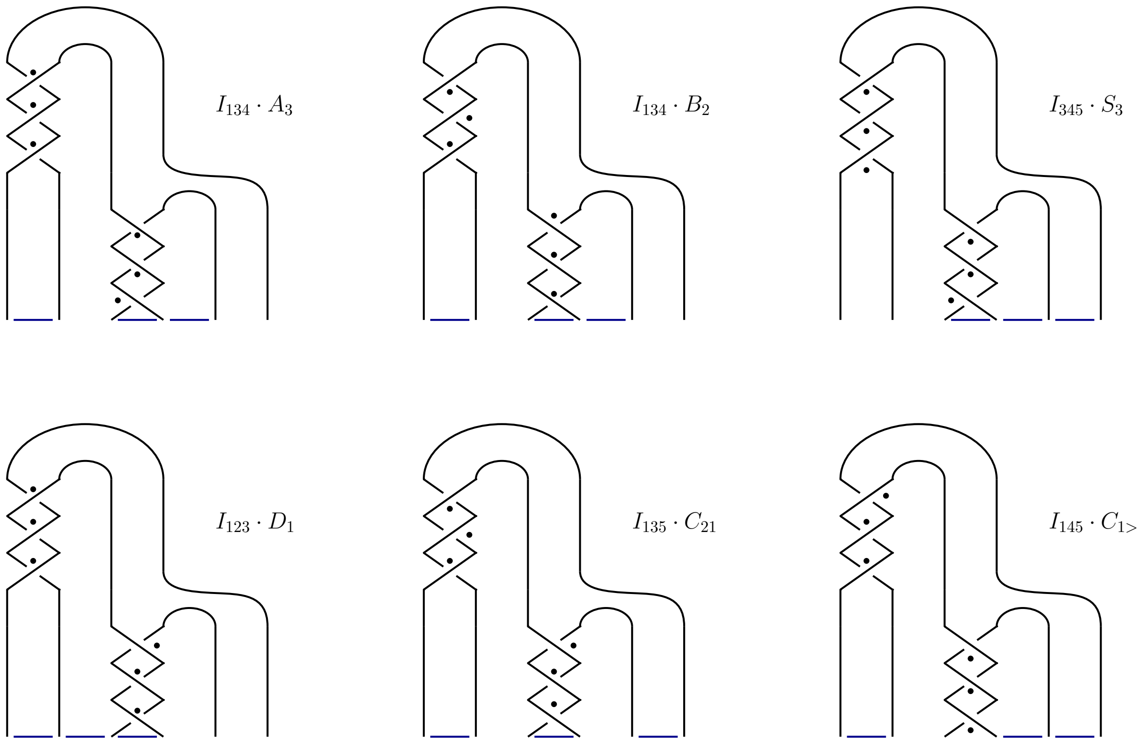


Figure 5.8: *Diagrams depicting the different categories into which Kauffman states representing the generators of* $\left(\mathcal{N}^3\right)^{2b+1} \boxtimes X$ *may fall. As described above, there are also* $>$ *states, such as the* $C_{1>}$ *state pictured. The associated idempotents to each generator have been shown.*

generators of this module.

$$\{A_1, A_>, D_1\} \quad \cup$$

$$\{B_1, \ldots, B_{2c+1}, B_>\} \quad \cup$$

$$\{C_{11}, \ldots, C_{2c+1,1}, C_{>,1}, C_{1,>}, \ldots, C_{2c+1,>}, C_{>,>}, S_1\}.$$

The idempotents force the position of Kauffman states in the tensored DA-bimodule $\mathcal{N}$, or rather restrict the positions in which they might be. Using the idempotents noted in Figure 5.7, since 3 belongs to the idempotents of both B_k and $B_>$, non-zero tensor products will have the generator N in the $\mathcal{N}$ tensor coordinate. In a similar way: $I_{124} \cdot A$ forces either E, S or W in the tensor product; $I_{145} \cdot C_k$ forces E or S; and $I_{245} \cdot C_>$ forces E, S or W.

One then has the following calculation of the differential map d in $\mathcal{N} \boxtimes X$, which is also presented as a weighted directed graph in Figure 5.9. Matching terms are omitted, for ease of notation.

$$\begin{aligned}
d(A_1) &= d(W \otimes A) \\
&= \delta^1_2(W, R_3U_1) \otimes B_1 + \delta^1_2(W, C_{45}) \otimes A + \delta^1_3(W, R_3U_1, R_5R_4) \otimes C_1 \\
&= U_1U_4 \otimes N \otimes B_1 + L_3U_5 \otimes S \otimes A + R_5R_4U_1 \otimes S \otimes C_1 \\
&= U_1U_4 \otimes B_1 + L_3U_5 \otimes A_> \otimes R_5R_4U_1 \otimes C_{1,>} \\
d(A_>) &= d(S \otimes A) \\
&= \delta^1_1(S) \otimes A \\
&= R_3 \otimes W \otimes A + L_4 \otimes E \otimes A \\
&= R_3 \otimes A_1 + L_4 \otimes D_1 \\
d(D_1) &= d(E \otimes A) \\
&= \delta^1_2(E, C_{13}) \otimes A + \delta^1_2(E, R_3U_1) \otimes B_1 \\
&= R_4U_1 \otimes S \otimes A + R_4R_3U_1 \otimes N \otimes B_1 \\
&= R_4U_1 \otimes A_> + R_4R_4U_1 \otimes B_1 \\
d(B_1) &= d(N \otimes B_1) \\
&= \delta^1_2(N, L_3) \otimes A + \delta^1_2(N, U_2) \otimes B_2 + \delta^1_2(N, \otimes R_5R4) \otimes C_1 \\
&= 1 \otimes W \otimes A + U_2 \otimes N \otimes B_2 + R_5 \otimes E \otimes C_1 \\
&= 1 \otimes A_1 + U_2 \otimes B_2 + R_5 \otimes C_{11}. \\
d(B_{2r}) &= d(N \otimes B_{2r}) \\
&= \delta^1_2(N, U_1) \otimes B_{2r+1} + \delta^1_2(N, R_5R_4) \otimes C_{2r} \\
&= U_1 \otimes N \otimes B_{2r+1} + R_5 \otimes E \otimes C_{2r} \\
&= U_1 \otimes B_{2r+1} + R_5 \otimes C_{2r,1} \\
d(B_{2r+1}) &= d(N \otimes B_{2r+1}) \\
&= \delta^1_2(N, U_3) \otimes B_{2r} + \delta^1_2(N, U_2) \otimes B_{2r+2} \\
&\qquad + \delta^1_2(N, R_5R_4) \otimes C_{2r+1} + \delta^1_3(N, U_3, R_5R_4) \otimes C_{2r} \\
&= U_4 \otimes N \otimes B_{2r} + U_2 \otimes N \otimes B_{2r+2} + R_5 \otimes E \otimes C_{2r+1} + R_5R_4 \otimes S \otimes C_{2r} \\
&= U_4 \otimes B_{2r} + U_2 \otimes B_{2r+2} + R_5 \otimes C_{2r+1} + R_5R_4 \otimes C_{2r,>} \\
d(B_{2c+1}) &= d(N \otimes B_{2c+1}) \\
&= \delta^1_2(N, U_3) \otimes B_{2c} + \delta^1_2(N, R_5R_4) \otimes C_{2c+1} + \delta^1_2(N, R_2) \otimes B_> \\
&\qquad + \delta^1_3(N, U_3, R_5R_4) \otimes C_{2c}
\end{aligned}$$

$$
\begin{aligned}
&= U_4 \otimes N \otimes B_{2c} + R_5 \otimes E \otimes C_{2c+1} + R_2 \otimes N \otimes B_> + R_5R_4 \otimes S \otimes C_{2c} \\
&= U_4 \otimes B_{2c} + R_5 \otimes C_{2c+1,1} + R_2 \otimes B_> + R_5R_4 \otimes C_{2c,>} \\
d(B_>) &= d(N \otimes B_>) = \delta_2^1(N, R_5R_4) \otimes C_> \\
&= R_5 \otimes E \otimes C_> = R_5 \otimes C_{>,1} \\
d(C_{11}) &= d(E \otimes C1) \\
&= \delta_2^1(E, L_4L_5) \otimes B_1 + \delta_2^1(E, U_2) \otimes C_2 + \delta_3^1(E, L_4L_5, L_3) \otimes A + \delta_2^1(E, C_{13}) \otimes C_1 \\
&= L_5U_3 \otimes N \otimes B_1 + U_2 \otimes N \otimes C_2 + L_3L_5 \otimes S \otimes A + R_4U_1 \otimes S \otimes C_1 \\
&= L_5U_3 \otimes B_1 + U_2 \otimes C_{21} + L_3L_5 \otimes A_> + R_4U_1 \otimes C_{1,>} \\
d(C_{2r,1}) &= d(E \otimes C_{2r}) \\
&= \delta_2^1(E, U_6) \otimes C_{2r-1} + \delta_2^1(E, U_1) \otimes C_{2r+1} + \delta_2^1(E, L_4L_5) \otimes B_{2r} \\
&\qquad + \delta_2^1(E, C_{13}) \otimes C_{2r} \\
&= U_6 \otimes E \otimes C_{2r-1} + U_1 \otimes E \otimes C_{2r+1} + L_5U_3 \otimes N \otimes B_{2r} + R_4U_1 \otimes S \otimes C_{2r} \\
&= U_6 \otimes C_{2r-1,1} + U_1 \otimes C_{2r+1,1} + L_5U_3 \otimes B_{2r} + R_4U_1 \otimes C_{2r,>} \\
d(C_{2r+1,1}) &= d(E \otimes C_{2r+1}) \\
&= \delta_2^1(E, U_2) \otimes C_{2r+2} + \delta_2^1(E, C_{13}) \otimes C_{2r+1} + \delta_2^1(E, L_4L_5) \otimes B_{2r+1} \\
&= U_2 \otimes E \otimes C_{2r+2} + R_4U_1 \otimes S \otimes C_{2r+1} + L_5U_3 \otimes N \otimes B_{2r+1} \\
&= U_2 \otimes C_{2r+2,1} + R_4U_1 \otimes C_{2r+1,>} + L_5U_3 \otimes B_{2r+1} \\
d(C_{2c+1,1}) &= d(E \otimes C_{2c+1}) \\
&= \delta_2^1(E, R_2) \otimes C_> + \delta_2^1(E, C_{13}) \otimes C_{2c+1} + \delta_2^1(E, L_4L_5) \otimes B_{2c+1} \\
&= R_2 \otimes E \otimes C_> + R_4U_1 \otimes S \otimes C_{2c+1} + L_5U_3 \otimes N \otimes B_{2c+1} \\
&= R_2 \otimes C_{>,1} + R_4U_1 \otimes C_{2c+1,>} + L_5U_3 \otimes B_{2c+1} \\
d(C_{>,1}) &= d(E \otimes C_>) \\
&= \delta_2^1(E, L_2U_6) \otimes C_{2c+1} + \delta_2^1(E, L_4L_5) \otimes B_> + \delta_2^1(E, C_{13}) \otimes C_> \\
&= L_2U_6 \otimes E \otimes C_{2c+1} + L_5U_3 \otimes N \otimes B_> + R_4U_1 \otimes (I_{245} \cdot C_>) \\
&= L_2U_6 \otimes C_{2c+1,1} + L_5U_3 \otimes B_> + 0 \otimes C_> \\
d(C_{2r+1,>}) &= d(S \otimes C_{2r+1}) \\
&= \delta_1^1(S) \otimes C_{2r+1} + \delta_2^1(S, U_2) \otimes C_{2r+2} \\
&= L_4 \otimes E \otimes C_{2r+1} + U_2 \otimes S \otimes C_{2r+2} \\
&= L_4 \otimes C_{2r+1,1} + U_2 \otimes C_{2r+2,>}
\end{aligned}
$$

$$
\begin{aligned}
d(C_{2r,>}) &= d(S \otimes C_{2r}) \\
&= \delta_1^1(S) \otimes C_{2r} + \delta_2^1(S, U_6) \otimes C_{2r-1} + \delta_2^1(S, U_1) \otimes C_{2r+1} \\
&= L_4 \otimes E \otimes C_{2r} + U_6 \otimes S \otimes C_{2r-1} + U_1 \otimes S \otimes C_{2r+1} \\
&= L_4 \otimes C_{2r,1} + U_6 \otimes C_{2r-1,>} + U_1 \otimes C_{2r+1,>} \\
d(C_{>,>}) &= d(S \otimes C_>) \\
&= \delta_1^1(S) \otimes C_> + \delta_2^1(S, L_2U_6) \otimes C_{2c+1} \\
&= L_4 \otimes E \otimes C_> + R_3 \otimes W \otimes C_> + L_2U_6 \otimes S \otimes C_{2c+1} \\
&= L_4 \otimes C_{>,1} + R_3 \otimes S_1 + L_2U_6 \otimes C_{2c+1,>}. \\
d(S_1) &= d(W \otimes C_>) \\
&= \delta_2^1(W, C_{45}) \otimes C_> + \delta_2^1(W, L_4L_5) \otimes B_> \\
&= L_3U_5 \otimes S \otimes C_> + L_3L_4L_5 \otimes N \otimes B_> \\
&= L_3U_5 \otimes C_{>,>} + L_3L_4L_5 \otimes B_>.
\end{aligned}
$$

Note, the idempotents do restrict certain differentials being present. For example, in the term $d(C_{>,1})$ one has that $R_4U_1 \otimes C_> = 0$ since $I_{245} \cdot U_1 \cdot I_{245} = 0$. Moreover, except in the case of $C_{>,>}$, the term $\delta_1^1(S) = L_4 \otimes E$ by virtue of the idempotents enforcing that $R_3 \otimes W = 0$.

5.4.1 Inductive statement

Let P_{2b+1} be the inductive statement that the Type D structure for

$$\left(\mathcal{N}^3\right)^{2b+1} \boxtimes X$$

is as displayed in Figure 5.10. Note, that the idempotents do not change from the case with a single added negative crossing, and once more, the structure in the blue box can be copied and pasted for the remaining rows. Once more, since by construction all of the Type D structures are standard, the self-arrows have been suppressed.

5.4.2 Base case

The base case in this inductive proof is shown by calculating the Type D structure for $k = 1$, that is determining the bimodule $\left(\mathcal{N}^3\right)^{\boxtimes 3} \boxtimes X$ takes the form as demonstrated in the inductive statement and Figure 5.10.

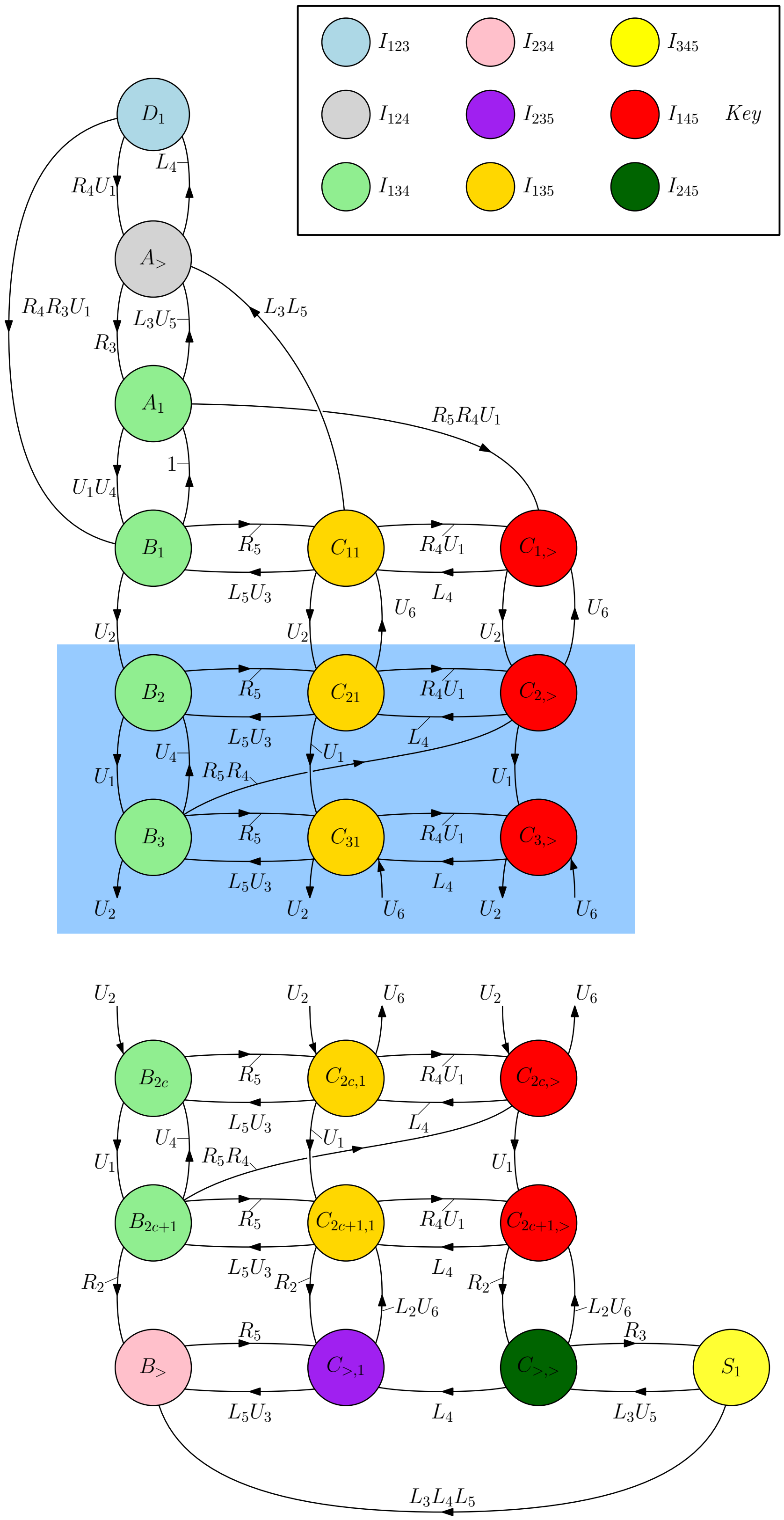


Figure 5.9: *Directed weighted graph describing the Type D structure $\mathcal{N} \boxtimes X$. Note that the boxed area is not dependent on the position of the marked state on the first set of crossings. Hence, to determine the full Type D structure, simply copy and paste this section until $r = c$.*

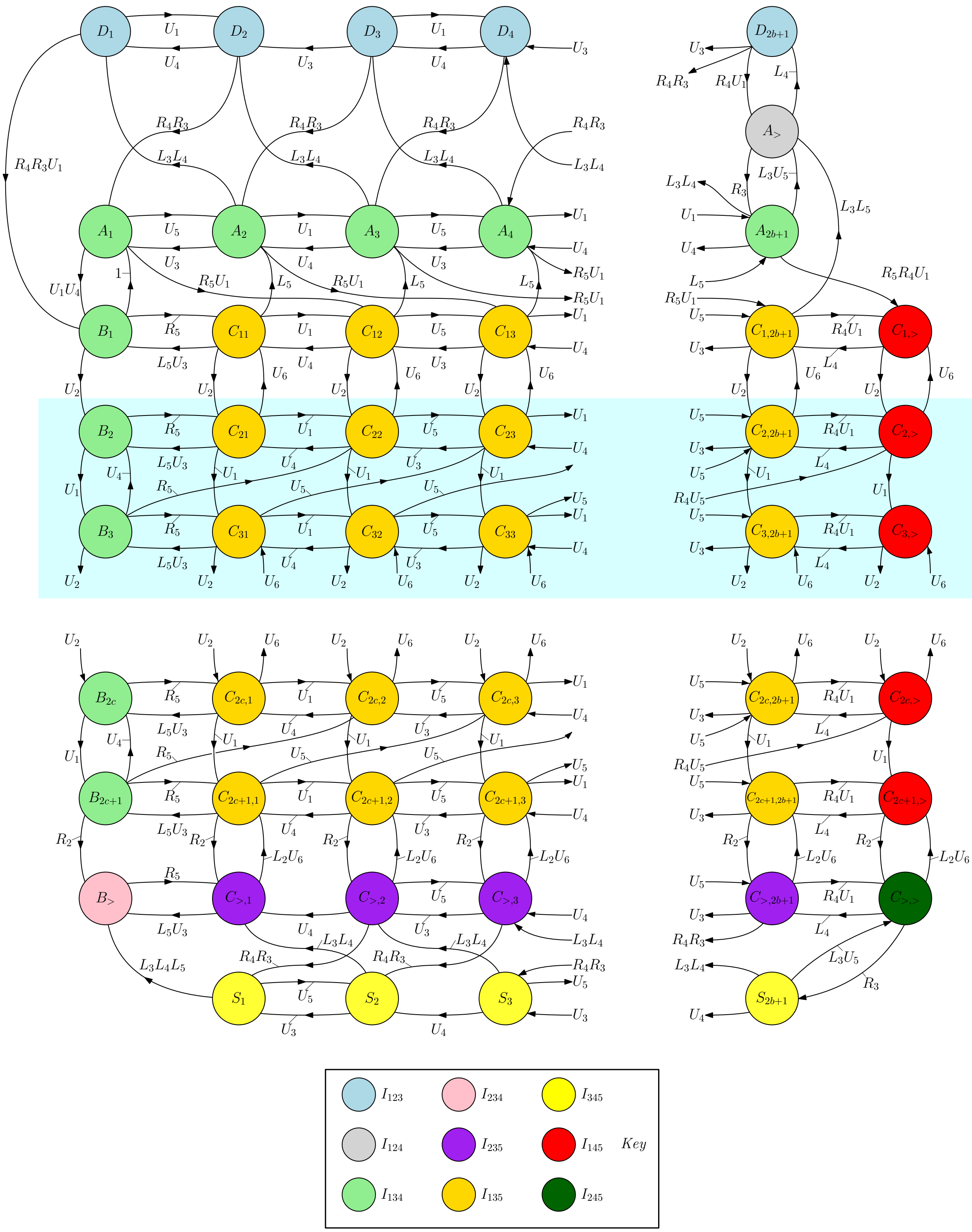


Figure 5.10: *Directed weighted graph displaying the Type D structure for the module $(\mathcal{N}^3)^{2b+1} \boxtimes X$. The blue highlighted box can be copied and pasted for the rows $(2k, 2k+1)$, as this structure is inherited from the first set of crossings.*

In the following, it is instructive to note that the number of B_ℓ states does not change, because there is only a single choice of generator for $\mathcal{N}^3$ corresponding to a cardinal direction that yields a non-trivial tensor product, namely $I_{134} \cdot N \cdot I_{134}$. Moreover, any algebra element with weight outside of $\operatorname{span}\{e_3, e_4\}$ commutes with the map $\delta^1 = \sum_k \delta^1_k$ in $\left(\mathcal{N}, \delta^1\right)$.

Since the Type D structure $\mathcal{N} \boxtimes X$ has already been determined, as shown in Figure 5.9. The determination of this base case then proceeds in two steps: determining $\left(\mathcal{N}^3\right)^{\boxtimes 2} \boxtimes X$, then $\left(\mathcal{N}^3\right)^{\boxtimes 3} \boxtimes X$. Determining that $P_1 \Rightarrow P_3$ requires tensoring by the DA-bimodule $\mathcal{N}^3$ twice. The taking of the second tensor product is nearly identical to the first, with some care needed only when considering the 'right' hand edge of the diagram: for example the map $\delta^1_2(E, C_{3p}) = R_4 U_p \otimes S$ is sometimes zero based upon the value of p and the associated idempotents of the generator in the Type D structure.

First tensor product with $\mathcal{N}$

The generators of the Type D structure are as displayed in Figure 5.8, and are simply enumerated by determining the possible cardinal generators of $\mathcal{N}$ that pair with generators of the Type D structure $\mathcal{N} \boxtimes X$. With a slight abuse of notation, these are:

$$
\begin{aligned}
A_1 &= N \otimes A_1 \\
A_2 &= W \otimes A_> \\
A_> &= S \otimes A_> \\
D_1 &= N \otimes D_1 \\
D_2 &= E \otimes A_> \\
B_k &= N \otimes B_k \\
B_> &= N \otimes B_> \\
C_{k1} &= N \otimes C_{k1} \\
C_{k2} &= E \otimes C_{k,>} \\
C_{k,>} &= S \otimes C_{k,>} \\
C_{>,1} &= N \otimes C_{>,1} \\
C_{>,2} &= E \otimes C_{>,>} \\
C_{>,>} &= S \otimes C_{>,>}
\end{aligned}
$$

$$S_1 = N \otimes S_1$$

$$S_2 = W \otimes C_{>,>}.$$

The idempotents of these generators are shown in the key of Figure 5.10.

The maps d in the Type D structure are then calculated as follows. It is important that one considers the idempotents and the algebra relations in $\mathcal{A}$, since in the calculation of $d(D_2)$ one has that the algebra coefficient of the map to $A_>$ must be zero, since $I_{123} \cdot R_4U_5 \cdot I_{124}$ has the same weight and idempotents as $I_{123} \cdot R_4R_5L_5 \cdot I_{124}$.

$$\begin{aligned}
d(A_1) &= \delta_2^1(N, U_1U_4) \otimes B_1 + \delta_2^1(N, L_3U_5) \otimes A_> + \delta_2^1(N, R_5R_4U_1) \otimes C_{1,>} \\
&= U_1U_3 \otimes N \otimes B_1 + U_5 \otimes W \otimes A_> + R_5U_1 \otimes E \otimes C_{1,>} \\
&= U_1U_3 \otimes B_1 + U_5 \otimes A_2 + R_5U_1 \otimes C_{12}. \\
d(A_2) &= \delta_2^1(W, R_3) \otimes A_1 + \delta_2^1(W, L_4) \otimes D_1 + \delta_2^1(W, C_{14}) \otimes A_> \\
&\quad + \delta_3^1(W, R_3, R_5R_4U_1) \otimes C_{1,>} \\
&= U_4 \otimes N \otimes A_1 + L_3L_4 \otimes N \otimes D_1 + U_1L_3 \otimes S \otimes A_> \\
&\quad + R_5R_4U_1 \otimes S \otimes C_{1,>} \\
&= U_4 \otimes A_1 + L_3L_4 \otimes D_1 + L_3U_1 \otimes A_> + R_5R_4U_1 \otimes C_{1,>}. \\
d(A_>) &= \delta_1^1(S) \otimes A_> \\
&= R_3 \otimes W \otimes A_> + L_4 \otimes E \otimes A_> \\
&= R_3 \otimes A_2 + L_4 \otimes D_2. \\
d(D_1) &= \delta_2^1(N, R_4R_3U_1) \otimes B_1 + \delta_2^1(N, R_4U_1) \otimes A_> \\
&= R_4R_3U_1 \otimes N \otimes B_1 + U_1 \otimes E \otimes A_> \\
&= R_4R_3U_1 \otimes B_1 + U_1 \otimes D_2. \\
d(D_2) &= \delta_2^1(E, L_4) \otimes D_1 + \delta_2^1(E, R_3) \otimes A_1 + \delta_2^1(E, C_{35}) \otimes A_> \\
&= U_1 \otimes N \otimes D_1 + R_4R_3 \otimes N \otimes A_1 + R_4U_5 \otimes S \otimes A_> \\
&= U_1 \otimes D_1 + R_4R_3 \otimes A_1 + I_{123} \cdot R_4U_5 \cdot I_{124} \otimes A_> \\
&= U_1 \otimes D_1 + R_4R_3 \otimes A_1 + \mathbf{0} \otimes A_>. \\
d(B_1) &= \delta_2^1(N, 1) \otimes A_1 + \delta_2^1(N, R_5) \otimes C_{11} + \delta_2^1(N, U_2) \otimes B_2 \\
&= 1 \otimes N \otimes A_1 + R_5 \otimes N \otimes C_{11} + U_2 \otimes N \otimes B_2 \\
&= 1 \otimes A_1 + R_5 \otimes C_{11} + U_2 \otimes B_2. \\
d(B_{2r}) &= \delta_2^1(N, R_5) \otimes C_{2r,1} + \delta_2^1(N, U_1) \otimes B_{2r+1}
\end{aligned}$$

$$
\begin{aligned}
&= R_5 \otimes N \otimes C_{2r,1} + U_1 \otimes N \otimes B_{2r+1} \\
&= R_5 \otimes C_{2r,1} + U_1 \otimes B_{2r+1}. \\
d(B_{2r+1}) &= \delta_2^1(N, U_4) \otimes B_{2r} + \delta_2^1(N, U_2) \otimes B_{2r+2} + \delta_2^1(N, R_5) \otimes C_{2r+1,1} \\
&\quad + \delta_2^1(N, R_5R_4) \otimes C_{2r,>} \\
&= U_3 \otimes N \otimes B_{2r} + U_2 \otimes N \otimes B_{2r+2} + R_5 \otimes N \otimes C_{2r+1,1} \\
&\quad + R_5 \otimes E \otimes C_{2r,>} \\
&= U_3 \otimes B_{2r} + U_2 \otimes B_{2r+2} + R_5 \otimes C_{2r+1,1} + R_5 \otimes C_{2r,2}. \\
d(B_{2c+1}) &= U_3 \otimes B_{2c} + R_5 \otimes C_{2c+1,1} + R_5 \otimes C_{2c,2} + \delta_2^1(N, R_2) \otimes B_> \\
&= U_3 \otimes B_{2c} + R_5 \otimes C_{2c+1,1} + R_5 \otimes C_{2c,2} + R_2 \otimes B_>. \\
d(B_>) &= \delta_2^1(N, R_5) \otimes C_{>,1} \\
&= R_5 \otimes N \otimes C_{>,1} = R_5 \otimes C_{>,1}. \\
d(C_{11}) &= \delta_2^1(N, L_3L_5) \otimes A_> + \delta_2^1(N, L_5U_3) \otimes B_1 + \delta_2^1(N, U_2) \otimes C_{21} \\
&\quad + \delta_2^1(N, R_4U_1) \otimes C_{1,>} \\
&= L_5 \otimes W \otimes A_> + L_5U_4 \otimes N \otimes B_1 + U_2 \otimes N \otimes C_{21} + U_1 \otimes E \otimes C_{1,>} \\
&= L_5 \otimes A_2 + L_5U_4 \otimes B_1 + U_2 \otimes C_{21} + U_1 \otimes C_{12}. \\
d(C_{12}) &= \delta_2^1(E, L_4) \otimes C_{11} + \delta_2^1(E, U_2) \otimes C_{2,>} + \delta_3^1(E, L_4, L_3L_5) \otimes A_> \\
&\quad + \delta_2^1(E, C_{35}) \otimes C_{1,>} \\
&= U_3 \otimes N \otimes C_{11} + U_2 \otimes E \otimes C_{2,>} + L_3L_5 \otimes S \otimes A_> + R_4U_5 \otimes S \otimes C_{1,>} \\
&= U_3 \otimes C_{11} + U_2 \otimes C_{22} + L_3L_5 \otimes A_> + R_4U_5 \otimes C_{1,>}. \\
d(C_{1,>}) &= \delta_1^1(S) \otimes C_{1,>} + \delta_2^1(S, U_2) \otimes C_{2,>} \\
&= L_4 \otimes E \otimes C_{1,>} + U_2 \otimes S \otimes C_{2,>} \\
&= L_4 \otimes C_{12} + U_2 \otimes C_{2,>}. \\
d(C_{2r,1}) &= \delta_2^1(N, U_6) \otimes C_{2r-1,1} + \delta_2^1(N, L_5U_3) \otimes B_{2r} + \delta_2^1(N, R_4U_1) \otimes C_{2r,>} \\
&\quad + \delta_2^1(N, U_1) \otimes C_{2r+1,1} \\
&= U_6 \otimes N \otimes C_{2r-1,1} + L_5U_4 \otimes N \otimes B_{2r} + U_1 \otimes E \otimes C_{2r,>} \\
&\quad + U_1 \otimes N \otimes C_{2r+1,1} \\
&= U_6 \otimes C_{2r-1,1} + L_5U_4 \otimes B_{2r} + U_1 \otimes C_{2r,2} + U_1 \otimes C_{2r+1,1}. \\
d(C_{2r,2}) &= \delta_2^1(E, L_4) \otimes C_{2r,1} + \delta_2^1(E, U_6) \otimes C_{2r-1,>} + \delta_2^1(E, U_1) \otimes C_{2r+1,>} \\
&\quad + \delta_2^1(E, C_{35}) \otimes C_{2r,>}
\end{aligned}
$$

$$
\begin{aligned}
&= U_3 \otimes N \otimes C_{2r,1} + U_6 \otimes E \otimes C_{2r-1,>} + U_1 \otimes E \otimes C_{2r+1,>} \\
&\quad + R_4U_5 \otimes S \otimes C_{2r,>} \\
&= U_3 \otimes C_{2r,1} + U_6 \otimes C_{2r-1,2} + U_1 \otimes C_{2r+1,2} + R_4U_5 \otimes C_{2r,>}. \\
d(C_{2r,>}) &= \delta_1^1(S) \otimes C_{2r,>} + \delta_2^1(S, U_6) \otimes C_{2r-1,>} + \delta_2^1(S, U_1) \otimes C_{2r+1,>} \\
&= L_4 \otimes E \otimes C_{2r,>} + U_6 \otimes S \otimes C_{2r-1,>} + U_1 \otimes S \otimes C_{2r+1,>} \\
&= L_4 \otimes C_{2r,2} + U_6 \otimes C_{2r-1,>} + U_1 \otimes C_{2r+1,>}. \\
d(C_{2r+1,1}) &= \delta_2^1(N, L_5U_3) \otimes B_{2r+1} + \delta_2^1(N, R_4U_1) \otimes C_{2r+1,>} + \delta_2^1(N, U_2) \otimes C_{2r+2,1} \\
&\quad + \delta_3^1(N, L_5U_3, R_5R_4) \otimes C_{2r,>} \\
&= L_5U_4 \otimes N \otimes B_{2r+1} + U_1 \otimes E \otimes C_{2r+1,>} + U_2 \otimes N \otimes C_{2r+2,1} \\
&\quad + L_5R_5R_4 \otimes S \otimes C_{2r,>} \\
&= L_5U_4 \otimes B_{2r+1} + U_1 \otimes C_{2r+1,2} + U_2 \otimes C_{2r+2,1} + R_4U_5 \otimes C_{2r,>}. \\
d(C_{2r+1,2}) &= \delta_2^1(E, L_4) \otimes C_{2r+1,1} + \delta_2^1(E, U_2) \otimes C_{2r+2,>} + \delta_2^1(E, C_{35}) \otimes C_{2r+1,>} \\
&= U_3 \otimes N \otimes C_{2r+1,1} + U_2 \otimes E \otimes C_{2r+2,>} + R_4U_5 \otimes S \otimes C_{2r+1,>} \\
&= U_3 \otimes C_{2r+1,1} + U_2 \otimes C_{2r+2,2} + R_4U_5 \otimes C_{2r+1,>}. \\
d(C_{2r+1,>}) &= \delta_1^1(S) \otimes C_{2r+1,>} + \delta_2^1(S, U_2) \otimes C_{2r+2,>} \\
&= L_4 \otimes E \otimes C_{2r+1,>} + U_2 \otimes S \otimes C_{2r+2,>} \\
&= L_4 \otimes C_{2r+1,2} + U_2 \otimes C_{2r+2,>}. \\
d(C_{2c+1,1}) &= \delta_2^1(N, L_5U_3) \otimes B_{2c+1} + \delta_2^1(N, R_2) \otimes C_{>,1} + \delta_2^1(N, R_4U^1) \otimes C_{2c+1,>} \\
&\quad + \delta_3^1(N, L_5U_3, R_5R_4) \otimes C_{2c,>} \\
&= L_5U_4 \otimes N \otimes B_{2c+1} + R_2 \otimes N \otimes C_{>,1} + U_1 \otimes E \otimes C_{2c+1,>} \\
&\quad + R_4U_5 \otimes S \otimes C_{2c,>} \\
&= L_5U_4 \otimes B_{2c+1} + R_2 \otimes C_{>,1} + U_1 \otimes C_{2c+1,2} + R_4U_5 \otimes C_{2c,>}. \\
d(C_{2c+1,2}) &= \delta_2^1(E, L_4) \otimes C_{2c+1,1} + \delta_2^1(E, R_2) \otimes C_{>,>} + \delta_2^1(E, C_{35}) \otimes C_{2c+1,>} \\
&= U_3 \otimes N \otimes C_{2c+1,1} + R_2 \otimes E \otimes C_{>,>} + R_4U_5 \otimes S \otimes C_{2c+1,>} \\
&= U_3 \otimes C_{2c+1,1} + R_2 \otimes C_{>,2} + R_4U_5 \otimes C_{2c+1,>}. \\
d(C_{2c+1,>}) &= \delta_1^1(S) \otimes C_{2c+1,>} + \delta_2^1(S, R_2) \otimes C_{>,>} \\
&= L_4 \otimes E \otimes C_{2c+1,>} + R_2 \otimes S \otimes C_{>,>} \\
&= L_4 \otimes C_{2c+1,2} + R_2 \otimes C_{>,>}. \\
d(C_{>,1}) &= \delta_2^1(N, L_5U_3) \otimes B_{>} + \delta_2^1(N, L_2U_6) \otimes C_{2c+1,1}
\end{aligned}
$$

$$\begin{aligned}
&= L_5U_4 \otimes N \otimes B_> + L_2U_6 \otimes N \otimes C_{2c+1,1} \\
&= L_5U_4 \otimes B_> + L_2U_6 \otimes C_{2c+1,1}. \\
d(C_{>,2}) &= \delta_2^1(E, L_4) \otimes C_{>,1} + \delta_2^1(E, L_2U_6) \otimes C_{2c+1,>} + \delta_2^1(E, R_3) \otimes S_1 \\
&\quad + \delta_2^1(E, C_{35}) \otimes C_{>,>} \\
&= U_3 \otimes N \otimes C_{>,1} + L_2U_6 \otimes E \otimes C_{2c+1,>} + R_4R_3 \otimes N \otimes S_1 \\
&\quad + R_4U_5 \otimes S \otimes C_{>,>} \\
&= U_3 \otimes C_{>,1} + L_2U_6 \otimes C_{2c+1,2} + R_4R_3 \otimes S_1 + R_4U_5 \otimes C_{>,>}. \\
d(C_{>,>}) &= \delta_1^1(S) \otimes C_{>,>} + \delta_2^1(S, L_2U_6) \otimes C_{2c+1,>} \\
&= L_4 \otimes E \otimes C_{>,>} + R_3 \otimes W \otimes C_{>,>} + L_2U_6 \otimes S \otimes C_{2c+1,>} \\
&= L_4 \otimes C_{>,2} + R_3 \otimes S_2 + L_2U_6 \otimes C_{2c+1,>}. \\
d(S_1) &= \delta_2^1(N, L_3L_4L_5) \otimes B_> + \delta_2^1(N, L_3U_5) \otimes C_{>,>} \\
&= L_3L_4L_5 \otimes N \otimes B_> + U_5 \otimes W \otimes C_{>,>} \\
&= L_3L_4L_4 \otimes B_> + U_5 \otimes S_2. \\
d(S_2) &= \delta_2^1(W, L_4) \otimes C_{>,1} + \delta_2^1(W, L_2U_6) \otimes C_{2c+1,>} + \delta_2^1(W, R_3) \otimes S_1 \\
&\quad + \delta_2^1(W, C_{14}) \otimes C_{>,>} \\
&= L_3L_4 \otimes N \otimes C_{>,1} + L_2U_6 \otimes W \otimes C_{2c+1,>} + U_4 \otimes N \otimes S_1 \\
&\quad + L_3U_1 \cdot I_{245} \otimes S \otimes C_{>,>} \\
&= L_3L_4 \otimes C_{>,1} + L_2U_6 \cdot I_{345} \otimes W \otimes C_{2c+1,>} + U_4 \otimes S_1 \\
&\quad + \mathbf{0} \otimes C_{>,>} \\
&= L_3L_4 \otimes C_{>,1} + \mathbf{0} \otimes C_{2c+1,>} + U_4 \otimes S_1.
\end{aligned}$$

While lengthy, this calculation is simply applying the DA-bimodule maps detailed in Section 4.4.2 and [46, 49]. The only nuance is as stated above, that the idempotents of the Type D generators sometimes force the calculated algebra coefficient to be 0. In other cases, one simply cannot have an upper Kauffman state as suggested by the DA-bimodule maps. For example $d(S \otimes C_{1,>})$ would contain the term $R_3 \otimes W \otimes C_{1,>}$ from $\delta_1^1(S) \otimes C_{1,>}$; yet no such upper Kauffman state can exist, as can be seen from the corresponding upper knot diagram.

There is a little subtlety here. In [49], Ozsváth-Szabó introduce their algebra $\mathcal{A}(n)$ in the most general sense, so that idempotents can include 0 and $2n$ as part of their n-

element subsets, unlike the definition presented in Definition 4.12. As noted, following [49, Prop. 8.2], the truncation of this ring of idempotents is made suitable for the construction of a knot invariant, since in a special knot diagram the *distinguished meridian* used in the construction of a Heegaard diagram from a knot projection [36] is placed on the global minimum.

Indeed, other truncations of the algebra can be made, as in the case of [27, Def. 3.16], and these truncations have suitable interpretations in terms of Heegaard diagrams for bordered sutured knot Floer homology and corresponding quiver algebra representations, see [27, 28].

Second tensor product with $\mathcal{N}$

The above calculation follows in nearly exactly the same way under the second tensor product with $\mathcal{N}$. Indeed, the roles of U_3 and U_4 are switched in the same way. Moreover, all of those states with 3 in the idempotent have the corresponding Kauffman state in $\mathcal{N}$ forced as N, as this region must be occupied. This simplifies the majority of the calculation, hence the calculation for B_k states will be omitted, as it is almost identical to the above. A visual description of these states is once more provided in Figure 5.8.

$$\begin{aligned}
d(A_1) &= \delta_2^1(N, U_1U_3) \otimes B_1 + \delta_2^1(N, U_5) \otimes A_2 + \delta_2^1(N, R_5U_1) \otimes C_{12}) \\
&= U_1U_4 \otimes N \otimes B_1 + U_5 \otimes N \otimes A_2 + R_5U_1 \otimes N \otimes C_{12} \\
&= U_1U_4 \otimes B_1 + U_5 \otimes A_2 + R_5U_1 \otimes C_{12}. \\
d(A_2) &= \delta_2^1(N, U_4) \otimes A_1 + \delta_2^1(N, L_3L_4) \otimes D_1 + \delta_2^1(N, L_3U_1) \otimes A_> \\
&\quad + \delta_2^1(N, R_5R_4U_1) \otimes C_{1,>} \\
&= U_3 \otimes N \otimes A_1 + L_3L_4 \otimes N \otimes D_1 + U_1 \otimes W \otimes A_> + R_5U_1 \otimes E \otimes C_{1,>} \\
&= U_3 \otimes A_1 + L_3L_4 \otimes D_1 + U_1 \otimes A_3 + R_5U_1 \otimes C_{13}. \\
d(A_3) &= \delta_2^1(W, R_3) \otimes A_2 + \delta_2^1(W, L_4) \otimes D_2 \\
&\quad + \delta_3^1(W, R_3, R_5R_4U_1) \otimes C_{1,>} + \delta_2^1(W, C_{45}) \otimes A_> \\
&= U_4 \otimes N \otimes A_2 + L_3L_4 \otimes N \otimes D_2 + R_5R_4U_1 \otimes S \otimes C_{1,>} \\
&\quad + L_3U_5 \otimes S \otimes A_> \\
&= U_4 \otimes A_2 + L_3L_4 \otimes D_2 + R_5R_4U_1 \otimes C_{1,>} + L_3U_5 \otimes A_>. \\
d(A_>) &= \delta_1^1(S) \otimes A_>
\end{aligned}$$

$$
\begin{aligned}
&= R_3 \otimes W \otimes A_> + L_4 \otimes E \otimes A_> \\
&= R_3 \otimes A_3 + L_4 \otimes D_3. \\
d(D_1) &= \delta_2^1(N, R_4R_3U_1) \otimes B_1 + \delta_2^1(N, U_1) \otimes D_2 \\
&= R_4R_3U_1 \otimes N \otimes B_1 + U_1 \otimes N \otimes D_2 \\
&= R_4R_3U_1 \otimes B_1 + U_1 \otimes D_2. \\
d(D_2) &= \delta_2^1(N, U_1) \otimes D_1 + \delta_2^1(N, R_4R_3) \otimes A_1 \\
&= U_1 \otimes N \otimes D_1 + R_4R_3 \otimes N \otimes A_1 \\
&= U_1 \otimes D_1 + R_4R_3 \otimes A_1. \\
d(D_3) &= \delta_2^1(E, R_3) \otimes A_2 + \delta_2^1(E, L_4) \otimes D_2 + \delta_2^1(E, C_{13}) \otimes A_> \\
&= R_4R_3 \otimes N \otimes A_2 + U_3 \otimes N \otimes D_2 + R_4U_1 \otimes S \otimes A_> \\
&= R_4R_3 \otimes A_2 + U_3 \otimes D_2 + R_4U_1 \otimes A_>. \\
d(C_{11}) &= \delta_2^1(N, L_5) \otimes A_2 + \delta_2^1(N, L_5U_4) \otimes B_1 + \delta_2^1(N, U_2) \otimes C_{21} + \delta_2^1(N, U_1) \otimes C_{12} \\
&= L_5 \otimes N \otimes A_2 + L_5U_3 \otimes N \otimes B_1 + U_2 \otimes N \otimes C_{21} + U_1 \otimes N \otimes C_{12} \\
&= L_5 \otimes A_2 + L_5U_3 \otimes B_1 + U_2 \otimes C_{21} + U_1 \otimes C_{12}. \\
d(C_{12}) &= \delta_2^1(N, U_3) \otimes C_{11} + \delta_2^1(N, U_2) \otimes C_{22} + \delta_2^1(N, L_3L_5) \otimes A_> \\
&\quad + \delta_2^1(N, R_4U_5) \otimes C_{1,>} \\
&= U_4 \otimes N \otimes C_{11} + U_2 \otimes N \otimes C_{22} + L_5 \otimes W \otimes A_> + U_5 \otimes E \otimes C_{1,>} \\
&= U_4 \otimes C_{11} + U_2 \otimes C_{22} + L_5 \otimes A_3 + U_5 \otimes C_{13}. \\
d(C_{13}) &= \delta_2^1(E, L_4) \otimes C_{12} + \delta_2^1(E, U_2) \otimes C_{2,>} + \delta_3^1(E, L_4, L_3L_5) \otimes A_> \\
&\quad + \delta_2^1(E, C_{13}) \otimes C_{1,>} \\
&= U_3 \otimes N \otimes C_{12} + U_2 \otimes E \otimes C_{2,>} + L_3L_5 \otimes S \otimes A_> + R_4U_1 \otimes S \otimes C_{1,>} \\
&= U_3 \otimes C_{12} + U_2 \otimes C_{2,3} + L_3L_5 \otimes A_> + R_4U_1 \otimes C_{1,>}. \\
d(C_{1,>}) &= \delta_1^1(S) \otimes C_{1,>} + \delta_2^1(S, U_2) \otimes C_{2,>} \\
&= L_4 \otimes E \otimes C_{1,>} + U_2 \otimes S \otimes C_{2,>} \\
&= L_4 \otimes C_{13} + U_2 \otimes C_{2,>}. \\
d(C_{2r,1}) &= \delta_2^1(N, U_6) \otimes C_{2r-1,1} + \delta_2^1(N, L_5U_4) \otimes B_{2r} + \delta_2^1(N, U_1) \otimes C_{2r,2} \\
&\quad + \delta_2^1(N, U_1) \otimes C_{2r+1,1} \\
&= U_6 \otimes N \otimes C_{2r-1,1} + L_5U_3 \otimes N \otimes B_{2r} + U_1 \otimes N \otimes C_{2r,2} \\
&\quad + U_1 \otimes N \otimes C_{2r+1,1}
\end{aligned}
$$

$$
\begin{aligned}
&= U_6 \otimes C_{2r-1,1} + L_5U_3 \otimes B_{2r} + U_1 \otimes C_{2r,2} + U_1 \otimes C_{2r+1,1}.\\
d(C_{2r,2}) &= \delta_2^1(N,U_3) \otimes C_{2r,1} + \delta_2^1(N,U_6) \otimes C_{2r-1,2} + \delta_2^1(N,U_1) \otimes C_{2r+1,2}\\
&\quad + \delta_2^1(N,R_4U_5) \otimes C_{2r,>}\\
&= U_4 \otimes N \otimes C_{2r,1} + U_6 \otimes N \otimes C_{2r-1,2} + U_1 \otimes N \otimes C_{2r+1,2}\\
&\quad + U_5 \otimes E \otimes C_{2r,>}\\
&= U_4 \otimes C_{2r,1} + U_6 \otimes C_{2r-1,2} + U_1 \otimes C_{2r+1,2} + U_5 \otimes C_{2r,3}.\\
d(C_{2r,3}) &= \delta_2^1(E,L_4) \otimes C_{2r,2} + \delta_2^1(E,U_6) \otimes C_{2r-1,>} + \delta_2^1(E,U_1) \otimes C_{2r+1,>}\\
&\quad + \delta_2^1(E,C_{13}) \otimes C_{2r,>}\\
&= U_3 \otimes N \otimes C_{2r,2} + U_6 \otimes E \otimes C_{2r-1,>} + U_1 \otimes E \otimes C_{2r+1,>}\\
&\quad + R_4U_1 \otimes S \otimes C_{2r,>}\\
&= U_3 \otimes C_{2r,2} + U_6 \otimes C_{2r-1,3} + U_1 \otimes C_{2r+1,3} + R_4U_1 \otimes C_{2r,>}.\\
d(C_{2r,>}) &= \delta_1^1(S) \otimes C_{2r,>} + \delta_2^1(S,U_6) \otimes C_{2r-1,>} + \delta_2^1(S,U_1) \otimes C_{2r+1,>}\\
&= L_4 \otimes E \otimes C_{2r,>} + U_6 \otimes S \otimes C_{2r-1,>} + U_1 \otimes S \otimes C_{2r+1,>}\\
&= L_4 \otimes C_{2r,3} + U_6 \otimes C_{2r-1,>} + U_1 \otimes C_{2r+1,>}.\\
d(C_{2r+1,1}) &= \delta_2^1(N,L_5U_4) \otimes B_{2r+1} + \delta_2^1(N,U_1) \otimes C_{2r+1,2}\\
&\quad + \delta_2^1(N,U_2) \otimes C_{2r+2,1} + \delta_2^1(N,R_4U_5) \otimes C_{2r,>}\\
&= L_5U_3 \otimes N \otimes B_{2r+1} + U_1 \otimes N \otimes C_{2r+1,2} + U_2 \otimes N \otimes C_{2r+2,1}\\
&\quad + U_5 \otimes E \otimes C_{2r,>}\\
&= L_5U_3 \otimes B_{2r+1} + U_1 \otimes C_{2r+1,2} + U_2 \otimes C_{2r+2,1} + U_5 \otimes C_{2r,3}.\\
d(C_{2r+1,2}) &= \delta_2^1(N,U_3) \otimes C_{2r+1,1} + \delta_2^1(N,U_2) \otimes C_{2r+2,2} + \delta_2^1(N,R_4U_5) \otimes C_{2r+1,>}\\
&\quad + \delta_3^1(N,U_3,R_4U_5) \otimes C_{2r,>}\\
&= U_4 \otimes N \otimes C_{2r+1,1} + U_2 \otimes N \otimes C_{2r+2,2} + U_5 \otimes E \otimes C_{2r+1,>}\\
&\quad + R_4U_5 \otimes S \otimes C_{2r,>}\\
&= U_4 \otimes C_{2r+1,1} + U_2 \otimes C_{2r+2,2} + U_5 \otimes C_{2r+1,3} + R_4U_5 \otimes C_{2r,>}.\\
d(C_{2r+1,3}) &= \delta_2^1(E,L_4) \otimes C_{2r+1,2} + \delta_2^1(E,U_2) \otimes C_{2r+2,>} + \delta_2^1(E,C_{13}) \otimes C_{2r+1,>}\\
&= U_3 \otimes N \otimes C_{2r+1,2} + U_2 \otimes E \otimes C_{2r+2,>} + R_4U_1 \otimes S \otimes C_{2r+1,>}\\
&= U_3 \otimes C_{2r+1,2} + U_2 \otimes C_{2r+2,3} + R_4U_1 \otimes C_{2r+1,>}.\\
d(C_{2r+1,>}) &= \delta_1^1(S) \otimes C_{2r+1,>} + \delta_2^1(S,U_2) \otimes C_{2r+2,>}\\
&= L_4 \otimes E \otimes C_{2r+1,>} + U_2 \otimes S \otimes C_{2r+2,>}
\end{aligned}
$$

$$= L_4 \otimes C_{2r+1,3} + U_2 \otimes C_{2r+2,>}.$$

For the sake of brevity, note that when $r = c$, the only change to the calculations for $d(C_{2r+1,k})$ where $k \in \{1, 2, 3, >\}$ involves changing the term $\delta_2^1(X, U_2) \otimes C_{2r+2,k}$ for the term $\delta_2^1(X, R_2) \otimes C_{>,k}$, which is equal to $R_2 \otimes X \otimes C_{>,k}$.

$$\begin{aligned}
d(C_{>,1}) &= \delta_2^1(N, L_5U_4) \otimes B_> + \delta_2^1(N, L_2U_6) \otimes C_{2c+1,1} \\
&= L_5U_3 \otimes N \otimes B_> + L_2U_6 \otimes N \otimes C_{2c+1,1} \\
&= L_5U_3 \otimes B_> + L_2U_6 \otimes C_{2c+1,1}. \\
d(C_{>,2}) &= \delta_2^1(N, U_3) \otimes C_{>,1} + \delta_2^1(N, L_2U_6) \otimes C_{2c+1,2} \\
&\quad + \delta_2^1(N, R_4R_3) \otimes S_1 + \delta_2^1(N, R_4U_5) \otimes C_{>,>} \\
&= U_4 \otimes N \otimes C_{>,1} + L_2U_6 \otimes N \otimes C_{2c+1,2} + R_4R_3 \otimes N \otimes S_1 \\
&\quad + U_5 \otimes E \otimes C_{>,>} \\
&= U_4 \otimes C_{>,1} + L_2U_6 \otimes C_{2c+1,2} + R_4R_3 \otimes S_1 + U_5 \otimes C_{>,3}. \\
d(C_{>,3}) &= \delta_2^1(E, L_4) \otimes C_{>,2} + \delta_2^1(E, R_3) \otimes S_2 + \delta_2^1(E, L_2U_6) \otimes C_{2c+1,>} \\
&\quad + \delta_2^1(E, C_{13}) \otimes C_{>,>} \\
&= U_3 \otimes N \otimes C_{>,2} + R_4R_3 \otimes N \otimes S_2 + L_2U_6 \otimes C_{2c+1,>} \\
&\quad + R_4U_1 \otimes I_{245} \cdot S \otimes C_{>,>} \\
&= U_3 \otimes C_{>,2} + R_4R_3 \otimes S_2 + L_2U_6 \otimes C_{2c+1,>}. \\
d(C_{>,>}) &= \delta_1^1(S) \otimes C_{>,>} + \delta_2^1(S, L_2U_6) \otimes C_{2c+1,>} \\
&= L_4 \otimes E \otimes C_{>,>} + R_3 \otimes W \otimes C_{>,>} + L_2U_6 \otimes S \otimes C_{2c+1,>} \\
&= L_4 \otimes C_{>,3} + R_3 \otimes S_3 + L_2U_6 \otimes C_{2c+1,>}. \\
d(S_1) &= \delta_2^1(N, L_3L_4L_5) \otimes B_> + \delta_2^1(N, U_5) \otimes S_2 \\
&= L_3L_4L_5 \otimes N \otimes B_> + U_5 \otimes N \otimes S_2 \\
&= L_3L_4L_5 \otimes B_> + U_5 \otimes S_2. \\
d(S_2) &= \delta_2^1(N, L_3L_4) \otimes C_{>,1} + \delta_2^1(N, U_4) \otimes S_1 \\
&= L_3L_4 \otimes N \otimes C_{>,1} + U_3 \otimes N \otimes S_1 \\
&= L_3L_4 \otimes C_{>,1} + U_3 \otimes S_1. \\
d(S_3) &= \delta_2^1(W, L_4) \otimes C_{>,2} + \delta_2^1(W, R_3) \otimes S_2 + \delta_2^1(W, L_2U_6) \otimes C_{2c+1,>} \\
&\quad + \delta_2^1(W, C_{45}) \otimes C_{>,>} \\
&= L_3L_4 \otimes N \otimes C_{>,2} + U_4 \otimes N \otimes S_2 + \mathbf{0} \otimes W \otimes C_{2c+1,>}
\end{aligned}$$

$$
\begin{aligned}
&+ L_3U_5 \otimes S \otimes C_{>,>} \\
&= L_3L_4 \otimes C_{>,2} + U_4 \otimes S_2 + L_3U_5 \otimes C_{>,>}.
\end{aligned}
$$

Hence, comparing to the form required in Figure 5.10, one can see that the base case is determined, namely that $\left(\mathcal{N}^3\right)^3 \boxtimes X$ has the form fitting the inductive statement.

Type D structure relations

Once more, it is a relatively simple calculation to verify that the Type D relations are satisfied, as presented in Figure 4.7. An example of this relation being satisfied is presented here, as the sum of the two different terms must be zero. Once more, the fact that algebra elements are uniquely determined by their idempotents and weight (Proposition 4.13) is used.

$$
\begin{aligned}
(\mu_1 \otimes Id_M) \circ d(I_{234} \cdot B_>) &= \mu_1 \left(C_{14} + C_{26} + C_{35}\right) \otimes B_> \\
&= I_{234} \cdot (U_1U_4 + U_2U_6 + U_3U_5) \otimes I_{234} \otimes B_> \\
&= U_3U_5 \otimes B_>. \\
(\mu_2 \otimes Id_M) \circ (Id_{\mathcal{A}} \otimes d) \circ d(B_>) &= (\mu_2 \otimes Id_M) \circ (Id_{\mathcal{A}} \otimes d) \left(R_5 \times C_{>,1}\right) \\
&= R_5L_2U_6 \otimes C_{2c+1,1} + R_5L_5U_3 \otimes B_> \\
&= I_{234} \cdot L_2R_5U_6 \cdot I_{135} \otimes C_{2c+1,1} + U_3U_5 \otimes B_> \\
&= \mathbf{0} \otimes C_{2c+1,1} + U_3U_5 \otimes B_> = U_3U_5 \otimes B_>.
\end{aligned}
$$

5.4.3 Inductive assumption and argument

Assume for inductive purposes that the P_{2k+1} holds, in other words that

$$\left(\mathcal{N}^3\right)^{2k+1} \boxtimes X$$

has the form as described by the directed graph Figure 5.10, with the self-arrows coming from the matching, i.e. $d(X) \ni (C_{14} + C_{26} + C_{35}) \otimes X$. Denote the Type D structure $\left(\mathcal{N}^3\right)^{2k+1} \boxtimes X$ by Y.

Remark 5.3 *Under taking the tensor product with $\mathcal{N} \boxtimes \mathcal{N}$, the 'right' hand side of the diagram is extended, i.e. one has new states A_{2k+2}, A_{2k+3}, D_{2k+2}, D_{2k+3}, $C_{r,2k+2}, C_{r,2k+3}$ and S_{2k+2}, S_{2k+3}. As can be seen in the calculation of the base case in Section 5.4.2, the left hand side of the diagram does not change. Tensoring once by $\mathcal{N}$ swaps the role of U_3 and U_4, which is undone by the second tensor product. Moreover, since 3 belongs to all* *of the idempotents on the 'left' hand side of the diagram, the only compatible generator in* $\mathcal{N} \boxtimes \mathcal{N}$ *is* $N \otimes N$.

It is important to verify that more complicated behaviour on the left hand side is not introduced by taking a tensor product with $\mathcal{N}$. *Usefully, this can be seen by checking that there is no involvement of the* $\delta_3^1(N, a_1, a_2)$ *arrow in the DA-bimodule. There is a finite list of viable coefficients for such an arrow to be present, as described by [49, p. 21], and one can easily check that taking the tensor product of the Type D structure with* $\mathcal{N}$ *does not include maps with these algebra coefficients.*

Hence, all the maps on the left hand side — i.e. originating from the states B_j *for all* j, *and the states* A_j, C_{ij}, D_j, S_j *with* $j < 2k+1$ *— arise from maps with* $\delta_2^1(N, a)$, *since* $\delta_1^1(N) = 0$, *and so every DA-bimodule map only takes a single algebra element as an input: i.e. only one step is taken in the Type D structure to calculate the tensor product. Since* $\delta_2^1(N, a)$ *maps are very simple for the algebra elements* a *featured in the diagram, only the calculation for the 'right' hand side of the diagram will be presented.*

Calculation for $\mathcal{N}^3 \boxtimes Y$

Starting at the right hand side of the diagram (Figure 5.10) consider that in Y, one has that

$$d_Y(A_{2k}) = U_1 \otimes A_{2k+1} + L_3L_4 \otimes D_{2k-1} + U_3 \otimes A_{2k-1} + R_5U_1 \otimes C_{2k+1,1}.$$

One can then see that after taking tensor product by N, the map $d_{\mathcal{N}\boxtimes Y}(N \otimes A_{2k})$ is described by

$$d(N \otimes A_{2k}) = U_1 \otimes A_{2k+1} + L_3L_4 \otimes D_{2k-1} + U_4 \otimes A_{2k-1} + R_5U_1 \otimes C_{2k+1}.$$

Taking the tensor product once more with $\mathcal{N}$, one yields $d_{\mathcal{N}\boxtimes\mathcal{N}\boxtimes Y}(N \otimes N \otimes A_{2k})$ as

$$d(N \otimes N \otimes A_{2k}) = U_1 \otimes A_{2k+1} + L_3L_4 \otimes D_{2k-1} + U_3 \otimes A_{2k-1} + R_5U_1 \otimes C_{2k+1,1}.$$

The calculation is nearly identical for $D_{2k}, C_{r,2k}$ and S_{2k}. Hence, start with the right hand edge of the diagram.

$$\begin{aligned} d(A_{2k+1}) &= d(N \otimes A_{2k+1}) \\ &= \delta_2^1(N, L_3U_5) \otimes A_> + \delta_2^1(N, U_4) \otimes A_{2k} + \delta_2^1(N, R_5R_4U_1) \otimes C_{1,>} \\ &\quad + \delta_2^1(N, L_3L_4) \otimes D_{2k} \end{aligned}$$

$$\begin{aligned}
&= U_5 \otimes W \otimes A_> + U_3 \otimes N \otimes A_{2k} + R_5U_1 \otimes E \otimes C_{1,>} \\
&\qquad + L_3L_4 \otimes N \otimes D_{2k} \\
&= U_5 \otimes A_{2k+2} + U_3 \otimes A_{2k} + R_5U_1 \otimes C_{1,2k+2} + L_3L_4 \otimes D_{2k}. \\
d(A_{2k+2}) &= d(W \otimes A_>) \\
&= \delta_2^1(W, R_3) \otimes A_{2k+1} + \delta_2^1(W, L_4) \otimes D_{2k+1} + \delta_2^1(W, C_{14}) \otimes A_> \\
&\qquad + \delta_3^1(W, R_3, R_5R_4U_1) \otimes C_{1,>} \\
&= U_4 \otimes N \otimes A_{2k+1} + L_3L_4 \otimes N \otimes D_{2k+1} + L_3U_1 \otimes S \otimes A_> \\
&\qquad + R_5R_4U_1 \otimes S \otimes C_{1,>} \\
&= U_4 \otimes A_{2k+1} + L_3L_4 \otimes D_{2k+1} + L_3U_1 \otimes A_> + R_5R_4U_1 \otimes C_{1,>}. \\
d(D_{2k+1}) &= d(N \otimes D_{2k+1}) \\
&= \delta_2^1(N, R_4R_3) \otimes A_{2k} + \delta_2^1(N, U_3) \otimes D_{2k} + \delta_2^1(N, R_4U_1) \otimes A_> \\
&= R_4R_3 \otimes N \otimes A_{2k} + U_4 \otimes N \otimes D_{2k} + U_1 \otimes E \otimes A_> \\
&= R_4R_3 \otimes A_{2k} + U_4 \otimes D_{2k} + U_1 \otimes D_{2k+2}. \\
d(D_{2k+2}) &= d(E \otimes A_>) \\
&= \delta_2^1(E, L_4) \otimes D_{2k+1} + \delta_2^1(E, R_3) \otimes A_{2k+1} + \delta_2^1(E, C_{35}) \otimes A_> \\
&= U_3 \otimes N \otimes D_{2k+1} + R_4R_3 \otimes N \otimes A_{2k+1} + R_4U_5 \otimes I_{124} \cdot S \otimes A_> \\
&= U_3 \otimes D_{2k+1} + R_4R_3 \otimes A_{2k+1} + \mathbf{0} \otimes A_>. \\
d(A_>) &= d(S \otimes A_>) \\
&= \delta_1^1(S) \otimes A_> = L_4 \otimes E \otimes A_> + R_3 \otimes W \otimes A_> \\
&= L_4 \otimes D_{2k+2} + R_3 \otimes A_{2k+2}. \\
d(C_{1,2k+1}) &= d(N \otimes C_{1,2k+1}) \\
&= \delta_2^1(N, U_3) \otimes C_{1,2k} + \delta_2^1(N, U_2) \otimes C_{2,2k+1} + \delta_2^1(N, R_4U_1) \otimes C_{1,>} \\
&\qquad + \delta_2^1(N, L_3L_5) \otimes A_> \\
&= U_4 \otimes N \otimes C_{1,2k} + U_2 \otimes N \otimes C_{2,2k+1} + U_1 \otimes E \otimes C_{1,>} + L_5 \otimes W \otimes A_> \\
&= U_4 \otimes C_{1,2k} + U_2 \otimes C_{2,2k+1} + U_1 \otimes C_{1,2k+2} + L_5 \otimes A_{2k+2}. \\
d(C_{1,2k+2}) &= d(E \otimes C_{1,>}) \\
&= \delta_2^1(E, L_4) \otimes C_{1,2k+1} + \delta_2^1(E, U_2) \otimes C_{2,>} + \delta_2^1(E, C_{35}) \otimes C_{1,>} \\
&\qquad + \delta_3^1(E, L_4, L_3L_5) \otimes A_> \\
&= U_3 \otimes N \otimes C_{1,2k+1} + U_2 \otimes E \otimes C_{2,>} + R_4U_5 \otimes S \otimes C_{1,>}
\end{aligned}$$

$$
\begin{aligned}
&+ L_3L_5 \otimes S \otimes A_> \\
&= U_3 \otimes C_{1,2k+1} + U_2 \otimes C_{2,2k+2} + R_4U_5 \otimes C_{1,>} + L_3L_5 \otimes A_>. \\
d(C_{1,>}) &= d(S \otimes C_{1,>}) \\
&= \delta^1_1(S) \otimes C_{1,>} + \delta^1_2(S, U_2) \otimes C_{2,>} \\
&= L_4 \otimes E \otimes C_{1,>} + U_2 \otimes S \otimes C_{2,>} \\
&= L_4 \otimes C_{1,2k+2} + U_2 \otimes C_{2,>}. \\
d(C_{2r,2k+1}) &= d(N \otimes C_{2r,2k+1}) \\
&= \delta^1_2(N, U_3) \otimes C_{2r,2k} + \delta^1_2(N, U_6) \otimes C_{2r-1,2k+1} + \delta^1_2(N, U_1) \otimes C_{2r+1,2k+1} \\
&\quad + \delta^1_2(N, R_4U_1) \otimes C_{2r,>} \\
&= U_4 \otimes N \otimes C_{2r,2k} + U_6 \otimes N \otimes C_{2r-1,2k+1} + U_1 \otimes N \otimes C_{2r+1,2k+1} \\
&\quad + U_1 \otimes E \otimes C_{2r,>} \\
&= U_4 \otimes C_{2r,2k} + U_6 \otimes C_{2r-1,2k+1} + U_1 \otimes C_{2r+1,2k+1} + U_1 \otimes C_{2r,2k+2}. \\
d(C_{2r,2k+2}) &= d(E \otimes C_{2r,>}) \\
&= \delta^1_2(E, L_4) \otimes C_{2r,2k+1} + \delta^1_2(E, U_1) \otimes C_{2r+1,>} + \delta^1_2(E, U_6) \otimes C_{2r-1,>} \\
&\quad + \delta^1_2(E, C_{35}) \otimes C_{2r,>} \\
&= U_3 \otimes N \otimes C_{2r,2k+1} + U_1 \otimes E \otimes C_{2r+1,>} + U_6 \otimes E \otimes C_{2r-1,>} \\
&\quad + R_4U_5 \otimes S \otimes C_{2r,>} \\
&= U_3 \otimes C_{2r,2k+1} + U_1 \otimes C_{2r+1,2k+2} + U_6 \otimes C_{2r-1,2k+2} + R_4U_5 \otimes C_{2r,>}.
\end{aligned}
$$

$$
\begin{aligned}
d(C_{2r,>}) &= d(S \otimes C_{2r,>}) \\
&= \delta^1_1(S) \otimes C_{2r,>} + \delta^1_2(S, U_1) \otimes C_{2r+1,>} + \delta^1_2(S, U_6) \otimes C_{2r-1,>} \\
&= L_4 \otimes E \otimes C_{2r,>} + U_1 \otimes S \otimes C_{2r+1,>} + U_6 \otimes S \otimes C_{2r-1,>} \\
&= L_4 \otimes C_{2r,2k+2} + U_1 \otimes C_{2r+1,>} + U_6 \otimes C_{2r-1,>}. \\
d(C_{2r+1,2k+1}) &= d(N \otimes C_{2r+1,2k+1}) \\
&= \delta^1_2(N, U_3) \otimes C_{2r+1,2k} + \delta^1_2(N, U_2) \otimes C_{2r+2,2k+1} \\
&\quad + \delta^1_2(N, R_4U_1) \otimes C_{2r+1,>} + \delta^1_3(N, U_3, R_4U_5) \otimes C_{2r,>} \\
&= U_4 \otimes N \otimes C_{2r+1,2k} + U_2 \otimes N \otimes C_{2r+2,2k+1} + U_1 \otimes E \otimes C_{2r+1,>} \\
&\quad + R_4U_5 \otimes S \otimes C_{2r,>} \\
&= U_4 \otimes C_{2r+1,2k} + U_2 \otimes C_{2r+2,2k+1} + U_1 \otimes C_{2r+1,2k+2}
\end{aligned}
$$

$$
\begin{aligned}
&\quad + R_4U_5 \otimes C_{2r,>}.\\
d(C_{2r+1,2k+2}) &= d(E \otimes C_{2r+1,>})\\
&= \delta_2^1(E, L_4) \otimes C_{2r+1,2k+1} + \delta_2^1(E, U_2) \otimes C_{2r+2,>}\\
&\quad + \delta_2^1(E, C_{35}) \otimes C_{2r+1,>}\\
&= U_3 \otimes N \otimes C_{2r+1,2k+1} + U_2 \otimes E \otimes C_{2r+2,>} + R_4U_5 \otimes S \otimes C_{2r+1,>}\\
&= U_3 \otimes C_{2r+1,2k+1} + U_2 \otimes C_{2r+2,2k+2} + R_4U_5 \otimes C_{2r+1,>}.\\
d(C_{2r+1,>}) &= d(S \otimes C_{2r+1,>})\\
&= \delta_1^1(S) \otimes C_{2r+1,>} + \delta_2^1(S, U_2) \otimes C_{2r+2,>}\\
&= L_4 \otimes E \otimes C_{2r+1,>} + U_2 \otimes S \otimes C_{2r+2,>}\\
&= L_4 \otimes C_{2r+1,2k+2} + U_2 \otimes C_{2r+2,>}.
\end{aligned}
$$

When $r = c$, the states in consideration are $C_{2c+1,\ell}$ at the bottom of Figure 5.10. The only difference of the calculation of the $d_{\mathcal{N}\boxtimes Y}$ for these states rather than the calculation of $d_{\mathcal{N}\boxtimes Y}(C_{2r+1}, \ell)$ above is that one substitutes the 'downward' arrow with the associated algebra element U_2 for the algebra element R_2. This is exactly as was remarked in the verification of the base case in Section 5.4.2. Since both of these algebra elements have weight outside of the span $\{e_3, e_4\}$, the changing U_2 for R_2 in the above gives the required result.

Continuing with the calculation, one has that:

$$
\begin{aligned}
d(C_{>,2k+1}) &= d(N \otimes C_{>,2k+1})\\
&= \delta_2^1(N, U_3) \otimes C_{>,2k} + \delta_2^1(N, L_2U_6) \otimes C_{2c+1,2k+1} + \delta_2^1(N, R_4U_1) \otimes C_{>,>}\\
&\quad + \delta_2^1(N, R_4R_3) \otimes S_{2k}\\
&= U_4 \otimes N \otimes C_{>,2k} + L_2U_6 \otimes N \otimes C_{2c+1,2k+1} + U_1 \otimes I_{235} \cdot E \otimes C_{>,>}\\
&\quad + R_4R_3 \otimes N \otimes S_{2k}\\
&= U_4 \otimes C_{>,2k} + L_2U_6 \otimes C_{2c+1,2k+1} + \mathbf{0} \otimes C_{>,2k+2} + R_4R_3 \otimes S_{2k}.\\
d(C_{>,2k+2}) &= d(E \otimes C_{>,>})\\
&= \delta_2^1(E, L_4) \otimes C_{>,2k+1} + \delta_2^1(E, L_2U_6) \otimes C_{2c+1,>} + \delta_2^1(E, R_3) \otimes S_{2k+1}\\
&\quad + \delta_2^1(E, C_{35}) \otimes C_{>,>}\\
&= U_3 \otimes N \otimes C_{>,2k+1} + L_2U_6 \otimes E \otimes C_{2c+1,>} + R_4R_3 \otimes N \otimes S_{2k+1}\\
&\quad + R_4U_5 \otimes S \otimes C_{>,>}
\end{aligned}
$$

$$
\begin{aligned}
&= U_3 \otimes C_{>,2k+1} + L_2U_6 \otimes C_{2c+1,2k+2} + R_4R_3 \otimes S_{2k+1} + R_4U_5 \otimes C_{>,>}.\\
d(C_{>,>}) &= d(S \otimes C_{>,>})\\
&= \delta^1_1(S) \otimes C_{>,>} + \delta^1_2(S, L_2U_6) \otimes C_{2c+1,>}\\
&= R_3 \otimes W \otimes C_{>,>} + L_4 \otimes E \otimes C_{>,>} + L_2U_6 \otimes S \otimes C_{2c+1,>}\\
&= R_3 \otimes S_{2k+2} + L_4 \otimes C_{>,2k+2} + L_2U_6 \otimes C_{2c+1,>}.\\
d(S_{2k+1}) &= d(N \otimes S_{2k+1})\\
&= \delta^1_2(N, U_4) \otimes S_{2k} + \delta^1_2(N, L_3L_4) \otimes C_{>,2k} + \delta^1_2(N, L_3U_5) \otimes C_{>,>}\\
&= U_3 \otimes N \otimes S_{2k} + L_3L_4 \otimes N \otimes C_{>,2k} + U_5 \otimes W \otimes C_{>,>}\\
&= U_3 \otimes S_{2k} + L_3L_4 \otimes C_{>,2k} + U_5 \otimes S_{2k+2}.\\
d(S_{2k+2}) &= \delta^1_2(W, R_3) \otimes S_{2k+1} + \delta^1_2(W, L_4) \otimes C_{>,2k+1} + \delta^1_2(W, L_2U_6) \otimes C_{2c+1,>}\\
&\quad + \delta^1_2(W, C_{14}) \otimes C_{>,>}\\
&= U_4 \otimes N \otimes S_{2k+1} + L_3L_4 \otimes N \otimes C_{>,2k+1} + L_2U_6 \otimes \mathbf{0} \otimes C_{2c+1,>}\\
&\quad + L_3U_1 \otimes I_{245} \cdot S \otimes C_{>,>}\\
&= U_4 \otimes S_{2k+1} + L_3L_4 \otimes C_{>,2k+1} + \mathbf{0} \otimes C_{>,>}\\
&= U_4 \otimes S_{2k+1} + L_3L_4 \otimes C_{>,2k+1}.
\end{aligned}
$$

5.4.4 Calculation for $\mathcal{N}^3 \boxtimes \mathcal{N}^3 \boxtimes Y$

As before, the only elements that have slightly more complicated maps in the Type D structure to calculate are those at the end of the strand, or equivalently the right hand side of the diagram. The calculation is similar to the above.

$$
\begin{aligned}
d(A_{2k+2}) &= d(N \otimes A_{2k+2})\\
&= \delta^1_2(N, U_4) \otimes A_{2k+1} + \delta^1_2(N, L_3L_4) \otimes D_{2k+1} + \delta^1_2(N, L_3U_1) \otimes A_{>}\\
&\quad + \delta^1_2(N, R_5R_4U_1) \otimes C_{1,>}\\
&= U_3 \otimes N \otimes A_{2k+1} + L_3L_4 \otimes N \otimes D_{2k+1} + U_1 \otimes W \otimes A_{>}\\
&\quad + R_5U_1 \otimes E \otimes C_{1,>}\\
&= U_3 \otimes A_{2k+1} + L_3L_4 \otimes D_{2k+1} + U_1 \otimes A_{2k+3} + R_5U_1 \otimes C_{1,2k+3}.\\
d(A_{2k+3}) &= \delta^1_2(W, L_4) \otimes D_{2k+2} + \delta^1_2(W, R_3) \otimes A_{2k+2} + \delta^1_2(W, C_{45}) \otimes A_{>}\\
&\quad + \delta^1_3(W, R_3, R_5R_4U_1) \otimes C_{1,>}\\
&= L_3L_4 \otimes N \otimes D_{2k+2} + U_4 \otimes N \otimes A_{2k+2} + L_3U_5 \otimes S \otimes A_{>}
\end{aligned}
$$

$$
\begin{aligned}
&\quad + R_5R_4U_1 \otimes S \otimes C_{1,>} \\
&= L_3L_4 \otimes D_{2k+2} + U_4 \otimes A_{2k+2} + L_3U_5 \otimes A_> + R_5R_4U_1 \otimes C_{1,>}. \\
d(D_{2k+2}) &= d(N \otimes D_{2k+2}) \\
&= \delta_2^1(N, U_3) \otimes D_{2k+1} + \delta_2^1(N, R_4R_3) \otimes A_{2k+1} \\
&= U_4 \otimes N \otimes D_{2k+1} + R_4R_3 \otimes N \otimes A_{2k+1} \\
&= U_4 \otimes D_{2k+1} + R_4R_3 \otimes A_{2k+1}. \\
d(D_{2k+3}) &= d(E \otimes A_>) \\
&= \delta_2^1(E, L_4) \otimes D_{2k+2} + \delta_2^1(E, R_3) \otimes A_{2k+2} + \delta_2^1(E, C_{13}) \otimes A_> \\
&= U_3 \otimes N \otimes D_{2k+2} + R_4R_3 \otimes N \otimes A_{2k+2} + R_4U_1 \otimes S \otimes A_> \\
&= U_3 \otimes D_{2k+2} + R_4R_3 \otimes A_{2k+2} + R_4U_1 \otimes A_>. \\
d(A_>) &= d(S \otimes A_>) \\
&= \delta_1^1(S) \otimes A_> \\
&= L_4 \otimes E \otimes A_> + R_3 \otimes W \otimes A_> \\
&= L_4 \otimes D_{2k+3} + R_3 \otimes A_{2k+3}. \\
d(C_{1,2k+2}) &= d(N \otimes C_{1,2k+2}) \\
&= \delta_2^1(N, U_3) \otimes C_{1,2k+1} + \delta_2^1(N, U_2) \otimes C_{2,2k+2} + \delta_2^1(N, R_4U_5) \otimes C_{1,>} \\
&\quad + \delta_2^1(N, L_3L_5) \otimes A_> \\
&= U_4 \otimes N \otimes C_{1,2k+1} + U_2 \otimes N \otimes C_{2,2k+2} + U_5 \otimes E \otimes C_{1,>} \\
&\quad + L_5 \otimes W \otimes A_> \\
&= U_4 \otimes C_{1,2k+1} + U_2 \otimes C_{2,2k+2} + U_5 \otimes C_{1,2k+3} + L_5 \otimes A_{2k+3}. \\
d(C_{1,2k+3}) &= d(E \otimes C_{1,>}) \\
&= \delta_2^1(E, L_4) \otimes C_{1,2k+2} + \delta_2^1(E, U_2) \otimes C_{2,>} + \delta_2^1(E, C_{13}) \otimes C_{1,>} \\
&\quad + \delta_3^1(E, L_4, L_3L_5) \otimes A_> \\
&= U_3 \otimes N \otimes C_{1,2k+2} + U_2 \otimes E \otimes C_{2,>} + R_4U_1 \otimes S \otimes C_{1,>} \\
&\quad + L_3L_5 \otimes S \otimes A_> \\
&= U_3 \otimes C_{1,2k+2} + U_2 \otimes C_{2,2k+3} + R_4U_1 \otimes C_{1,>} + L_3L_5 \otimes A_>. \\
d(C_{1,>}) &= d(S \otimes C_{1,>}) \\
&= \delta_1^1(S) \otimes C_{1,>} + \delta_2^1(S, U_2) \otimes C_{2,>} \\
&= L_4 \otimes E \otimes C_{1,>} + U_2 \otimes S \otimes C_{2,>}
\end{aligned}
$$

$$
\begin{aligned}
&= L_4 \otimes C_{1,2k+3} + U_2 \otimes C_{2,>}. \\
d(C_{2r,2k+2}) &= d(N \otimes C_{2r,2k+2}) \\
&= \delta_2^1(N,U_3) \otimes C_{2r,2k+1} + \delta_2^1(N,U_1) \otimes C_{2r+1,2k+2} + \delta_2^1(N,U_6) \otimes C_{2r-1,2k+2} \\
&\quad + \delta_2^1(N,R_4U_5) \otimes C_{2r,>} \\
&= U_4 \otimes N \otimes C_{2r,2k+1} + U_1 \otimes N \otimes C_{2r+1,2k+2} + U_6 \otimes N \otimes C_{2r-1,2k+2} \\
&\quad + U_5 \otimes E \otimes C_{2r,>} \\
&= U_4 \otimes C_{2r,2k+1} + U_1 \otimes C_{2r+1,2k+2} + U_6 \otimes C_{2r-1,2k+2} + U_5 \otimes C_{2r,2k+3}. \\
d(C_{2r,2k+3}) &= d(E \otimes C_{2r,>}) \\
&= \delta_2^1(E,L_4) \otimes C_{2r,2k+2} + \delta_2^1(E,U_1) \otimes C_{2r+1,>} + \delta_2^1(E,U_6) \otimes C_{2r-1,>} \\
&\quad + \delta_2^1(E,C_{13}) \otimes C_{2r,>} \\
&= U_3 \otimes N \otimes C_{2r,2k+2} + U_1 \otimes E \otimes C_{2r+1,>} + U_6 \otimes E \otimes C_{2r-1,>} \\
&\quad + R_4U_1 \otimes S \otimes C_{2r,>} \\
&= U_3 \otimes C_{2r,2k+2} + U_1 \otimes C_{2r+1,2k+3} + U_6 \otimes C_{2r-1,2k+3} + R_4U_1 \otimes C_{2r,>}. \\
d(C_{2r,>}) &= d(S \otimes C_{2r,>}) \\
&= \delta_1^1(S) \otimes C_{2r,>} + \delta_2^1(S,U_1) \otimes C_{2r+1,>} + \delta_2^1(S,U_6) \otimes C_{2r-1,>} \\
&= L_4 \otimes E \otimes C_{2r,>} + U_1 \otimes S \otimes C_{2r+1,>} + U_6 \otimes S \otimes C_{2r-1,>} \\
&= L_4 \otimes C_{2r,2k+3} + U_1 \otimes C_{2r+1,>} + U_6 \otimes C_{2r-1,>}. \\
d(C_{2r+1,2k+2}) &= d(N \otimes C_{2r+1,2k+2}) \\
&= \delta_2^1(N,U_3) \otimes C_{2r+1,2k+1} + \delta_2^1(N,U_2) \otimes C_{2r+2,2k+2} \\
&\quad + \delta_2^1(N,R_4U_5) \otimes C_{2r+1,>} + \delta_3^1(N,U_3,R_4U_5) \otimes C_{2r,>} \\
&= U_4 \otimes N \otimes C_{2r+1,2k+1} + U_2 \otimes N \otimes C_{2r+2,2k+2} + U_5 \otimes E \otimes C_{2r+1,>} \\
&\quad + R_4U_5 \otimes S \otimes C_{2r,>} \\
&= U_4 \otimes C_{2r+1,2k+1} + U_2 \otimes C_{2r+2,2k+2} + U_5 \otimes C_{2r+1,2k+3} + R_4U_5 \otimes C_{2r,>}. \\
d(C_{2r+1,2k+3}) &= d(E \otimes C_{2r+1,>}) \\
&= \delta_2^1(E,L_4) \otimes C_{2r+1,2k+2} + \delta_2^1(E,U_2) \otimes C_{2r+2,>} + \delta_2^1(E,C_{13}) \otimes C_{2r+1,>} \\
&= U_3 \otimes N \otimes C_{2r+1,2k+2} + U_2 \otimes E \otimes C_{2r+2,>} + R_4U_1 \otimes S \otimes C_{2r+1,>} \\
&= U_3 \otimes C_{2r+1,2k+2} + U_2 \otimes C_{2r+2,2k+3} + R_4U_1 \otimes C_{2r+1,>}. \\
d(C_{2r+1,>}) &= d(S \otimes C_{2r+1,>}) \\
&= \delta_1^1(S) \otimes C_{2r+1,>} + \delta_2^1(S,U_2) \otimes C_{2r+2,>}
\end{aligned}
$$

$$
\begin{aligned}
&= L_4 \otimes E \otimes C_{2r+1,>} + U_2 \otimes S \otimes C_{2r+2,>} \\
&= L_4 \otimes C_{2r+1,2k+3} + U_2 \otimes C_{2r+2,>}.
\end{aligned}
$$

As before, the difference between the calculations for $C_{2c+1,\ell}$ and $C_{2r+1,\ell}$ where $1 \leq r < c$ is simply changing the role of U_2 and R_2. Since the weight is outside the region of the strands present in the crossing, these algebra elements commute with the map δ_2^1. So the calculation will continue with $C_{>,\ell}$ and S_ℓ.

$$
\begin{aligned}
d(C_{>,2k+2}) &= d(N \otimes C_{>,2k+2}) \\
&= \delta_2^1(N, U_3) \otimes C_{>,2k+1} + \delta_2^1(N, L_2U_6) \otimes C_{2c+1,2k+2} \\
&\quad + \delta_2^1(N, R_4R_3) \otimes S_{2k+1} + \delta_2^1(N, R_4U_5) \otimes C_{>,>} \\
&= U_4 \otimes N \otimes C_{>,2k+1} + L_2U_6 \otimes N \otimes C_{2c+1,2k+2} \\
&\quad + R_4R_3 \otimes N \otimes S_{2k+1} + U_5 \otimes E \otimes C_{>,>} \\
&= U_4 \otimes C_{>,2k+1} + L_2U_6 \otimes C_{2c+1,2k+2} + R_4R_3 \otimes S_{2k+1} + U_5 \otimes C_{>,2k+3}. \\
d(C_{>,2k+3}) &= d(E \otimes C_{>,>}) \\
&= \delta_2^1(E, R_3) \otimes S_{2k+2} + \delta_2^1(E, L_4) \otimes C_{>,2k+2} + \delta_2^1(E, L_2U_6) \otimes C_{2c+1,>} \\
&\quad + \delta_2^1(E, C_{13}) \otimes C_{>,>} \\
&= R_4R_3 \otimes N \otimes S_{2k+2} + U_3 \otimes N \otimes C_{>,2k+2} + L_2U_6 \otimes E \otimes C_{2c+1,>} \\
&\quad + R_4U_1 \otimes S \otimes C_{>,>} \\
&= R_4R_3 \otimes S_{2k+2} + U_3 \otimes C_{>,2k+2} + L_2U_6 \otimes C_{2c+1,2k+3} + R_4U_1 \otimes C_{>,>}. \\
d(C_{>,>}) &= d(S \otimes C_{>,>}) \\
&= \delta_1^1(S) \otimes C_{>,>} + \delta_2^1(S, L_2U_6) \otimes C_{2c+1,>} \\
&= R_3 \otimes W \otimes C_{>,>} + L_4 \otimes E \otimes C_{>,>} + L_2U_6 \otimes S \otimes C_{>,>} \\
&= R_3 \otimes S_{2k+3} + L_4 \otimes C_{>,2k+3} + L_2U_6 \otimes C_{2c+1,>}. \\
d(S_{2k+2}) &= d(N \otimes S_{2k+2}) \\
&= \delta_2^1(N, U_4) \otimes S_{2k+1} + \delta_2^1(N, L_3L_4) \otimes C_{>,2k+1} \\
&= U_3 \otimes N \otimes S_{2k+1} + L_3L_4 \otimes N \otimes C_{>,2k+1} \\
&= U_3 \otimes S_{2k+1} + L_3L_4 \otimes C_{>,2k+1}. \\
d(S_{2k+3}) &= d(W \otimes C_{>,>}) \\
&= \delta_2^1(W, L_4) \otimes C_{>,2k+2} + \delta_2^1(W, R_3) \otimes S_{2k+2} + \delta_2^1(W, L_2U_6) \otimes C_{2c+1,>} \\
&\quad + \delta_2^1(W, C_{45}) \otimes C_{>,>}
\end{aligned}
$$

$$= L_3L_4 \otimes N \otimes C_{>,2k+2} + U_4 \otimes N \otimes S_{2k+2} + L_2U_6 \otimes \mathbf{0} \otimes C_{2c+1,>}$$
$$+ L_3U_5 \otimes S \otimes C_{>,>}$$
$$= L_3L_4 \otimes C_{>,2k+2} + U_4 \otimes S_{2k+2} + L_3U_5 \otimes C_{>,>}.$$

This completes the calculation of the Type-D structure $\left(\mathcal{N}^3 \boxtimes \mathcal{N}^3 \boxtimes Y, d\right)$, and note that this agrees with the weighted, directed graph in Figure 5.10. Hence, since the calculation started from the inductive assumption that P_{2k+1} held, one has that $P_{2k+1} \Rightarrow P_{2(k+1)+1}$. Thus, by mathematical induction, the Type D structure of $\left(\mathcal{N}^3\right)^{2b+1} \boxtimes X$ is as described by Figure 5.10, completing the inductive proof.

Remark 5.4 *Note, the behaviour of the Type D structure does not depend on b or c, merely the dimensions of the weighted, directed graph. As will be determined later, the numerical invariants extracted from $\mathcal{C}(D)$ will depend on whether $b \leq c$ or $b > c$, but the Type D structure determined in Figure 5.10 does not.*

5.5 The third set of crossings

The next step in determining the Type D structure associated to the special upper knot diagram of three strand pretzel knots – shown in Figure 5.1 – is to take the box-tensor product of the Type D structure yielded above and $2a$ copies of the DA-bimodule $\mathcal{P}^5$. The bimodule $\mathcal{P}^5$ for a positive crossing has already been considered: it is merely a relabelling of the bimodule $\mathcal{P}^1$.

As noted by Ozsváth-Szabó in the definition in [49], algebra elements with weights outside of the set $span\{e_5, e_6\}$ commute with the map δ^1_k. Furthermore, since a crossing between the fifth and sixth strands is at the 'edge' of the knot diagram, and so incident to the exterior region also incident to the global maximum, it is enforced by the truncation of the algebra that 6 does not belong to the idempotent of any generator of the Type D structure or DA-bimodule. Hence, the permissible generators in the $\mathcal{P}^5$ tensor coordinate are N, W and S, since the generator E would have 6 in its incoming idempotent.

This simplifies the calculation, as does the fact that the D, C and S-states only have a non-zero tensor product with one element of the set $\{N, W, S\}$.

Specifically, since the generators C_{ij} shown in Figure 5.8 have associated idempotent I_{135}, one must have that the $\mathcal{P}$-tensor coordinate in any product with $I_{135} \cdot C_{ij}$ must be $I_{135} \cdot$

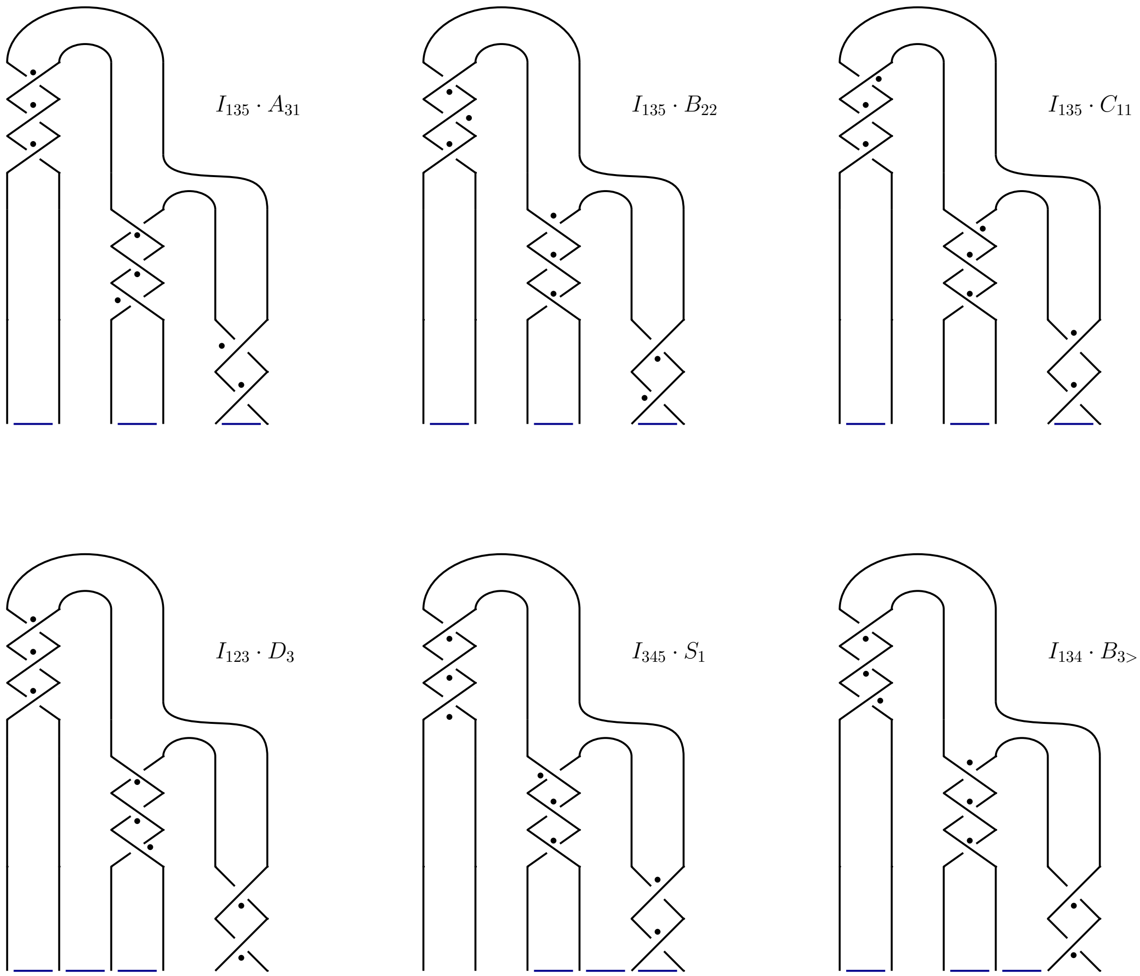


Figure 5.11: *Diagram depicting some of the categories into which Kauffman states corresponding to generators of $\left(\mathcal{P}^5\right)^{2a} \boxtimes Y$ may lie. States with $>$ as an index indicate that the marked point in the vacant interior region lies 'below' the current upper knot diagram.*

$N \cdot I_{135}$. Likewise, the states of type S and D have associated idempotents $I_{123} \cdot D_i$ and $I_{345} \cdot S_k$. As a consequence, the only non-zero tensor product in which these states appears are $I_{123} \cdot S \cdot I_{123} \otimes I_{123} \cdot D_i$ and $I_{345} \cdot N \cdot I_{345} \otimes I_{345} \cdot S_k$.

Denote the Type D structure $\left(\mathcal{N}^3\right)^{2b+1} \boxtimes X$ by Y. The possible states and their associated idempotents in $\left(\mathcal{P}^5\right)^{2a} \boxtimes Y$ are displayed in Figure 5.11, where the indices in A_{jk}, B_{ik} and C_{ij} denote the position of the marked points in the two interior regions. Where only one index is present — for example in the case of D_k — the index gives the position of the marked point in the region interior region that is *not* a value in the associated idempotent.

The states that gain an extra index are the B_{ik} states and A_{jk}. Hence, if one were to describe the associated Type D structure as a weighted directed graph as in Figure 5.10, the B-states and A-states would now be squares of two dimensions, rather than lines of one. Such a graph is difficult to display, and it is simpler to describe the Type D structure separately based upon the type of state to which the map d in $\left(\mathcal{P}^5\right)^{\boxtimes} \boxtimes Y$ is applied.

5.5.1 C-states and S-states

The C-states, since they have only one permissible tensor product with $\mathcal{P}^5$ have a form that is easy to determine inductively. Let P_{2k} be the statement that the states C_{ij} and S_j in the Type D structure $\left(\mathcal{P}^5\right)^{2k} \boxtimes Y$ have maps as displayed in Figure 5.12. The DA-bimodule $\left(\mathcal{P}^5, \delta_k^1\right)$ is defined exactly as in Section 4.4.2, noting that elements with weights outside of the strands concerned commute with the maps.

Base case: $k = 1$

As in the calculation presented in Section 5.4.3, the arrows for the states C_{ij} depend on the parity of i and j, and differ at the end of the strands. So, in $\mathcal{P}^5 \boxtimes Y$, one has the following.

$$\begin{aligned}
d(N \otimes C_{11}) &= d(C_{11}) \\
&= \delta_2^1(N, L_5U_3) \otimes B_1 + \delta_2^1(N, L_5) \otimes A_2 + \delta_2^1(N, U_1) \otimes C_{12} \\
&\quad + \delta_2^1(N, U_2) \otimes C_{21} \\
&= U_3U_6 \otimes W \otimes B_1 + U_6 \otimes W \otimes A_2 + U_1 \otimes N \otimes C_{12} \\
&\quad + U_2 \otimes N \otimes C_{21} \\
&= U_3U_6 \otimes B_{11} + U_6 \otimes A_{21} + U_1 \otimes C_{12} + U_2 \otimes C_{21}. \\
d(N \otimes C_{1,2r}) &= d(C_{1,2r}) \\
&= \delta_2^1(N, U_4) \otimes C_{1,2r-1} + \delta_2^1(N, L_5) \otimes A_{2r+1} + \delta_2^1(N, U_5) \otimes C_{1,2r+1} \\
&\quad + \delta_2^1(N, U_2) \otimes C_{2,2r} \\
&= U_4 \otimes N \otimes C_{1,2r-1} + U_6 \otimes W \otimes A_{2r+1} + U_6 \otimes N \otimes C_{1,2r+1} \\
&\quad + U_2 \otimes N \otimes C_{2,2r} \\
&= U_4 \otimes C_{1,2r-1} + U_6 \otimes A_{31} + U_6 \otimes C_{1,2r+1} + U_2 \otimes C_{2,2r}. \\
d(N \otimes C_{1,2r+1}) &= d(C_{1,2r+1}) \\
&= \delta_2^1(N, U_3) \otimes C_{1,2r} + \delta_2^1(N, L_5) \otimes A_{2r+2} + \delta_2^1(N, U_1) \otimes C_{1,2r+2} \\
&\quad + \delta_2^1(N, U_2) \otimes C_{2,2r+1}
\end{aligned}$$

$$
\begin{aligned}
&= U_3 \otimes N \otimes C_{1,2r} + U_6 \otimes W \otimes A_{2r+2} + U_1 \otimes N \otimes C_{1,2r+2} \\
&\quad + U_2 \otimes N \otimes C_{2,2r+2} \\
&= U_3 \otimes C_{1,2r} + U_6 \otimes A_{2r+2} + U_1 \otimes C_{1,2r+2} + U_2 \otimes C_{2,2r+1}. \\
d(N \otimes C_{1,2b+1}) &= d(C_{1,2b+1}) \\
&= \delta_2^1(N, U_3) \otimes C_{1,2b} + \delta_2^1(N, L_3L_5) \otimes A_> + \delta_2^1(N, R_4U_1) \otimes C_{1,>} \\
&\quad + \delta_2^1(N, U_2) \otimes C_{2,2b+1} \\
&= U_3 \otimes N \otimes C_{1,2b} + L_3U_6 \otimes W \otimes A_> + R_4U_1 \otimes N \otimes C_{1,>} \\
&\quad + U_2 \otimes N \otimes C_{2,2b+1} \\
&= U_3 \otimes C_{1,2b} + L_3U_6 \otimes A_{>,1} + R_4U_1 \otimes C_{1,>} + U_2 \otimes C_{2,2b+1}. \\
d(N \otimes C_{2\ell+1,>}) &= d(C_{2\ell+1,>}) \\
&= \delta_2^1(N, L_4) \otimes C_{2\ell+1,2b+1} + \delta_2^1(N, U_2) \otimes C_{2\ell+2,>} \\
&= L_4 \otimes N \otimes C_{2\ell+1,2b+1} + U_2 \otimes N \otimes C_{2\ell+2,>} \\
&= L_4 \otimes C_{2\ell+1,2b+1} + U_2 \otimes C_{2\ell+2,>}. \\
d(N \otimes C_{2\ell,1}) &= d(C_{2\ell,1}) \\
&= \delta_2^1(N, L_5U_3) \otimes B_{2r} + \delta_2^1(N, U_6) \otimes C_{2r-1,1} + \delta_2^1(N, U_1) \otimes C_{2r,2} \\
&\quad + \delta_2^1(N, U_1) \otimes C_{2r+1,1} \\
&= U_3U_6 \otimes W \otimes B_{2r} + U_5 \otimes N \otimes C_{2r-1,1} + U_1 \otimes N \otimes C_{2r,2} \\
&\quad + U_1 \otimes N \otimes C_{2r+1,1} \\
&= U_3U_6 \otimes B_{2r,1} + U_5 \otimes C_{2r-1,1} + U_1 \otimes C_{2r,2} + U_1 \otimes C_{2r+1,1}. \\
d(N \otimes C_{2\ell,2r}) &= d(C_{2\ell,2r}) \\
&= \delta_2^1(N, U_4) \times C_{2\ell,2r-1} + \delta_2^1(N, U_6) \otimes C_{2\ell-1,2r} \\
&\quad + \delta_2^1(N, U_5) \otimes C_{2\ell,2r+1} + \delta_2^1(N, U_1) \otimes C_{2\ell+1,2r} \\
&= U_4 \otimes N \otimes C_{2\ell,2r-1} + U_5 \otimes N \otimes C_{2\ell-1,2r} + U_6 \otimes N \otimes C_{2\ell,2r+1} \\
&\quad + U_1 \otimes N \otimes C_{2\ell+1,2r} \\
&= U_4 \otimes C_{2\ell,2r-1} + U_5 \otimes C_{2\ell-1,2r} + U_6 \otimes C_{2\ell,2r+1} + U_1 \otimes C_{2\ell+1,2r}. \\
d(N \otimes C_{2\ell,2r+1}) &= d(C_{2\ell,2r+1}) \\
&= \delta_2^1(N, U_3) \otimes C_{2\ell,2r} + \delta_2^1(N, U_6) \otimes C_{2\ell-1,2r+1} \\
&\quad + \delta_2^1(N, U_1) \otimes C_{2\ell,2r+2} + \delta_2^1(N, U_1) \otimes C_{2\ell+1,2r+1} \\
&= U_3 \otimes N \otimes C_{2\ell,2r} + U_5 \otimes N \otimes C_{2\ell-1,2r+1}
\end{aligned}
$$

$$
\begin{aligned}
&\quad + U_1 \otimes N \otimes C_{2\ell,2r+2} + U_1 \otimes N \otimes C_{2\ell+1,2r+1} \\
&= U_3 \otimes C_{2\ell,2r} + U_5 \otimes C_{2\ell-1,2r+1} + U_1 \otimes C_{2\ell,2r+2} + U_1 \otimes C_{2\ell+1,2r+1}. \\
d(N \otimes C_{2\ell,2b+1}) &= d(C_{2\ell,2b+1}) \\
&= \delta_2^1(N,U_3) \otimes C_{2\ell,2b} + \delta_2^1(N,U_6) \otimes C_{2\ell-1,2b+1} \\
&\quad + \delta_2^1(N,R_4U_1) \otimes C_{2\ell,>} + \delta_2^1(N,U_1) \otimes C_{2\ell+1,2b+1} \\
&= U_3 \otimes N \otimes C_{2\ell,2b} + U_5 \otimes N \otimes C_{2\ell-1,2b+1} + R_4U_1 \otimes N \otimes C_{2\ell,>} \\
&\quad + U_1 \otimes N \otimes C_{2\ell+1,2b+1} \\
&= U_3 \otimes C_{2\ell,2b} + U_5 \otimes C_{2\ell-1,2b+1} + R_4U_1 \otimes C_{2\ell,>} + U_1 \otimes C_{2\ell+1,2b+1}. \\
d(N \otimes C_{2\ell,>}) &= d(C_{2\ell,>}) \\
&= \delta_2^1(N,L_4) \otimes C_{2\ell,2b+1} + \delta_2^1(N,U_6) \otimes C_{2\ell-1,>} + \delta_2^1(N,U_1) \otimes C_{2\ell+1,>} \\
&= L_4 \otimes N \otimes C_{2\ell,2b+1} + U_5 \otimes N \otimes C_{2\ell-1,>} + U_1 \otimes N \otimes C_{2\ell+1,>} \\
&= L_4 \otimes C_{2\ell,2b+1} + U_5 \otimes C_{2\ell-1,>} + U_1 \otimes C_{2\ell+1,>}. \\
d(N \otimes C_{2\ell+1,1}) &= d(C_{2\ell+1,1}) \\
&= \delta_2^1(N,L_5U_3) \otimes B_{2\ell+1} + \delta_2^1(N,U_5) \otimes C_{2\ell,3} \\
&\quad + \delta_2^1(N,U_1) \otimes C_{2\ell+1,2} + \delta_2^1(N,U_2) \otimes C_{2\ell+2,1} \\
&= U_3U_6 \otimes W \otimes B_{2\ell+1} + U_6 \otimes N \otimes C_{2\ell,3} + U_1 \otimes N \otimes C_{2\ell+1,2} \\
&\quad + U_2 \otimes N \otimes C_{2\ell+2,1} \\
&= U_3U_6 \otimes B_{2\ell+1,1} + U_6 \otimes C_{2\ell,3} + U_1 \otimes C_{2\ell+1,2} + U_2 \otimes C_{2\ell+2,1}. \\
d(N \otimes C_{2\ell+1,2r}) &= d(C_{2\ell+1,2r}) \\
&= \delta_2^1(N,U_4) \otimes C_{2\ell+1,2r-1} + \delta_2^1(N,U_5) \otimes C_{2\ell,2r+2} \\
&\quad + \delta_2^1(N,U_5) \otimes C_{2\ell+1,2r+1} + \delta_2^1(N,U_2) \otimes C_{2\ell+2,2r} \\
&= U_4 \otimes N \otimes C_{2\ell+1,2r-1} + U_6 \otimes N \otimes C_{2\ell,2r+2} + U_6 \otimes N \otimes C_{2\ell+1,2r+1} \\
&\quad + U_2 \otimes N \otimes C_{2\ell+2,2r} \\
&= U_4 \otimes C_{2\ell+1,2r-1} + U_6 \otimes C_{2\ell,2r+2} + U_6 \otimes C_{2\ell+1,2r+1} + U_2 \otimes C_{2\ell+2,2r}. \\
d(N \otimes C_{2\ell+1,2r+1}) &= d(C_{2\ell+1,2r+1}) \\
&= \delta_2^1(N,U_3) \otimes C_{2\ell+1,2r} + \delta_2^1(N,U_5) \otimes C_{2\ell,2r+3} \\
&\quad + \delta_2^1(N,U_1) \otimes C_{2\ell+1,2r+2} + \delta_2^1(N,U_2) \otimes C_{2\ell+2,2r+1} \\
&= U_3 \otimes N \otimes C_{2\ell+1,2r} + U_6 \otimes N \otimes C_{2\ell,2r+3} + U_1 \otimes N \otimes C_{2\ell+1,2r+2} \\
&\quad + U_2 \otimes N \otimes C_{2\ell+2,2r+1}
\end{aligned}
$$

$$
\begin{aligned}
&= U_3 \otimes C_{2\ell+1,2r} + U_5 \otimes C_{2\ell,2r+2} + U_1 \otimes C_{2\ell+1,2r+2} + U_2 \otimes C_{2\ell+2,2r+1}.\\
d(N \otimes C_{2\ell+1,2b}) &= d(C_{2\ell+1,2b})\\
&= \delta_2^1(N, U_4) \otimes C_{2\ell+1,2b-1} + \delta_2^1(N, R_4U_5) \otimes C_{2\ell,>}\\
&\quad + \delta_2^1(N, U_5) \otimes C_{2\ell+1,2b+1} + \delta_2^1(N, U_2) \otimes C_{2\ell+2,2b}\\
&= U_4 \otimes N \otimes C_{2\ell+1,2b-1} + R_4U_6 \otimes N \otimes C_{2\ell,>} + U_6 \otimes N \otimes C_{2\ell+1,2b+1}\\
&\quad + U_2 \otimes N \otimes C_{2\ell+2,2b}\\
&= U_4 \otimes C_{2\ell+1,2b-1} + R_4U_6 \otimes C_{2\ell,>} + U_6 \otimes C_{2\ell+1,2b+1} + U_2 \otimes C_{2\ell+2,2b}.\\
d(N \otimes C_{2\ell+1,2b+1}) &= d(C_{2\ell+1,2b+1})\\
&= \delta_2^1(N, U_3) \otimes C_{2\ell+1,2b} + \delta_2^1(N, R_4U_1) \otimes C_{2\ell+1,>}\\
&\quad + \delta_2^1(N, U_2) \otimes C_{2\ell+2,2b+1}\\
&= U_3 \otimes N \otimes C_{2\ell+1,2b} + R_4U_1 \otimes N \otimes C_{2\ell+1,>} + U_2 \otimes N \otimes C_{2\ell+2,2b+1}\\
&= U_3 \otimes C_{2\ell+1,2b} + R_4U_1 \otimes C_{2\ell+1,>} + U_2 \otimes C_{2\ell+2,2b+1}.\\
d(N \otimes C_{2\ell+1,>}) &= d(C_{2\ell+1,>})\\
&= \delta_2^1(N, L_4) \otimes C_{2\ell+1,2b+1} + \delta_2^1(N, U_2) \otimes C_{2\ell+2,>}\\
&= L_4 \otimes N \otimes C_{2\ell+1,2b+1} + U_2 \otimes N \otimes C_{2\ell+2,>}\\
&= L_4 \otimes C_{2\ell+1,2b+1} + U_2 \otimes C_{2\ell+2,>}.
\end{aligned}
$$

As in the inductive proof presented in Section 5.4.3, the only change in the above calculation when $\ell = c$ is switching U_2 for R_2. Since the weight of both elements is outside of the $span\{e_5, e_6\}$, both commute with the map $\delta_2^1(N, -)$, so a practically identical calculation to the above yields the following.

$$
\begin{aligned}
d(C_{2c+1,1}) &= U_3U_6 \otimes B_{2c+1,1} + U_6 \otimes C_{2c,3}\\
&\quad + U_1 \otimes C_{2c+1,2} + R_2 \otimes C_{>,1}.\\
d(C_{2c+1,2r}) &= U_4 \otimes C_{2c+1,2r-1} + U_6 \otimes C_{2c,2r+2}\\
&\quad + U_6 \otimes C_{2c+1,2r+1} + R_2 \otimes C_{>,2r}.\\
d(C_{2c+1,2r+1}) &= U_3 \otimes C_{2c+1,2r} + U_6 \otimes C_{2c,2r+3}\\
&\quad + U_1 \otimes C_{2c+1,2r+2} + R_2 \otimes C_{>,2r+1}.\\
d(C_{2c+1,2b}) &= U_4 \otimes C_{2c+1,2b-1} + R_4U_6 \otimes C_{2c,>}\\
&\quad + U_6 \otimes C_{2c+1,2b+1} + R_2 \otimes C_{>,2b}\\
d(C_{2c+1,2b+1}) &= U_3 \otimes C_{2c+1,2b} + R_4U_1 \otimes C_{2c+1,>} + R_2 \otimes C_{>,2b+1}.
\end{aligned}
$$

$$d(C_{2c+1,>}) = L_4 \otimes C_{2c+1,2b+1} + R_2 \otimes C_{>,>}.$$

Continuing with the calculation, the map for the states $C_{>,k}$ and S_k is very similar.

$$\begin{aligned}
d(N \otimes C_{>,1}) &= d(C_{>,1}) \\
&= \delta_2^1(N, L_5U_3) \otimes B_> + \delta_2^1(N, L_2U_6) \otimes C_{2c+1,1} \\
&= U_3U_6 \otimes W \otimes B_> + L_2U_5 \otimes N \otimes C_{2c+1,1} \\
&= U_3U_6 \otimes B_{>,1} + L_2U_6 \otimes C_{2c+1,1}. \\
d(N \otimes C_{>,2r}) &= d(C_{>,2r}) \\
&= \delta_2^1(N, U_4) \otimes C_{>,2r-1} + \delta_2^1(N, L_2U_6) \otimes C_{2c+1,2r} \\
&\quad + \delta_2^1(N, U_5) \otimes C_{>,2r+1} + \delta_2^1(N, R_4R_3) \otimes S_{2r-1} \\
&= U_4 \otimes N \otimes C_{>,2r-1} + L_2U_5 \otimes N \otimes C_{2c+1,2r} + U_6 \otimes C_{>,2r+1} \\
&\quad + R_4R_3 \otimes N \otimes S_{2r-1} \\
&= U_4 \otimes C_{>,2r-1} + L_2U_5 \otimes C_{2c+1,2r} + U_6 \otimes C_{>,2r+1} + R_4R_3 \otimes S_{2r-1}. \\
d(N \otimes C_{>,2r+1}) &= d(C_{>,2r+1}) \\
&= \delta_2^1(N, U_3) \otimes C_{>,2r} + \delta_2^1(N, L_2U_6) \otimes C_{2c+1,2r+1} + \delta_2^1(N, R_4R_3) \otimes S_{2r} \\
&= U_3 \otimes N \otimes C_{>,2r} + L_2U_5 \otimes N \otimes C_{2c+1,2r+1} + R_4R_3 \otimes N \otimes S_{2r} \\
&= U_3 \otimes C_{>,2r} + L_2U_5 \otimes C_{2c+1,2r+1} + R_4R_3 \otimes S_{2r}. \\
d(N \otimes C_{>,>}) &= d(C_{>,>}) \\
&= \delta_2^1(N, L_4) \otimes C_{>,2b+1} + \delta_2^1(N, L_2U_6) \otimes C_{2c+1,>} + \delta_2^1(N, R_3) \otimes S_{2b+1} \\
&= L_4 \otimes N \otimes C_{>,2b+1} + L_2U_5 \otimes N \otimes C_{2c+1,>} + R_3 \otimes N \otimes S_{2b+1} \\
&= L_4 \otimes C_{>,2b+1} + L_2U_5 \otimes C_{2c+1,>} + R_3 \otimes S_{2b+1}. \\
d(N \otimes S_1) &= d(S_1) \\
&= \delta_2^1(N, L_3L_4L_5) \otimes B_> + \delta_2^1(N, U_5) \otimes S_2 \\
&= L_3L_4U_6 \otimes W \otimes B_> + U_6 \otimes N \otimes S_2 \\
&= L_3L_4U_6 \otimes B_{>,1} + U_6 \otimes S_2. \\
d(N \otimes S_{2r}) &= d(S_{2r}) \\
&= \delta_2^1(N, U_3) \otimes S_{2r-1} + \delta_2^1(N, L_3L_4) \otimes C_{>,2r-1} \\
&= U_3 \otimes N \otimes S_{2r-1} + L_3L_4 \otimes N \otimes C_{>,2r-1} \\
&= U_3 \otimes S_{2r-1} + L_3L_4 \otimes C_{>,2r-1}.
\end{aligned}$$

$$\begin{aligned} d(N \otimes S_{2r+1}) &= d(S_{2r+1}) \\ &= \delta_2^1(N, U_4) \otimes S_{2r} + \delta_2^1(N, L_3L_4) \otimes C_{>,2r} + \delta_2^1(N, U_5) \otimes S_{2r+2} \\ &= U_4 \otimes N \otimes S_{2r} + L_3L_4 \otimes N \otimes C_{>,2r} + U_6 \otimes N \otimes S_{2r+2} \\ &= U_4 \otimes S_{2r} + L_3L_4 \otimes C_{>,2r} + U_6 \otimes S_{2r+2}. \\ d(N \otimes S_{2b+1}) &= d(S_{2b+1}) \\ &= \delta_2^1(N, U_4) \otimes S_{2b} + \delta_2^1(N, L_3L_4) \otimes C_{>,2b} + \delta_2^1(N, L_3U_5) \otimes C_{>,>} \\ &= U_4 \otimes N \otimes S_{2b} + L_3L_4 \otimes N \otimes C_{>,2b} + L_3U_6 \otimes N \otimes C_{>,>} \\ &= U_4 \otimes S_{2b} + L_3L_4 \otimes C_{>,2b} + L_3U_6 \otimes C_{>,>}. \end{aligned}$$

The above calculates the map d for the Type D structure $\mathcal{P}^5 \boxtimes Y$ with domain in the states C_{ij} or S_k. In order to complete the base case, one must take the box-tensor product with $\mathcal{P}^5$ a second time. But, note that every algebra element $a \in \mathcal{A}$ appearing in $d : \mathcal{P}^5 \boxtimes Y \to \mathcal{A} \otimes \mathcal{P}^5 \boxtimes Y$ has only integer weight in $span\{e_5, e_6\}$.

Then, observing that since all of the states $I_{135} \cdot B_{i,1}$, $I_{235} \cdot B_{>,1}$, $I_{135} \cdot C_{ij}$ and $I_{345} \cdot S_\ell$ have 5 in their associated idempotents, the $\mathcal{P}^5$-tensor coordinate in $\mathcal{P}^5 \boxtimes \left(\mathcal{P}^5 \boxtimes Y\right)$ must be N. Hence, since in $\mathcal{P}^5$ only algebra inputs $a \in \mathcal{A}$ of half-integer weight result in $N \to W$ or $N \to E$, it is clear that the image of d for the states above must fix N in the $\mathcal{P}^5$ tensor coordinate. So the effect of taking the second tensor product with $\mathcal{P}^5$ is to simply swap U_5 and U_6. This yields a Type D structure matching that in Figure 5.12, and so P_2, the base case, holds.

Induction

Assume for inductive purposes that the statement P_{2k} is true, so that the Type D structure $\left(\mathcal{P}^5\right)^{2k} \boxtimes Y$ has maps and states as shown in Figure 5.12.

Then, the same argument can be applied as when studying the base case. Since all of the states shown in Figure 5.11 have 5 in their idempotent, the only generator of $\mathcal{P}^5$ to produce a non-zero tensor product under taking the box-tensor product of the DA-bimodule and Type D structure is N.

Since there are no algebra elements featuring L_5, R_5, L_6 and R_6, every algebra element commutes with the maps δ_k^1, switching the role of e_5 and e_6. Then, taking the box-tensor product with $\mathcal{P}^5$ twice thus leaves the module and associated map unchanged, hence

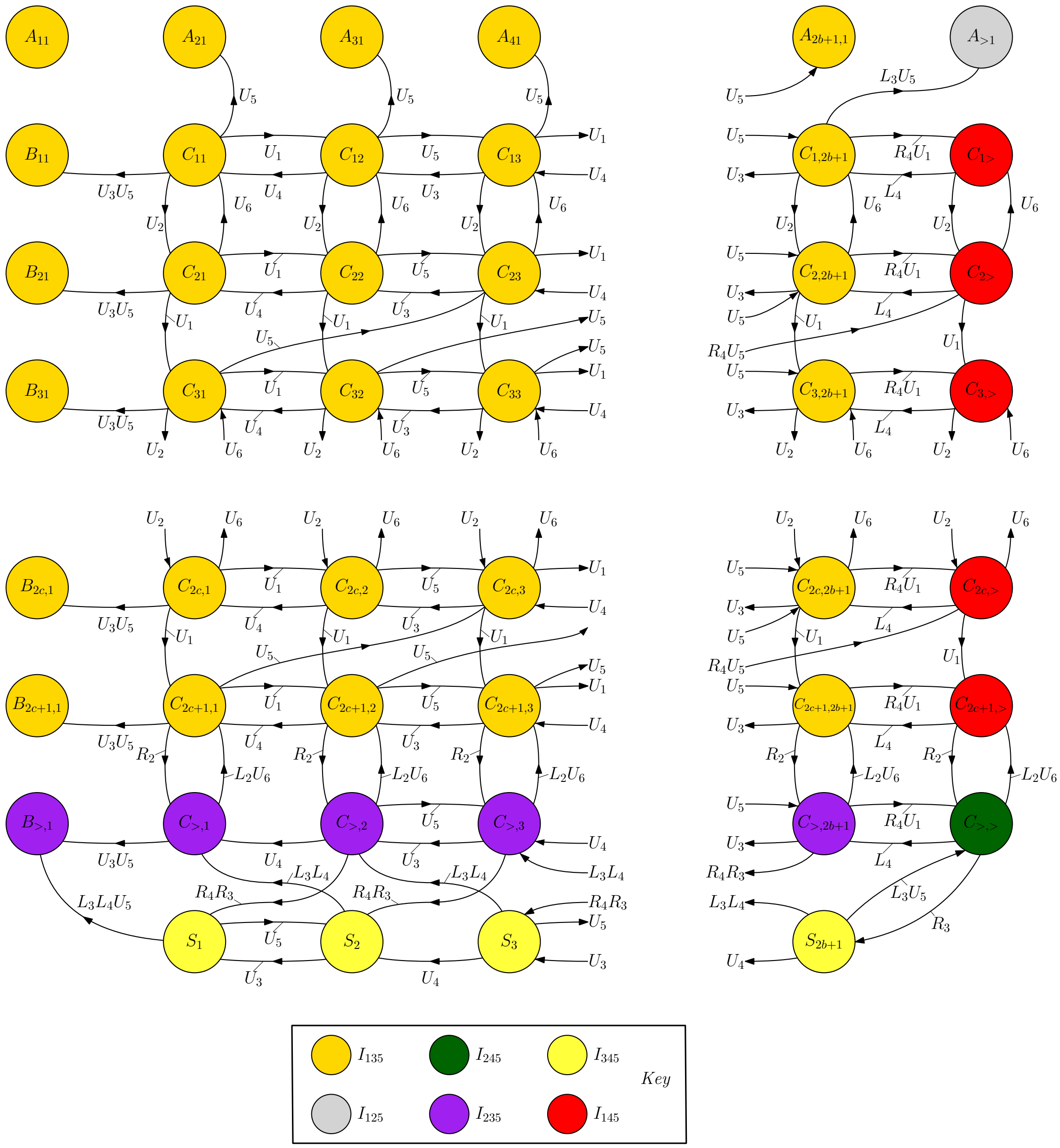


Figure 5.12: *This is a weighted, direted graph demonstrating the maps from C_{ij} and S_k states in the Type D structure $\left(\mathcal{P}^5\right)^{2k} \boxtimes Y$, where $k \geq 1$.*

$P_{2k} \Rightarrow P_{2k+2}$. Consequently, the statement P_{2n} is true by mathematical induction.

5.5.2 A-states and D-states

As seen when calculating the map in the Type D structure with domain in the C_{ij} and S_k states, since 5 does not belong to the idempotent of $I_{134} \cdot A_k \in (Y, d)$, there is more than one valid cardinal generator in $\mathcal{P}^5$ that yields a non-zero tensor product in $\mathcal{P}^5 \boxtimes Y$: namely $I_{135} \cdot W \cdot I_{134}$ and $I_{134} \cdot S \cdot I_{134}$. But, as noted above, this is not the case for states $I_{123} \cdot D_k$, which only has a valid tensor product with $I_{123} \cdot S \cdot I_{123}$.

The generators for Type D structure for $\left(\mathcal{P}^5\right)^{2k} \boxtimes Y$ that are A_{jk} and D_k states are featured in Figure 5.14, with the associated maps and algebraic elements. So, define P_{2k} as the statement that the maps and generators are as displayed in Figure 5.14.

Base case: $k = 1$

The Type D structure $\mathcal{P}^5 \boxtimes Y$ has the following generators that are either A_{jk} states or D_k states: i.e. have either A_j or D_k as their Y-tensor coordinate.

$$\begin{aligned} A_{j1} &= I_{135} \cdot W \otimes A_j \\ A_{j,>} &= I_{134} \cdot S \otimes A_j \\ A_{>,>} &= I_{124} \cdot S \otimes A_> \\ D_k &= I_{123} \cdot S \otimes D_k. \end{aligned}$$

Then, the calculation of the map d in the Type D structure $\mathcal{P}^5 \boxtimes Y$ is as follows.

$$\begin{aligned} d(A_{11}) &= \delta_2^1(W, U_5) \otimes A_2 + \delta_2^1(W, U_1U_4) \otimes B_1 + \delta_1^1(W) \otimes A_1 \\ &\quad + \delta_2^1(W, R_5U_1) \otimes C_{12} \\ &= U_6 \otimes W \otimes A_2 + U_1U_4 \otimes W \otimes B_1 + L_5 \otimes S \otimes A_1 \\ &\quad + U_1 \otimes N \otimes C_{12} \\ &= U_6 \otimes A_{21} + U_1U_4 \otimes B_{11} + L_5 \otimes A_{1,>} + U_1 \otimes C_{12}. \\ d(A_{2r,1}) &= \delta_2^1(W, U_3) \otimes A_{2r-1} + \delta_2^1(W, L_3L_4) \otimes D_{2r-1} + \delta_2^1(W, U_1) \otimes A_{2r+1} \\ &\quad + \delta_2^1(W, R_5U_1) \otimes C_{1,2r+1} + \delta_1^1(W) \otimes A_{2r} \\ &= U_3 \otimes W \otimes A_{2r-1} + L_3L_4 \otimes I_{135} \cdot W \otimes D_{2r-1} + U_1 \otimes W \otimes A_{2r+1} \\ &\quad + U_1 \otimes N \otimes C_{1,2r+1} + L_5 \otimes S \otimes A_{2r} \end{aligned}$$

$$
\begin{aligned}
&= U_3 \otimes A_{2r-1,1} + \mathbf{0} \otimes D_{2r-1} + U_1 \otimes A_{2r+1,1} \\
&\quad + U_1 \otimes C_{1,2r+1} + L_5 \otimes A_{2r,>}. \\
d(A_{2r+1,1}) &= \delta_2^1(W, U_4) \otimes A_{2r} + \delta_2^1(W, L_3L_4) \otimes D_{2r} + \delta_2^1(W, U_5) \otimes A_{2r+2} \\
&\quad + \delta_2^1(W, R_5U_1) \otimes C_{1,2r+1} + \delta_1^1(W) \otimes A_{2r+1} \\
&= U_4 \otimes W \otimes A_{2r} + \mathbf{0} \otimes D_{2r} + U_6 \otimes W \otimes A_{2r+2} \\
&\quad + U_1 \otimes N \otimes C_{1,2r+2} + L_5 \otimes S \otimes A_{2r+1} \\
&= U_4 \otimes A_{2r,1} + U_6 \otimes A_{2r+2,1} + U_1 \otimes C_{1,2r+2} + L_5 \otimes A_{2r+1,>}. \\
d(A_{2b+1,1}) &= \delta_2^1(W, U_4) \otimes A_{2b} + \delta_2^1(W, L3U_5) \otimes A_> + \delta_2^1(W, R_5R_4U_1) \otimes C_{1,>} \\
&\quad + \delta_1^1(W) \otimes A_{2b+1} \\
&= U_4 \otimes W \otimes A_{2b} + L_3U_6 \otimes W \otimes A_> + R_4U_1 \otimes N \otimes C_{1,>} \\
&\quad + L_5 \otimes S \otimes A_{2b+1} \\
&= U_4 \otimes A_{2b,1} + L_3U_6 \otimes A_{>,1} + R_4U_1 \otimes C_{1,>} + L_5 \otimes A_{2b+1,>}. \\
d(A_{>,1}) &= \delta_2^1(W, R_3) \otimes A_{2b+1} + \delta_2^1(W, L_4) \otimes D_{2b+1} + \delta_1^1(W) \otimes A_> \\
&= R_3 \otimes W \otimes A_{2b+1} + L_4 \otimes W \otimes D_{2b+1} + L_5 \otimes S \otimes A_> \\
&= R_3 \otimes A_{2b+1,1} + \mathbf{0} \otimes D_{2b+1} + L_5 \otimes A_{>>}.
\end{aligned}
$$

The matching terms in the outgoing algebra of the Type D structure are C_{14}, C_{26} and C_{35}. These are algebraic inputs to the maps $\delta_2^1(S, -)$, and so one yields the following.

$$
\begin{aligned}
d(A_{1,>}) &= \delta_2^1(S, U_1U_4) \otimes B_1 + \delta_2^1(S, U_5) \otimes A_2 + \delta_2^1(S, R_5U_1) \otimes C_{12} \\
&\quad + \delta_2^1(S, C_{26}) \otimes A_1 \\
&= U_1U_4 \otimes S \otimes B_1 + U_6 \otimes I_{134} \cdot S \otimes A_2 + \mathbf{0} \otimes C_{12} + R_5U_2 \otimes W \otimes A_1 \\
&= U_1U_4 \otimes B_{1,>} + \mathbf{0} \otimes A_{2,>} + R_5U_2 \otimes A_{11}. \\
d(A_{2r,>}) &= \delta_2^1(S, U_3) \otimes A_{2r-1,>} + \delta_2^1(S, L_3L_4) \otimes D_{2r-1} + \delta_2^1(S, U_1) \otimes A_{2r+1} \\
&\quad + \delta_2^1(S, R_5U_1) \otimes C_{1,2r+1} + \delta_2^1(S, C_{26}) \otimes A_{2r} \\
&= U_3 \otimes S \otimes A_{2r-1} + L_3L_4 \otimes S \otimes D_{2r-1} + U_1 \otimes S \otimes A_{2r+1} \\
&\quad + \mathbf{0} \otimes C_{1,2r+1} + R_5U_2 \otimes W \otimes A_{2r} \\
&= U_3 \otimes A_{2r-1,>} + L_3L_4 \otimes D_{2r-1} + U_1 \otimes A_{2r+1,>} + R_5U_2 \otimes A_{2r,1}. \\
d(A_{2r+1,>}) &= \delta_2^1(S, U_4) \otimes A_{2r} + \delta_2^1(S, L_3L_4) \otimes D_{2r} + \delta_2^1(S, U_5) \otimes A_{2r+2} \\
&\quad + \delta_2(S, R_5U_1) \otimes C_{1,2r+2} + \delta_2^1(S, C_{26}) \otimes A_{2r+1} \\
&= U_4 \otimes S \otimes A_{2r} + L_3L_4 \otimes S \otimes D_{2r} + I_{134} \cdot U_6 \otimes S \otimes A_{2r+2}
\end{aligned}
$$

$$
\begin{aligned}
&+ \mathbf{0} \otimes C_{1,2r+2} + R_5U_2 \otimes W \otimes A_{2r+1} \\
&= U_4 \otimes A_{2r,>} + L_3L_4 \otimes D_{2r} + R_5U_2 \otimes A_{2r+1,1}. \\
d(A_{2b+1,>}) &= \delta_2^1(S, U_4) \otimes A_{2b} + \delta_2^1(S, L_3L_4) \otimes D_{2b} + \delta_2^1(S, C_{26}) \otimes A_{2b+1} \\
&= U_4 \otimes S \otimes A_{2b} + L_3L_4 \otimes S \otimes D_{2b} + R_5U_2 \otimes W \otimes A_{2b+1} \\
&= U_4 \otimes A_{2b,>} + L_3L_4 \otimes D_{2b} + R_5U_2 \otimes A_{2b+1,1}. \\
d(A_{>,>}) &= \delta_2(S, R_3) \otimes A_{2b+1} + \delta_2^1(S, L_4) \otimes D_{2b+1} + \delta_2^1(S, C_{26}) \otimes A_> \\
&= R_3 \otimes S \otimes A_{2b+1} + L_4 \otimes S \otimes D_{2b+1} + R_5U_2 \otimes W \otimes A_> \\
&= R_3 \otimes A_{2b+1,>} + L_4 \otimes D_{2b+1} + R_5U_2 \otimes A_{>,1}. \\
d(D_1) &= \delta_2^1(S, R_4R_3U_1) \otimes B_1 + \delta_2^1(S, U_1) \otimes D_2 \\
&= R_4R_3U_1 \otimes S \otimes B_1 + U_1 \otimes S \otimes D_2 \\
&= R_4R_3U_1 \otimes B_{1,>} + U_1 \otimes D_2. \\
d(D_{2r}) &= \delta_2^1(S, U_4) \otimes D_{2r-1} + \delta_2^1(S, R_4R_3) \otimes A_{2r-1} \\
&= U_4 \otimes S \otimes D_{2r-1} + R_4R_3 \otimes S \otimes A_{2r-1} \\
&= U_4 \otimes D_{2r-1} + R_4R_3 \otimes A_{2r-1,>}. \\
d(D_{2r+1}) &= \delta_2^1(S, U_3) \otimes D_{2r} + \delta_2^1(S, U_1) \otimes D_{2r+2} + \delta_2^1(S, R_4R_3) \otimes A_{2r} \\
&= U_3 \otimes S \otimes D_{2r} + U_1 \otimes S \otimes D_{2r+2} + R_4R_3 \otimes S \otimes A_{2r} \\
&= U_3 \otimes D_{2r} + U_1 \otimes D_{2r+2} + R_4R_3 \otimes A_{2r,>}. \\
d(D_{2b+1}) &= \delta_2^1(S, U_3) \otimes D_{2b} + \delta_2^1(S, R_4U_1) \otimes A_> + \delta_2^1(S, R_4R_3) \otimes A_{2b} \\
&= U_3 \otimes S \otimes D_{2b} + R_4U_1 \otimes S \otimes A_> + R_4R_3 \otimes S \otimes A_{2b} \\
&= U_3 \otimes D_{2b} + R_4U_1 \otimes A_{>,>} + R_4R_3 \otimes A_{2b}.
\end{aligned}
$$

This does not yet verify the base case, but do note that because of the idempotents I_{134} associated to states $A_{j,>}$, the 'horizontal' arrows in the picture differ between the bottom row A_{j1} and $A_{j,>}$.

Taking the second tensor product with $\mathcal{P}^5$, and noting that if $5 \in I_x$ for any associated idempotent with a generator of the Type D structure $\left(\mathcal{P}^5\right) \boxtimes Y$, the only non-zero tensor product would be with the generator $N \in \mathcal{P}^5$. Hence, the module $\left(\mathcal{P}^5\right)^2 \boxtimes Y$ has the following generators.

$$A_{k1} = I_{135} \cdot N \otimes A_{k1}$$

$$A_{>,1} = I_{125} \cdot N \otimes A_{>,1}$$

$$A_{k2} = I_{135} \cdot W \otimes A_{k,>}$$

$$A_{>>} = I_{124} \cdot S \otimes A_{>,>}$$

$$D_r = I_{123} \cdot S \otimes D_r.$$

The corresponding map d for the Type D structure $\left(\mathcal{P}^5\right)^2 \boxtimes Y$ are then as follows.

$$\begin{aligned}
d(A_{11}) &= \delta_2^1(N, U_6) \otimes A_{21} + \delta_2^1(N, U_1U_4) \otimes B_{11} \\
&\quad + \delta_2^1(N, L_5) \otimes A_{1,>} + \delta_2^1(N, U_1) \otimes C_{12} \\
&= U_5 \otimes N \otimes A_{21} + U_1U_4 \otimes N \otimes B_{11} + U_6 \otimes W \otimes A_{1,>} \\
&\quad + U_1 \otimes N \otimes C_{12} \\
&= U_5 \otimes A_{21} + U_1U_4 \otimes B_{11} + U_6 \otimes A_{12} + U_1 \otimes C_{12}. \\
d(A_{2r,1}) &= \delta_2^1(N, U_3) \otimes A_{2r-1,1} + \delta_2^1(N, U_1) \otimes A_{2r+1,1} \\
&\quad + \delta_2^1(N, U_1) \otimes C_{1,2r+1} + \delta_2^1(N, L_5) \otimes A_{2r,>} \\
&= U_3 \otimes N \otimes A_{2r-1,1} + U_1 \otimes N \otimes A_{2r+1,1} \\
&\quad + U_1 \otimes N \otimes C_{1,2r+1} + U_6 \otimes W \otimes A_{2r,>} \\
&= U_3 \otimes A_{2r-1,1} + U_1 \otimes A_{2r+1,1} + U_1 \otimes C_{1,2r+1} + U_6 \otimes A_{2r,2}. \\
d(A_{2r+1,1}) &= \delta_2^1(N, U_4) \otimes A_{2r,1} + \delta_2^1(N, U_6) \otimes A_{2r+2,1} + \delta_2^1(N, U_1) \otimes C_{1,2r+2} \\
&\quad + \delta_2^1(N, L_5) \otimes A_{2r+1,>} \\
&= U_4 \otimes N \otimes A_{2r,1} + U_5 \otimes N \otimes A_{2r+2,1} + U_1 \otimes N \otimes C_{1,2r+2} \\
&\quad + U_6 \otimes W \otimes A_{2r+1,>} \\
&= U_4 \otimes A_{2r,1} + U_5 \otimes A_{2r+2,1} + U_1 \otimes C_{1,2r+2} + U_6 \otimes A_{2r+1,2}. \\
d(A_{2b+1,1}) &= \delta_2^1(N, U_4) \otimes A_{2b,1} + \delta_2^1(N, L_3U_6) \otimes A_{>,1} + \delta_2^1(N, R_4U_1) \otimes C_{1,>} \\
&\quad + \delta_2^1(N, L_5) \otimes A_{2b+1,>} \\
&= U_4 \otimes N \otimes A_{2b,1} + L_3U_5 \otimes N \otimes A_{>,1} + R_4U_1 \otimes N \otimes C_{1,>} \\
&\quad + U_6 \otimes W \otimes A_{2b+1,>} \\
&= U_4 \otimes A_{2b,1} + L_3U_5 \otimes A_{>,1} + R_4U_1 \otimes C_{1,>} + U_6 \otimes A_{2b+1,2}. \\
d(A_{>,1}) &= \delta_2^1(N, R_3) \otimes A_{2b+1,1} + \delta_2^1(N, L_5) \otimes A_{>,>} \\
&= R_3 \otimes N \otimes A_{2b+1,1} + U_6 \otimes W \otimes A_{>,>} \\
&= R_3 \otimes A_{2b+1,1} + U_6 \otimes A_{>,2}. \\
d(A_{12}) &= \delta_2^1(W, U_1U_4) \otimes B_{1,>} + \delta_2^1(W, R_5U_2) \otimes A_{11} + \delta_1^1(W) \otimes A_{1,>}
\end{aligned}$$

$$
\begin{aligned}
&= U_1U_4 \otimes W \otimes B_{1,>} + U_2 \otimes N \otimes A_{11} + L_5 \otimes S \otimes A_{1,>} \\
&= U_1U_4 \otimes B_{12} + U_2 \otimes A_{11} + L_5 \otimes A_{1,>}. \\
d(A_{2r,2}) &= \delta_2^1(W, U_3) \otimes A_{2r-1,>} + \delta_2^1(W, L_3L_4) \otimes D_{2r-1} + \delta_2^1(W, U_1) \otimes A_{2r+1,>} \\
&\quad + \delta_2^1(W, R_5U_2) \otimes A_{2r,1} + \delta_1^1(W) \otimes A_{2r,>} \\
&= U_3 \otimes W \otimes A_{2r-1,>} + L_3L_4 \otimes W \otimes D_{2r-1} + U_1 \otimes W \otimes A_{2r+1,>} \\
&\quad + U_2 \otimes N \otimes A_{2r,1} + L_5 \otimes S \otimes A_{2r,>} \\
&= U_3 \otimes A_{2r-1,2} + \mathbf{0} \otimes D_{2r-1} + U_1 \otimes A_{2r+1,2} + U_2 \otimes A_{2r,1} + L_5 \otimes A_{2r,>}. \\
d(A_{2r+1,2}) &= \delta_2^1(W, U_4) \otimes A_{2r,>} + \delta_2^1(W, L_3L_4) \otimes D_{2r} + \delta_2^1(W, R_5U_2) \otimes A_{2r+1,1} \\
&\quad + \delta_1^1(W) \otimes A_{2r+1,>} \\
&= U_4 \otimes W \otimes A_{2r,>} + \mathbf{0} \otimes D_{2r} + U_2 \otimes N \otimes A_{2r+1,1} \\
&\quad + L_5 \otimes S \otimes A_{2r+1,>} \\
&= U_4 \otimes A_{2r,2} + U_2 \otimes A_{2r+1,1} + L_5 \otimes A_{2r+1,>}. \\
d(A_{2b+1,2}) &= \delta_2^1(W, U_4) \otimes A_{2b,>} + \delta_2^1(W, L_3L_4) \otimes D_{2b} + \delta_2^1(W, R_5U_2) \otimes A_{2b+1,1} \\
&\quad + \delta_1^1(W) \otimes A_{2b+1,>} \\
&= U_4 \otimes W \otimes A_{2b,>} + \mathbf{0} \otimes D_{2b} + U_2 \otimes N \otimes A_{2b+1,1} \\
&\quad + L_5 \otimes S \otimes A_{2b+1,>} \\
&= U_4 \otimes A_{2b,2} + U_2 \otimes A_{2b+1,1} + L_5 \otimes A_{2b+1,>}. \\
d(A_{>,2}) &= \delta_2^1(W, R_3) \otimes A_{2b+1,>} + \delta_2^1(W, R_5U_2) \otimes A_{>,1} + \delta_2^1(W, L_4) \otimes D_{2b+1} \\
&\quad + \delta_1^1(W) \otimes A_{>,>} \\
&= R_3 \otimes W \otimes A_{2b+1,>} + U_2 \otimes N \otimes A_{>,1} + \mathbf{0} \otimes D_{2b+1} \\
&\quad + L_5 \otimes S \otimes A_{>,>} \\
&= R_3 \otimes A_{2b+1,2} + U_2 \otimes A_{>,1} + L_5 \otimes A_{>,>}. \\
d(A_{1,>}) &= \delta_2^1(S, U_1U_4) \otimes B_{1,>} + \delta_2(S, C_{36}) \otimes A_{1,>} + \delta_3^1(S, R_5U_2, U_6) \otimes A_{21} \\
&= U_1U_4 \otimes S \otimes B_{1,>} + R_5U_3 \otimes W \otimes A_{1,>} + R_5U_2 \otimes N \otimes A_{21} \\
&= U_1U_4 \otimes B_{1,>} + R_5U_3 \otimes A_{12} + R_5U_2 \otimes A_{21}. \\
d(A_{2r,>}) &= \delta_2^1(S, U_3) \otimes A_{2r-1,>} + \delta_2^1(S, L_3L_4) \otimes D_{2r-1} + \delta_2^1(S, U_1) \otimes A_{2r+1,>} \\
&\quad + \delta_2^1(S, C_{36}) \otimes A_{2r,>} \\
&= U_3 \otimes S \otimes A_{2r-1,>} + L_3L_4 \otimes S \otimes D_{2r-1} + U_1 \otimes S \otimes A_{2r+1,>} \\
&\quad + R_5U_3 \otimes W \otimes A_{2r,>}
\end{aligned}
$$

$$
\begin{aligned}
&= U_3 \otimes A_{2r-1,>} + L_3L_4 \otimes D_{2r-1} + U_1 \otimes A_{2r+1,>} + R_5U_3 \otimes A_{2r,2}.\\
d(A_{2r+1,>}) &= \delta_2^1(S, U_4) \otimes A_{2r,>} + \delta_2^1(S, L_3L_4) \otimes D_{2r} + \delta_2^1(S, C_{36}) \otimes A_{2r+1,>}\\
&\quad + \delta_3^1(S, R_5U_2, U_6) \otimes A_{2r+2,1}\\
&= U_4 \otimes S \otimes A_{2r,>} + L_3L_4 \otimes S \otimes D_{2r} + R_5U_3 \otimes W \otimes A_{2r+1,>}\\
&\quad + R_5U_2 \otimes N \otimes A_{2r+2,1}\\
&= U_4 \otimes A_{2r,>} + L_3L_4 \otimes D_{2r} + R_5U_3 \otimes A_{2r+1,2} + R_5U_2 \otimes A_{2r+2,1}.\\
d(A_{2b+1,>}) &= \delta_2^1(S, U_4) \otimes A_{2b,>} + \delta_2^1(S, L_3L_4) \otimes D_{2b} + \delta_2^1(S, C_{36}) \otimes A_{2b+1,>}\\
&\quad + \delta_3^1(S, R_5U_2, L_3U_6) \otimes A_{>,1}\\
&= U_4 \otimes S \otimes A_{2b,>} + L_3L_4 \otimes S \otimes D_{2b} + R_5U_3 \otimes W \otimes A_{2b+1,>}\\
&\quad + R_5L_3U_2 \otimes N \otimes A_{>,1}\\
&= U_4 \otimes A_{2b,>} + L_3L_4 \otimes D_{2b} + R_5U_3 \otimes A_{2b+1,2}\\
&\quad + R_5L_3U_2 \otimes A_{>,1}.\\
d(A_{>,>}) &= \delta_2^1(S, C_{36}) \otimes A_{>,>} + \delta_2^1(S, R_3) \otimes A_{2b+1,>} + \delta_2^1(S, L_4) \otimes D_{2b+1}\\
&= R_5U_3 \otimes W \otimes A_{>,>} + R_3 \otimes S \otimes A_{2b+1,>} + L_4 \otimes S \otimes D_{2b+1}\\
&= R_5U_3 \otimes A_{>,2} + R_3 \otimes A_{2b+1,>} + L_4 \otimes D_{2b+1}.\\
d(D_1) &= \delta_2^1(S, R_4R_3U_1) \otimes B_{1,>} + \delta_2^1(S, U_1) \otimes D_2\\
&= R_4R_3U_1 \otimes S \otimes B_{1,>} + U_1 \otimes S \otimes D_2\\
&= R_4R_3U_1 \otimes B_1 + U_1 \otimes D_2.\\
d(D_{2r}) &= \delta_2^1(S, U_4) \otimes D_{2r-1} + \delta_2^1(S, R_4R_3) \otimes A_{2r-1}\\
&= U_4 \otimes S \otimes D_{2r-1} + R_4R_3 \otimes S \otimes A_{2r-1}\\
&= U_4 \otimes D_{2r-1} + R_4R_3 \otimes A_{2r-1}.\\
d(D_{2r+1}) &= \delta_2^1(S, U_1) \otimes D_{2r+2} + \delta_2^1(S, U_3) \otimes D_{2r} + \delta_2^1(S, R_4R_3) \otimes A_{2r}\\
&= U_1 \otimes S \otimes D_{2r+2} + U_3 \otimes S \otimes D_{2r} + R_4R_3 \otimes S \otimes A_{2r}\\
&= U_1 \otimes D_{2r+2} + U_3 \otimes D_{2r} + R_4R_3 \otimes A_{2r}.\\
d(D_{2b+1}) &= \delta_2^1(S, U_3) \otimes D_{2b} + \delta_2^1(S, R_4U_1) \otimes A_{>} + \delta_2^1(S, R_4R_3) \otimes A_{2b}\\
&= U_3 \otimes S \otimes D_{2b} + R_4U_1 \otimes S \otimes A_{>} + R_4R_3 \otimes S \otimes A_{2b}\\
&= U_3 \otimes D_{2b} + R_4U_1 \otimes A_{>,>} + R_4R_3 \otimes A_{2b}.
\end{aligned}
$$

As can be seen through checking Figure 5.14, this matches the required form for Type D structures according to the inductive statement, and so the inductive statement is true for

$k = 1$, i.e. the base case P_2 holds.

Remark 5.5 *Before continuing by making the inductive assumption, note that under taking the next box tensor product with the DA-bimodule $\mathcal{P}^5$, one would have that*

$$\begin{aligned} d(A_{2r+1,3}) &= d(W \otimes A_{2r+1,>}) \\ &\ni \delta_2^1(W, R_5U_2) \otimes A_{2r+2,1} \\ &= U_2 \otimes N \otimes A_{2r+2,1}. \end{aligned}$$

This gives the red U_2 arrows highlighted in Figure 5.14.

Induction

Assume for inductive purposes that P_{2k} holds, i.e. that that the Type D structure $\left(\mathcal{P}^5\right)^{2k} \boxtimes Y$ has the form specified in Figure 5.14 when restricting to A and D states.

The effect of taking the tensor product with $(\mathcal{P})^2$ leaves all arrows at the bottom of Figure 5.14 unaffected. This is because all of the gold states — those with associated idempotent I_{135}, see Figure 5.11 — only have a valid tensor product with $N \otimes N$ in the $\left(\mathcal{P}^5\right)^2$ tensor coordinate.

For every state $A_{\ell,r}$ with $r \leq 2k-1$, the only algebra elements with weights in $span\{e_5, e_6\}$ are U_5 and U_6. Since $\delta_2^1(N, U_5) = U_6 \otimes N$, and $\delta_2^1(N, U_6) = U_5 \otimes N$, taking the tensor product with $\mathcal{P}^5$ twice has no effect on this part of the Type D structure. More formally, in the module

$$\left(\mathcal{P}^5\right)^2 \boxtimes \left(\left(\mathcal{P}^5\right)^{2k} \boxtimes Y\right),$$

those generators with $A_{\ell,r}$ in the right tensor coordinate have maps to the other generators with the right tensor coordinate as described in Figure 5.14, since every arrow is preserved (as $\delta_2^1(N, a) \neq 0$ for the algebra elements a in Figure 5.14), and U_5 and U_6 switch twice.

Furthermore, for the D-states, since the only valid tensor product is with $S \otimes S$, and the arrows from the D_ℓ states have weights outside $span\{e_5, e_6\}$, every algebra element commutes with the map δ_2^1. Consequently, since $B_{1,>} \in \left(\mathcal{P}^5\right)^{2k+2} \boxtimes Y$ is equal to $S \otimes S \otimes B_{1,>}$, taking this tensor product twice yields identical generators with maps with identical weights.

Then, consider the states $A_{\ell,2k}$ in $\left(\mathcal{P}^5\right)^{2k} \boxtimes Y$. Using the fact that $\delta_2^1(N, L_5) = U_6 \otimes W$, and that $\delta_2^1(W, R_5U_3) = U_3 \otimes N$, taking the tensor product with $\mathcal{P}^5$ once has the same effect as in the first part of the proof of the base case. Namely:

$$\begin{aligned} d(A_{\ell,2k}) &= d(N \otimes A_{\ell,2k}) \\ &\ni \delta_2^1(N, L_5) \otimes A_{\ell,>} \\ &= U_6 \otimes W \otimes A_{\ell,>} = U_6 \otimes A_{\ell,2k+1}. \end{aligned}$$

The remaining arrows from this state exhibit the same behaviour as the other A_{jk} states, since all other algebra elements on the edges starting at the vertex corresponding to $A_{\ell,2k}$ have weight outside $span\{e_5, e_6\}$.

Likewise, for the states $A_{\ell,2k+1} = W \otimes A_{\ell,>}$, the only edges in Figure 5.14 with weight in $span\{e_5, e_6\}$ are the R_5U_3 weighted edge to $A_{\ell,2k}$ and R_5U_2 weighted edge to $A_{\ell+1,2k-1}$. Every other edge has an algebra element that commutes with δ_2^1. The only edges left to consider are thus:

$$\begin{aligned} d(A_{\ell,2k+1}) &= d(W \otimes A_{\ell,>}) \\ &\ni \delta_2^1(W, R_5U_3) \otimes A_{\ell,2k} + \delta_2^1(W, R_5U_2) \otimes A_{\ell+1,2k-1} + \delta_1^1(W) \otimes A_{\ell,>} \\ &= U_3 \otimes N \otimes A_{\ell,2k} + U_2 \otimes N \otimes A_{\ell+1,2k-1} + L_5 \otimes S \otimes A_{\ell,>} \\ &= U_3 \otimes A_{\ell,2k} + U_2 \otimes A_{\ell+1,2k-1} + L_5 \otimes A_{\ell,>}. \end{aligned}$$

Similarly, the same edges are of concern in the calculation of $d(A_{\ell,>})$ for $A_{\ell,>} = S \otimes A_{\ell,>} \in \mathcal{P}^5 \boxtimes \left(\mathcal{P}^5\right)^{2k} \boxtimes Y$. The calculation is nearly identical in form to the calculation of $\left(\mathcal{P}^5\right)^2 \boxtimes Y$ from $\mathcal{P}^5 \boxtimes Y$. There are two cases, depending on whether ℓ is even or odd.

$$\begin{aligned} d(A_{2r+1,>}) &= d(S \otimes A_{2r+1,>}) \\ &\ni \delta_2^1(S, C_{26}) \otimes A_{2r+1,>} + \delta_3^1(S, R_5U_2, U_6) \otimes A_{2r+2,2k} \\ &= R_5U_2 \otimes W \otimes A_{2r+1,>} + R_5U_2 \otimes N \otimes A_{2r+2,2k} \\ &= R_5U_2 \otimes A_{2r+1,2k+1} + R_5U_2 \otimes A_{2r+2,2k}. \\ d(A_{2r,>}) &= d(S \otimes A_{2r,>}) \\ &\ni \delta_2^1(S, C_{26}) \otimes A_{2r,>} \\ &= R_5U_2 \otimes W \otimes A_{2r,>} \\ &= R_5U_2 \otimes A_{2r,2k+1}. \end{aligned}$$

All other edges from these vertices have weights outside the span where δ^1 has an action that does not commute with the algebra element.

The Type D structure for these states is then shown in Figure 5.13.

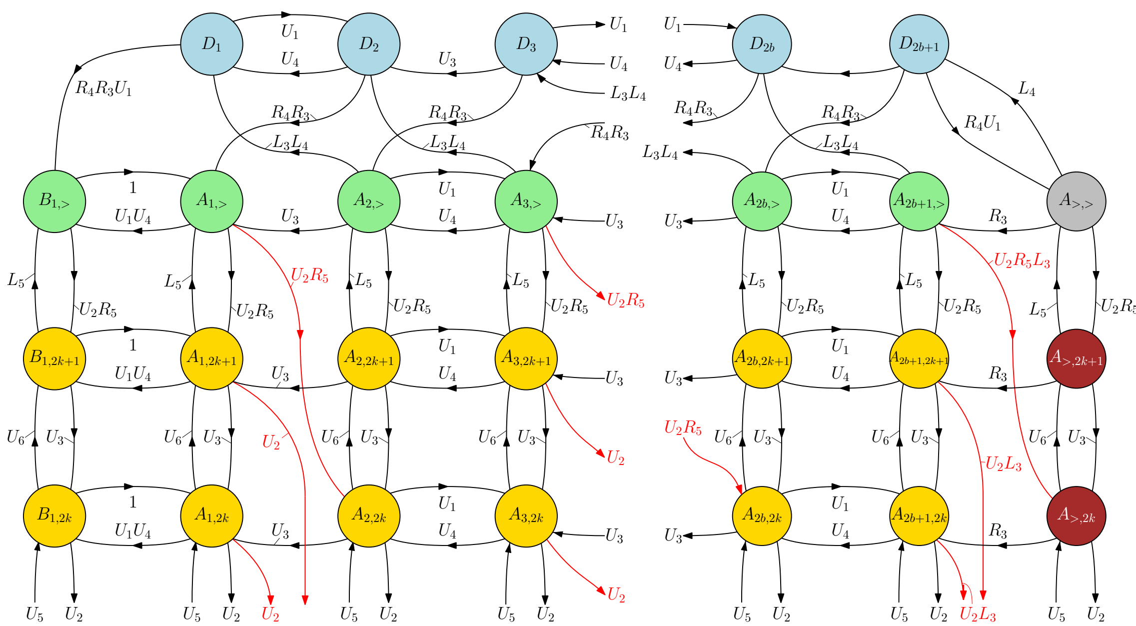


Figure 5.13: *The weighted directed graph for the Type D structure showing the intermediate stage of the inductive proof of the statement P_{2k}.*

Continuing with the inductive proof, using the fact that the matching terms in the outgoing algebra of the Type D structure $\left(\mathcal{P}^5\right)^{2k+1} \boxtimes Y$ are now C_{14}, C_{25} and C_{36}, the maps

$$\delta_2^1(S, C_{36}) = R_5U_3 \otimes W$$

$$\delta_2^1(W, R_5U_2) = U_2 \otimes N$$

$$\delta_2^1(N, L_5) = U_6 \otimes W$$

$$\delta_2^1(N, U_5) = U_6 \otimes N$$

$$\delta_2^1(N, U_6) = U_5 \otimes N,$$

applied within the tensor products $d(A_{\ell,2k+1}) = d(N \otimes A_{\ell,2k+1})$, $d(A_{\ell,2k+2}) = d(W \otimes A_{\ell,>})$ and $d(A_{\ell,>}) = d(S \otimes A_{\ell,>})$ yield the maps in the bimodule $\left(\mathcal{P}^5\right)^{2k+2} \boxtimes Y$, which matches the form described in Figure 5.14.

So, P_{2k} implies P_{2k+2}, and so by induction, since P_2 holds, P_{2n} holds for all $n \in \mathbb{N}$. This completes the determination of the Type D structure for these states by induction.

Remark 5.6 *Note, applying Proposition 4.13, the algebra elements $I_{134} \cdot R_5U_2 \cdot I_{135}$ and*

$I_{134} \cdot U_2R_5 \cdot I_{135}$ are equal because they have the same weight and idempotents. From the original definition of $\delta_2^1(S, C_{p6})$ in [49, Sec. 3.2], the form given by Ozsváth-Szabó is that the result is $U_pR_5 \otimes W \in \mathcal{A} \otimes \mathcal{P}^5$. But, in the above process p is either 2 or 3, and with either value, $U_pR_5 = [I_{134}, I_{135}, \frac{1}{2}e_5 + e_p] = R_5U_p$.

5.5.3 B-states

The remaining states to consider in the determination of the Type D structure for $\left(\mathcal{P}^5\right)^{2k} \boxtimes Y$ are the B_{ik} states. Using the recently proven equivalence between the Ozsváth-Szabó bordered construction of [49] and the classical knot Floer homology in the formulation of [60, 62], Lemma 2.10 suggests that these are the states which may not lie in the kernel of the chain complex $\mathcal{C}(D)$, where D is the diagram for $P(2a, -2b-1, 2c+1)$ with upper knot diagrams as displayed in Figure 5.11.

Indeed, as will be shown, the maps d in the Type D structure $\left(\mathcal{P}^5\right)^{2k} \boxtimes Y$ starting at B_{ik}-states do have representatives with 1 as the associated algebraic weight. The behaviour of these states is slightly more complex than can be easily seen in a weighted graph similar to Figure 5.12 and Figure 5.14.

In light of this, using the forms for the states B_{ik} as shown in Figure 5.11, the following lemma determines the less interesting behaviour of the B_{ik} states. Recall that $d(X) \ni b \otimes Y$ denotes that there is some weighted edge from X to Y with weight b in the corresponding graph for the Type D structure.

Lemma 5.7 *Where B_{ik} is of the form as described in Figure 5.11, the map $d(B_{ik})$ contains the following terms.*

$$
\begin{aligned}
d(B_{11}) &\ni 1 \otimes C_{11} + 1 \otimes A_{11} + U_6 \otimes B_{12} + U_2 \otimes B_{21}.\\
d(B_{2r,1}) &\ni 1 \otimes C_{2r,1} + U_1 \otimes B_{2r+1,1} + U_6 \otimes B_{2r,2}.\\
d(B_{2r+1,1}) &\ni 1 \otimes C_{2r+1,1} + 1 \otimes C_{2r,2} + U_2 \otimes B_{2r+2,1} + U_6 \otimes B_{2r+1,2} + U_4 \otimes B_{2r,1}.\\
d(B_{2c+1,1}) &\ni 1 \otimes C_{2c+1,1} + 1 \otimes C_{2c,2} + R_2 \otimes B_{>,1} + U_6 \otimes B_{2c+1,2} + U_4 \otimes B_{2c,1}.\\
d(B_{1,2r}) &\ni 1 \otimes A_{1,2r} + U_2 \otimes B_{1,2r-1} + U_2 \otimes B_{2,2r} + U_5 \otimes B_{1,2r+1}.\\
d(B_{1,2r+1}) &\ni 1 \otimes A_{1,2r+1} + U_3 \otimes B_{1,2r} + U_2 \otimes B_{2,2r+1} + U_6 \otimes B_{1,2r+2}.\\
d(B_{1,2a}) &\ni 1 \otimes A_{1,2a} + U_2 \otimes B_{1,2a-1} + U_2 \otimes B_{2,2a} + L_5 \otimes B_{1,>}.\\
d(B_{2r,2p}) &\ni U_3 \otimes B_{2r-1,2p-2} + U_2 \otimes B_{2r,2p-1} + U_1 \otimes B_{2r+1,2p} + U_5 \otimes B_{2r,2p+1}.\\
d(B_{2r,2p+1}) &\ni U_3 \otimes B_{2r-1,2p-1} + U_3 \otimes B_{2r,2p} + U_1 \otimes B_{2r+1,2p+1} + U_6 \otimes B_{2r,2p+2}.
\end{aligned}
$$

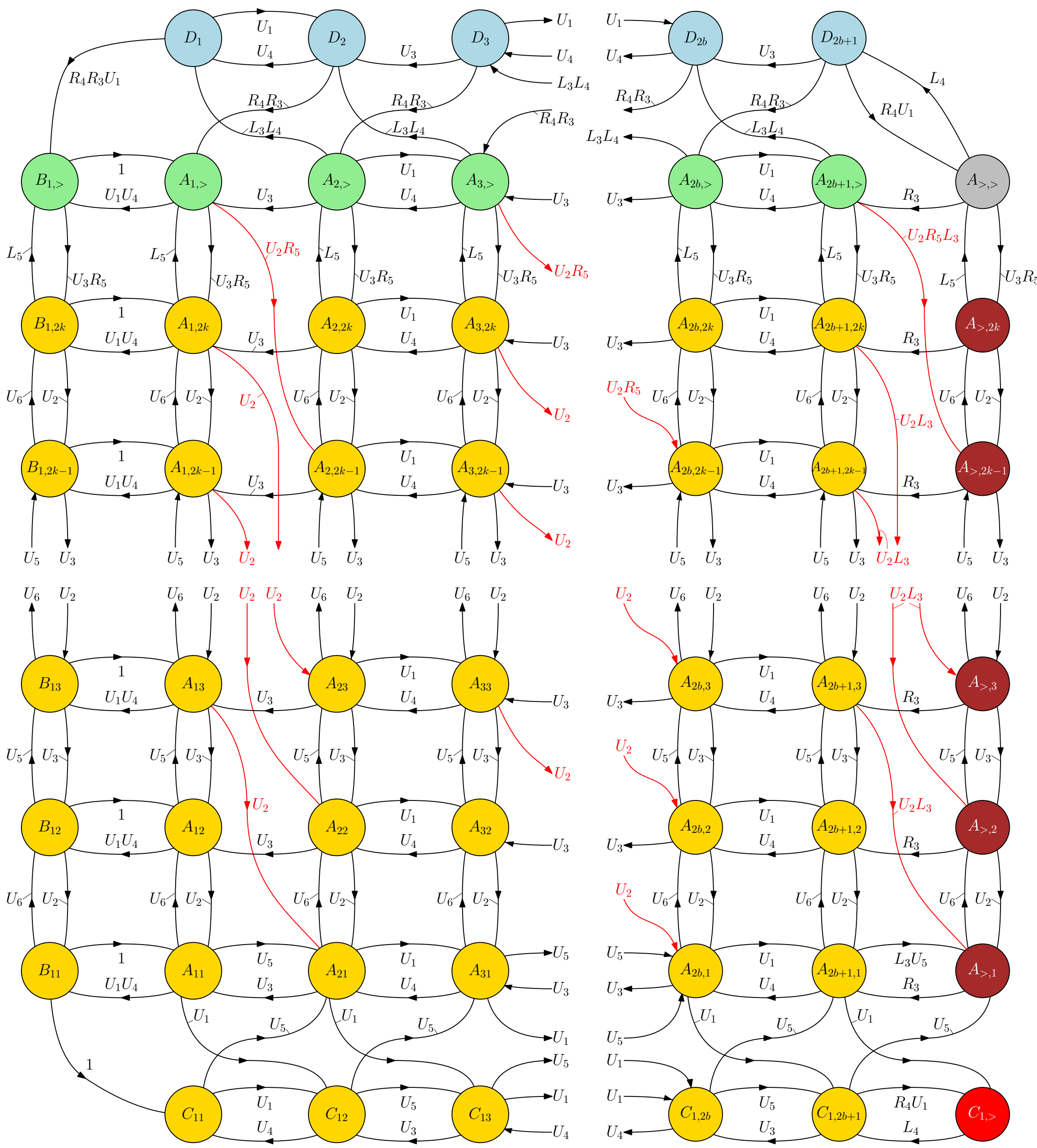


Figure 5.14: *Weighted, directed graph describing the Type D structure for the module $(\mathcal{P}^5)^{2k} \boxtimes Y$, where $k \geq 1$, restricting to the case where the domain is either an A-state or a D-state. Red arcs are highlighted to show that they have slightly unusual behaviour relative to the other maps.*

$$d(B_{2r,2a}) \ni U_3 \otimes B_{2r-1,2a-2} + U_2 \otimes B_{2r,2a-1} + U_1 \otimes B_{2r+1,2a} + L_5 \otimes B_{2r,>}.$$

$$d(B_{2r+1,2p}) \ni U_2 \otimes B_{2r+1,2p-1} + U_2 \otimes B_{2r+2,2p} + U_5 \otimes B_{2r+1,2p+1} + U_4 \otimes B_{2r-1,2p}.$$

$$d(B_{2r+1,2p+1}) \ni U_3 \otimes B_{2r+1,2p} + U_2 \otimes B_{2r+2,2p+1} + U_6 \otimes B_{2r+1,2p+2} + U_4 \otimes B_{2r,2p+1}.$$

$$d(B_{2c+1,2p}) \ni U_2 \otimes B_{2c+1,2p-1} + R_2 \otimes B_{>,2p} + U_5 \otimes B_{2c+1,2p+1} + U_4 \otimes B_{2c,2p}.$$

$$d(B_{2c+1,2p+1}) \ni U_3 \otimes B_{2c+1,2p} + R_2 \otimes B_{>,2p+1} + U_6 \otimes B_{2c+1,2p+2} + U_4 \otimes B_{2c,2p+1}.$$

$$d(B_{2r+1,2a}) \ni U_2 \otimes B_{2r+1,2a-1} + U_2 \otimes B_{2r+2,2a} + L_5 \otimes B_{2r+1,>} + U_4 \otimes B_{2r,2a}.$$

$$d(B_{2c+1,2a}) \ni U_2 \otimes B_{2c+1,2a-1} + R_2 \otimes B_{>,2a} + L_5 \otimes B_{2c+1,>} + U_4 \otimes B_{2c,2a}.$$

$$d(B_{1,>}) \ni 1 \otimes A_{1,>} + R_5U_3 \otimes B_{1,2a} + U_2 \otimes B_{2,>}.$$

$$d(B_{2r,>}) \ni R_5U_3 \otimes B_{2r-1,2a-1} + R_5U_3 \otimes B_{2r,2a} + U_1 \otimes B_{2r+1,>}.$$

$$d(B_{2r+1,>}) \ni R_5U_3 \otimes B_{2r+1,2a} + U_2 \otimes B_{2r+2,>} + U_4 \otimes B_{2r,>}.$$

$$d(B_{2c+1,>}) \ni R_5U_3 \otimes B_{2c+1,2a} + R_2 \otimes B_{>,>} + U_4 \otimes B_{2c,>}.$$

$$d(B_{>,1}) \ni 1 \otimes C_{>,1} + U_6 \otimes B_{>,2}.$$

$$d(B_{>,2}) \ni L_2 \otimes C_{2c+1,1} + U_2 \otimes B_{>,1} + U_5 \otimes B_{>,3}.$$

$$d(B_{>,2r}) \ni L_2U_3 \otimes B_{2c+1,2r-2} + U_2 \otimes B_{>,2r-1} + U_5 \otimes B_{>,2r+1}.$$

$$d(B_{>,2r+1}) \ni L_2U_3 \otimes B_{2c+1,2r-1} + U_3 \otimes B_{>,2r} + U_6 \otimes B_{>,2r+2}.$$

$$d(B_{>,2a}) \ni L_2U_3 \otimes B_{2c+1,2a-2} + U_2 \otimes B_{>,2a-1} + L_5 \otimes B_{>,>}.$$

$$d(B_{>,>}) \ni L_2R_5U_3 \otimes B_{2c+1,2a-1} + R_5U_3 \otimes B_{>,2a}.$$

Proof of Lemma 5.7

The proof will be by induction, so let Q_{2t} be the statement that the Type D structure $\left(\mathcal{P}^5\right)^{2t} \boxtimes Y$ has maps as described by the above.

Base case: k = 1

Using Figure 5.10, note that the idempotents associated with the B-states are $I_{134} \cdot B_r$ and $I_{234} \cdot B_>$. Similar to the determination of the maps for the states A_{jk}, in $\mathcal{P}^5 \boxtimes Y$ one thus has only states

$$B_{r,1} = I_{135} \cdot W \otimes B_r$$

$$B_{r,>} = I_{134} \cdot S \otimes B_r$$

$$B_{>,1} = I_{235} \cdot W \otimes B_>$$

$$B_{>,>} = I_{234} \cdot S \otimes B_>.$$

The map d for the Type D structure $\mathcal{P}^5 \boxtimes Y$ is then given by the following.

$$
\begin{aligned}
d(B_{1,1}) &\ni \delta_2^1(W,1)\otimes A_1 + \delta_2^1(W,R_5)\otimes C_{11} \\
&\quad + \delta_2^1(W,U_2)\otimes B_2 + \delta_1^1(W)\otimes B_1 \\
&= 1\otimes W\otimes A_1 + 1\otimes N\otimes C_{11} \\
&\quad + U_2\otimes W\otimes B_2 + L_5\otimes S\otimes B_1 \\
&= 1\otimes A_1 + 1\otimes C_{11} + U_2\otimes B_{21} + L_5\otimes B_{1,>}. \\
d(B_{2r,1}) &\ni \delta_2^1(W,R_5)\otimes C_{2r,1} + \delta_2^1(W,U_1)\otimes B_{2r+1} + \delta_1^1(W)\otimes B_{2r} \\
&= 1\otimes N\otimes C_{2r,1} + U_1\otimes W\otimes B_{2r+1} + L_5\otimes S\otimes B_{2r} \\
&= 1\otimes C_{2r,1} + U_1\otimes B_{2r+1,1} + L_5\otimes B_{2r,>}. \\
d(B_{2r+1,1}) &\ni \delta_2^1(W,U_4)\otimes B_{2r} + \delta_2^1(W,R_5)\otimes C_{2r,2} + \delta_2^1(W,R_5)\otimes C_{2r+1,1} \\
&\quad + \delta_2^1(W,U_2)\otimes B_{2r+2} + \delta_1^1(W)\otimes B_{2r+1} \\
&= U_4\otimes W\otimes B_{2r} + 1\otimes N\otimes C_{2r,2} + 1\otimes N\otimes C_{2r+1,1} \\
&\quad + U_2\otimes W\otimes B_{2r+2} + L_5\otimes S\otimes B_{2r+1} \\
&= U_4\otimes B_{2r,1} + 1\otimes C_{2r,2} + 1\otimes C_{2r+1,1} + U_2\otimes B_{2r+2,1} + L_5\otimes B_{2r+1,>}. \\
d(B_{2c+1,1}) &\ni \delta_2^1(W,U_4)\otimes B_{2c} + \delta_2^1(W,R_5)\otimes C_{2c,2} + \delta_2^1(W,R_5)\otimes C_{2c+1,1} \\
&\quad + \delta_2^1(W,R_2)\otimes B_{>} + \delta_1^1(W)\otimes B_{2c+1} \\
&= U_4\otimes W\otimes B_{2c} + 1\otimes N\otimes C_{2c,2} + 1\otimes N\otimes C_{2c+1,1} \\
&\quad + R_2\otimes W\otimes B_{>} + L_5\otimes S\otimes B_{2c+1} \\
&= U_4\otimes B_{2c,1} + 1\otimes C_{2c,2} + 1\otimes C_{2c+1,1} + +R_2\otimes B_{>,1} + L_5\otimes B_{2c+1,>}. \\
d(B_{>,1}) &\ni \delta_2^1(W,R_5)\otimes C_{>,1} + \delta_2^1(S,C_{26})\otimes B_{>} \\
&= 1\otimes N\otimes C_{>,1} + R_5U_2\otimes W\otimes B_{>} \\
&= 1\otimes C_{>,1} + R_5U_2\otimes B_{>,1}. \\
d(B_{1,>}) &\ni \delta_2^1(S,1)\otimes C_{11} + \delta_2^1(S,1)\otimes_1 + \delta_2^1(S,U_2)\otimes B_2 + \delta_2^1(S,C_{26})\otimes B_1 \\
&= 1\otimes S\otimes C_{11} + 1\otimes S\otimes A_1 + U_2\otimes S\otimes B_2 + R_5U_2\otimes W\otimes B_1 \\
&= \mathbf{0}\otimes C_{11} + 1\otimes A_{1,>} + U_2\otimes B_{2,>} + R_5U_2\otimes B_{11}. \\
d(B_{2r,>}) &\ni \delta_2^1(S,U_1)\otimes B_{2r+1} + \delta_2^1(S,C_{26})\otimes B_{2r} + \delta_3^1(S,R_5,U_6)\otimes C_{2r-1,1} \\
&= U_1\otimes S\otimes B_{2r+1} + R_5U_2\otimes W\otimes B_{2r} + R_5\otimes N\otimes C_{2r-1,1} \\
&= U_1\otimes B_{2r+1,>} + R_5U_2\otimes B_{2r,1} + R_5\otimes C_{2r-1,1}. \\
d(B_{2r+1,>}) &\ni \delta_2^1(S,U_4)\otimes B_{2r} + \delta_2^1(S,U_2)\otimes B_{2r+2} + \delta_2^1(S,C_{26})\otimes B_{2r+1}
\end{aligned}
$$

$$
\begin{aligned}
&= U_4 \otimes S \otimes B_{2r} + U_2 \otimes S \otimes B_{2r+2} + R_5 U_2 \otimes W \otimes B_{2r+1} \\
&= U_4 \otimes B_{2r,>} + U_2 \otimes B_{2r+2,>} + R_5 U_2 \otimes B_{2r+1,1}. \\
d(B_{2c+1,>}) &\ni \delta^1_2(S, U_4) \otimes B_{2c} + \delta^1_2(S, R_2) \otimes B_> + \delta^1_2(S, C_{26}) \otimes B_{2c+1} \\
&= U_4 \otimes S \otimes B_{2c} + R_2 \otimes S \otimes B_> + R_5 U_2 \otimes W \otimes B_{2c+1} \\
&= U_4 \otimes B_{2c,>} + R_2 \otimes B_{>,>} + R_5 U_2 \otimes B_{2c+1,1}. \\
d(B_{>,>}) &\ni \delta^1_2(S, C_{26}) \otimes B_> + \delta^1_3(S, R_5, L_2 U_6) \otimes C_{2c+1,1} \\
&= R_5 U_2 \otimes W \otimes B_> + R_5 L2 \otimes N \otimes C_{2c+1,1} \\
&= R_5 U_2 \otimes B_{>,1} + L_2 R_5 \otimes C_{2c+1,1}.
\end{aligned}
$$

Note that for any odd or even s, the state $B_{s,1} \in \left(\mathcal{P}^5\right) \boxtimes Y$ is such that the map $d(B_{s,1})$ contains terms with weight outside $span\{e_5, e_6\}$ which necessarily commute with $\delta^1_2(N, -)$. Since the associated idempotent to such states is $I_{135} \cdot B_{s,1}$, the only non-zero tensor product in $\left(\mathcal{P}^5\right)^2 \boxtimes Y$ is $N \otimes B_{s,1}$. If A is the collection of algebra elements in the image of d in $\mathcal{P}^5 \boxtimes Y$, and X_A the collection of module elements in $\mathcal{P}^5 \boxtimes Y$, then the calculation of $d(N \otimes B_{s,1})$ is given by

$$
\begin{aligned}
d(N \otimes B_{s,1}) &\ni \sum_{a \in A} \delta^1_2(N, a) \otimes X_a \\
&= \delta^1_2(N, L_5) \otimes B_{s,>} \quad + \sum_{\substack{a \in A \\ wt(a) \notin span\{e_5, e_6\}}} a \otimes N \otimes X_a \\
&= U_6 \otimes W \otimes B_{s,>} \quad + \sum_{\substack{a \in A \\ wt(a) \notin span\{e_5, e_6\}}} a \otimes X_a \\
&= U_6 \otimes B_{s,2} \quad + \sum_{\substack{a \in A \\ wt(a) \notin span\{e_5, e_6\}}} a \otimes X_a.
\end{aligned}
$$

The above calculation of $d(B_{s,1})$ for $B_{s,1} \in \mathcal{P}^5 \boxtimes Y$, and this observation, gives a calculation for $d(B_{s,1})$ with $B_{s,1} \in \left(\mathcal{P}^5\right)^2 \boxtimes Y$, and this agrees with the statement as in the lemma.

As an example, in $\mathcal{P}^5 \boxtimes Y$, it is calculated that

$$
d(B_{2r+1,1}) = U_4 \otimes B_{2r,1} + 1 \otimes C_{2r,2} + 1 \otimes C_{2r+1,1} + U_2 \otimes B_{2r+2,1} + L_5 \otimes B_{2r+1,>},
$$

and so in $\mathcal{P}^5 \boxtimes \left(\mathcal{P}^5 \boxtimes Y\right)$ one has that

$$
d(N \otimes B_{2r+1,1}) = \partial^1_2(N, U_4) \otimes B_{2r,1} + \partial^1_2(N, 1) \otimes C_{2r,2} + \partial^1_2(N, 1) \otimes C_{2r+1,1}
$$

$$
\begin{aligned}
&+ \partial_2^1(N, U_2) \otimes B_{2r+2,1} + \partial_2^1(N, L_5) \otimes B_{2r+1,>} \\
&= U_4 \otimes N \otimes B_{2r,1} + 1 \otimes N \otimes C_{2r,2} + 1 \otimes N \otimes C_{2r+1,1} \\
&\quad U_2 \otimes N \otimes B_{2r+2,1} + U_6 \otimes W \otimes B_{2r+1,>} \\
&= U_4 \otimes B_{2r,1} + 1 \otimes C_{2r,2} + 1 \otimes C_{2r+1,1} + U_2 \otimes B_{2r+2,1} + U_6 \otimes B_{2r+1,2}.
\end{aligned}
$$

Note that only the last term in each of the above has an algebraic input that is not outside $span\{e_5, e_6\}$, and all other algebra elements commute with $\partial_2^1(N, -)$.

Using the above calculation, and the calculation of $d(C_{r,1})$ in $\mathcal{P}^5 \boxtimes Y$ from Section 5.5.1, which states that $d(C_{r,1}) \ni U_3U_6 \otimes B_{r,1}$, the calculation of the remaining elements of $\mathcal{P}^5 \boxtimes \mathcal{P}^5 \boxtimes Y$ continues as follows.

$$
\begin{aligned}
d(B_{1,2}) &\ni \delta_2^1(W, 1) \otimes A_{1,>} + \delta_2^1(W, U_2) \otimes B_{2,>} \\
&\quad + \delta_2^1(W, R_5U_2) \otimes B_{11} + \delta_1^1(W) \otimes B_{1,>} \\
&= 1 \otimes W \otimes A_{1,>} + U_2 \otimes W \otimes B_{2,>} + U_2 \otimes N \otimes B_{11} \\
&\quad + L_5 \otimes S \otimes B_{1,>} \\
&= 1 \otimes A_{12} + U_2 \otimes B_{22} + U_2 \otimes B_{11} + L_5 \otimes B_{1,>}. \\
d(B_{2r,2}) &\ni \delta_2^1(W, U_1) \otimes B_{2r+1,>} + \delta_2^1(W, R_5U_2) \otimes B_{2r,1} \\
&\quad + \delta_2^1(W, R_5) \otimes C_{2r-1,1} + \delta_1^1(W) \otimes B_{2r,>} \\
&= U_1 \otimes W \otimes B_{2r+1,>} + U_2 \otimes N \otimes B_{2r,1} + 1 \otimes N \otimes C_{2r-1,1} \\
&\quad + L_5 \otimes S \otimes B_{2r,>} \\
&= U_1 \otimes B_{2r+1,2} + U_2 \otimes B_{2r,1} + 1 \otimes C_{2r-1,1} + L_5 \otimes B_{2r,>}. \\
d(B_{2r+1,2}) &\ni \delta_2^1(W, U_4) \otimes B_{2r,>} + \delta_2^1(W, U_2) \otimes B_{2r+2,>} \\
&\quad + \delta_2^1(W, R_5U_2) \otimes B_{2r+1,1} + \delta_1^1(W) \otimes B_{2r+1,>} \\
&= U_4 \otimes W \otimes B_{2r,>} + U_2 \otimes W \otimes B_{2r+2,>} + U_2 \otimes N \otimes B_{2r+1,1} \\
&\quad + L_5 \otimes S \otimes B_{2r+1,>} \\
&= U_4 \otimes B_{2r,2} + U_2 \otimes B_{2r+2,2} + U_2 \otimes B_{2r+1,1} + L_5 \otimes B_{2r+1,>}. \\
d(B_{2c+1,2}) &\ni \delta_2^1(W, U_4) \otimes B_{2c,>} + \delta_2^1(W, R_2) \otimes B_{>,>} \\
&\quad + \delta_2^1(W, R_5U_2) \otimes B_{2c+1,1} + \delta_1^1(W) \otimes B_{2c+1,>} \\
&= U_4 \otimes W \otimes B_{2c,>} + R_2 \otimes W \otimes B_{>,>} + U_2 \otimes N \otimes B_{2c+1,1} \\
&\quad + L_5 \otimes S \otimes B_{cr+1,>} \\
&= U_4 \otimes B_{2c,2} + R_2 \otimes B_{>,2} + U_2 \otimes B_{2c+1,1} + L_5 \otimes B_{2c+1,>}.
\end{aligned}
$$

$$
\begin{aligned}
d(B_{>,2}) &\ni \delta_2^1(W, R_5U_2) \otimes B_{>,1} + \delta_2^1(W, L_2R_5) \otimes C_{2c+1,1} + \delta_1^1(W) \otimes B_{>,>} \\
&= U_2 \otimes N \otimes B_{>,1} + L_2 \otimes N \otimes C_{2c+1,1} + L_5 \otimes S \otimes B_{>,>} \\
&= U_2 \otimes B_{>,1} + L_2 \otimes C_{2c+1,1} + L_5 \otimes B_{>,>}. \\
d(B_{1,>}) &= \delta_2^1(S, 1) \otimes A_{1,>} + \delta_2^1(S, U_2) \otimes B_{2,>} + \delta_2^1(S, C_{36}) \otimes B_{1,>} \\
&= 1 \otimes S \otimes A_{1,>} + U_2 \otimes S \otimes B_{2,>} + R_5U_3 \otimes W \otimes B_{1,>} \\
&= 1 \otimes A_{1,>} + U_2 \otimes B_{2,>} + R_5U_3 \otimes B_{12}. \\
d(B_{2r,>}) &\ni \delta_2^1(S, U_1) \otimes B_{2r+1,>} + \delta_2^1(S, C_{36}) \otimes B_{2r,>} + \delta_3^1(S, R_5, U_3U_6) \otimes B_{2r-1,1} \\
&= U_1 \otimes S \otimes B_{2r+1,>} + R_5U_3 \otimes W \otimes B_{2r,>} + R_5U_3 \otimes N \otimes B_{2r-1,1} \\
&= U_1 \otimes B_{2r+1,>} + R_5U_3 \otimes B_{2r,2} + R_5U_3 \otimes B_{2r-1,1}. \\
d(B_{2r+1,>}) &\ni \delta_2^1(S, U_4) \otimes B_{2r,>} + \delta_2^1(S, U_2) \otimes B_{2r+2,>} + \delta_2^1(S, C_{36}) \otimes B_{2r+1,>} \\
&= U_4 \otimes S \otimes B_{2r,>} + U_2 \otimes S \otimes B_{2r+2,>} + R_5U_3 \otimes W \otimes B_{2r+1,>} \\
&= U_4 \otimes B_{2r,>} + U_2 \otimes B_{2r+2,>} + R_5U_3 \otimes B_{2r+1,2}. \\
d(B_{>,>}) &\ni \delta_2^1(S, C_{36}) \otimes B_{>,>} + \delta_3^1(S, L_2R_5, U_3U_6) \otimes B_{2c+1,1} \\
&= R_5U_3 \otimes W \otimes B_{>,>} + L_2R_5U_3 \otimes N \otimes B_{2c+1,1} \\
&= R_5U_3 \otimes B_{>,2} + L_2R_5U_3 \otimes B_{2c+1,1}.
\end{aligned}
$$

Comparing the above to the form stated in Lemma 5.7, and using the fact that $a = 1$ here, one sees that the Type D structure maps are of the required form, and hence the base-case Q_2 holds.

Remark 5.8 *The full behaviour described in the lemma may appear to be absent here, since for example $B_{2r,2} = B_{2r,2a}$. In the statement of the lemma, one should make the assumption that the if a state does not exist by virtue of the available index being less than or equal to zero, then such a map does not exist. The lemma states that $d(B_{2r,2a}) \ni U_3 \otimes B_{2r-1,2a-2}$, however where $a = 1$, the states $B_{2r-1,0}$ does not exist.*

Inductive Assumption:

Assume for inductive purposes that the statement Q_{2a} holds. In a similar method to the determination of the map d for the C_{ij} states in Section 5.5.1, the states $B_{i,k}$ with $k < 2a$ have only U_5 and U_6 algebra terms in $d(B_{ik})$ within $span\{e_5, e_6\}$. The effect of taking the box-tensor product with $\mathcal{P}^5$ twice is to switch U_5 and U_6 coefficients once, and then switch back. Moreover, the associated idempotents for such states is I_{135} for $B_{i,k}$ with

$1 \leq i \leq 2c+1$ and $k < 2a$, and I_{235} for $B_{>,k}$, with $k < 2a$. Since 5 belongs to both idempotents, the only non-zero tensor product with $\mathcal{P} \boxtimes \mathcal{P}$ contains the term $N \otimes N$ in this tensor coordinate.

As a consequence, taking the box-tensor product with $\mathcal{P} \boxtimes \mathcal{P}$ yields the same number of states, with maps with the same algebra elements as weights. So, the behaviour at the end of the strand, where one can acquire new states is the only part left to study in the inductive process.

Figures 5.15 and 5.16 are useful in the determination of the Type D structure for $\mathcal{P} \boxtimes \mathcal{P} \boxtimes \big((\mathcal{P}^5)^{2t} \boxtimes Y\big)$, as they display the weighted graphs corresponding to the end of the strand after each tensor product with $\mathcal{P}$ is taken.

As displayed in Figure 5.15, the calculation of $d(B_{r,2a})$ involves algebra elements with weights outside of $span\{e_5, e_5\}$, excepting the term $\delta_2^1(N, L_5) \otimes B_{r,>}$. Evaluating this, one has that

$$\begin{aligned}
d(B_{1,2a}) &\ni \delta_2^1(N,1) \otimes A_{1,2a} + \delta_2^1(N,U_2) \otimes B_{1,2a-1} + \delta_2^1(N,U_2) \otimes B_{2,2a} \\
&\quad + \delta_2^1(N,L_5) \otimes B_{1,>} \\
&= 1 \otimes N \otimes A_{1,2a} + U_2 \otimes N \otimes B_{1,2a-1} + U_2 \otimes N \otimes B_{2,2a} \\
&\quad + U_6 \otimes W \otimes B_{1,>} \\
&= 1 \otimes A_{1,2a} + U_2 \otimes B_{1,2a-1} + U_2 \otimes B_{2,2a} + U_6 \otimes B_{1,2a+1}. \\
d(B_{2r,2a}) &\ni \delta_2^1(N, U_3 \otimes B_{2r-1,2a-2} + \delta_2^1(N,U_2) \otimes B_{2r,2a-1} + \delta_2^1(N,U_1) \otimes B_{2r+1,2a} \\
&\quad + \delta_2^1(N,L_5) \otimes B_{2r,>} \\
&= U_3 \otimes N \otimes B_{2r-1,2a-2} + U_2 \otimes N \otimes B_{2r,2a-1} + U_1 \otimes N \otimes B_{2r+1,2a} \\
&\quad + U_6 \otimes W \otimes B_{2r,>} \\
&= U_3 \otimes B_{2r-1,2a-2} + U_2 \otimes B_{2r,2a-1} + U_1 \otimes B_{2r+1,2a} + U_6 \otimes B_{2r,2a+1}. \\
d(B_{2r+1,2a}) &\ni \delta_2^1(N,U_2) \otimes B_{2r+1,2a-1} + \delta_2^1(N,U_2) \otimes B_{2r+2,2a} + \delta_2^1(N,L_5) \otimes B_{2r+1,>} \\
&\quad + \delta_2^1(N,U_4) \otimes B_{2r,2a} \\
&= U_2 \otimes N \otimes B_{2r+1,2a-1} + U_2 \otimes N \otimes B_{2r+2,2a} + U_6 \otimes W \otimes B_{2r+1,>} \\
&\quad + U_4 \otimes N \otimes B_{2r,2a} \\
&= U_2 \otimes B_{2r+1,2a-1} + U_2 \otimes B_{2r+2,2a} + U_6 \otimes B_{2r+1,2a+1} + U_4 \otimes B_{2r,2a}. \\
d(B_{2c+1,2a}) &\ni \delta_2^1(N,U_2) \otimes B_{2c+1,2a-1} + \delta_2^1(N,R_2) \otimes B_{>,2a} + \delta_2^1(N,L_5) \otimes B_{2c+1,>}
\end{aligned}$$

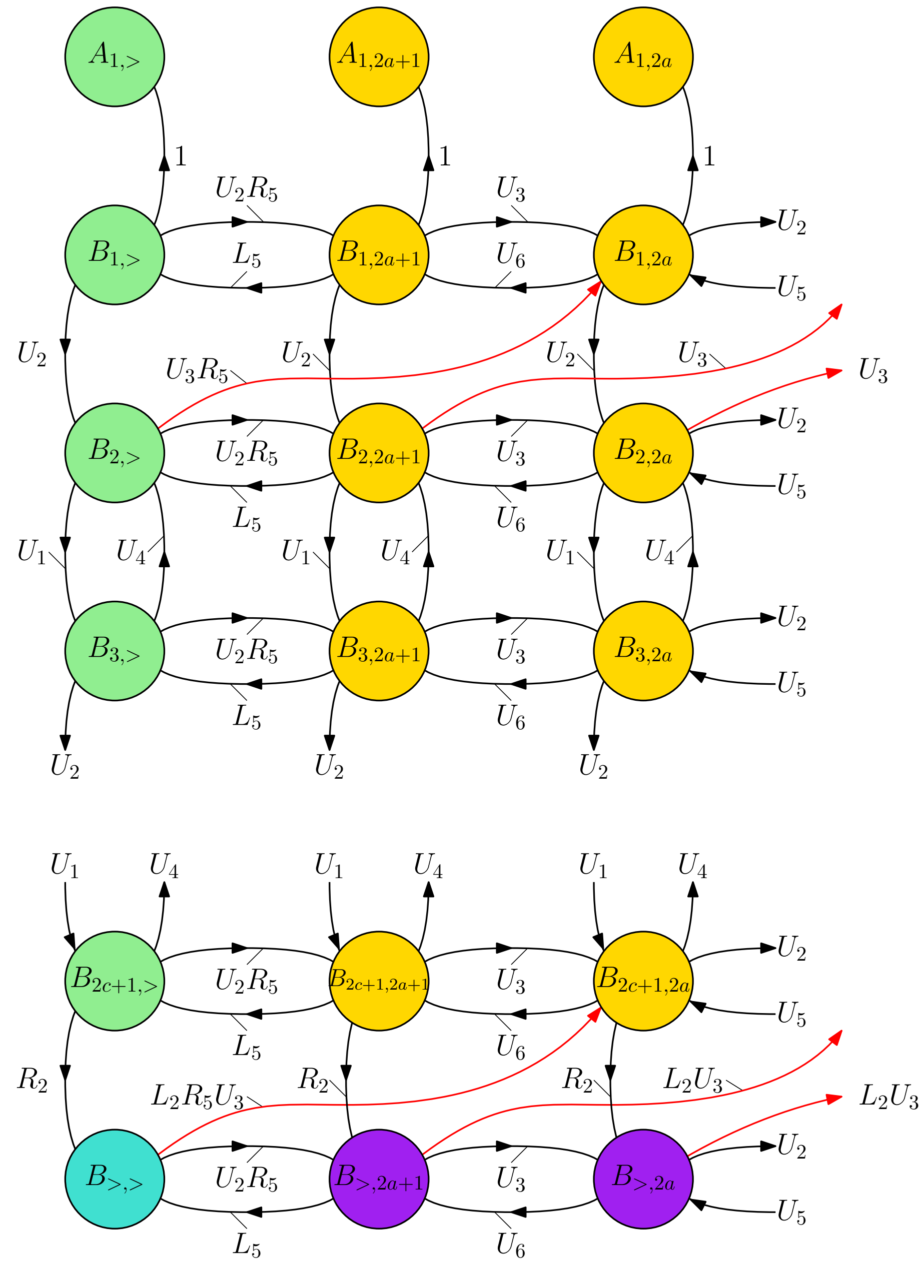


Figure 5.15: *Part of the Type D structure for $\mathcal{P}^5 \boxtimes \left((\mathcal{P}^5)^{2a} \boxtimes Y\right)$, represented by a weighted directed graph.*

$$
\begin{aligned}
&+ \delta_2^1(N, U_4) \otimes B_{2c,2a} \\
&= U_2 \otimes N \otimes B_{2c+1,2a-1} + R_2 \otimes N \otimes B_{>,2a} + U_6 \otimes W \otimes B_{2c+1,>} \\
&\quad + U_4 \otimes N \otimes B_{2c,2a} \\
&= U_2 \otimes B_{2r+1,2a-1} + R_2 \otimes B_{>,2a} + U_6 \otimes B_{2r+1,2a+1} + U_4 \otimes B_{2r,2a}. \\
d(B_{>,2a}) &\ni \delta_2^1(N, L_2U_3) \otimes B_{2c+1,2a-2} + \delta_2^1(N, U_2) \otimes B_{>,2a-1} + \delta_2^1(N, L_5) \otimes B_{>,>} \\
&= L_2U_3 \otimes N \otimes B_{2c+1,2a-2} + U_2 \otimes N \otimes B_{>,2a-1} + U_6 \otimes W \otimes B_{>,>}
\end{aligned}
$$

$$= L_2U_3 \otimes B_{2c+1,2a-2} + U_2 \otimes B_{>,2a-1} + U_6 \otimes B_{>,2a+1}.$$

For the sake of brevity, in the calculation of $d(W \otimes B_{r,>})$ and $d(S \otimes B_{r,>})$, only those calculations with algebra elements that have weight within $span\{e_5, e_6\}$ will be expanded. Hence, the calculation of $\mathcal{P} \boxtimes \left((\mathcal{P}^5)^{2a} \boxtimes Y\right)$ proceeds as follows.

$$\begin{aligned}
d(B_{1,2a+1}) &\ni \delta_2^1(W,1) \otimes A_{1,>} + \delta_2^1(W, R_5U_3) \otimes B_{1,2a} + \delta_2^1(W,U_2) \otimes B_{2,>} \\
&\quad + \delta_1^1(W) \otimes B_{1,>} \\
&= 1 \otimes A_{1,2a+1} + U_3 \otimes N \otimes B_{1,2a} + U_2 \otimes B_{2,2a+1} + L_5 \otimes S \otimes B_{1,>} \\
&= 1 \otimes A_{1,2a+1} + U_3 \otimes B_{1,2a} + U_2 \otimes B_{2,2a+1} + L_5 \otimes B_{1,>}. \\
d(B_{2r,2a+1}) &\ni \delta_2^1(W, R_5U_3) \otimes B_{2r-1,2a} + \delta_2^1(W, R_5U_3) \otimes B_{2r,2a} + \delta_2^1(W,U_1) \otimes B_{2r+1,>} \\
&\quad + \delta_1^1(W) \otimes B_{2r,>} \\
&= U_3 \otimes N \otimes B_{2r-1,2a} + U_3 \otimes N \otimes B_{2r,2a} + U_1 \otimes B_{2r,2a+1} \\
&\quad + L_5 \otimes S \otimes B_{2r,>} \\
&= U_3 \otimes B_{2r-1,2a} + U_3 \otimes B_{2r,2a} + U_1 \otimes B_{2r,2a+1} + L_5 \otimes B_{2r,>}. \\
d(B_{2r+1,2a+1}) &\ni \delta_2^1(W, R_5U_3) \otimes B_{2r+1,2a} + \delta_2^1(W,U_2) \otimes B_{2r+2,>} + \delta_2^1(W,U_4) \otimes B_{2r,>} \\
&\quad + \delta_1^1(W) \otimes B_{2r+1,>} \\
&= U_3 \otimes N \otimes B_{2r+1,2a} + U_2 \otimes B_{2r+2,2a+1} + U_4 \otimes B_{2r,2a+1} + L_5 \otimes S \otimes B_{2r+1,>} \\
&= U_3 \otimes B_{2r+1,2a} + U_2 \otimes B_{2r+2,2a+1} + U_4 \otimes B_{2r,2a+1} + L_5 \otimes B_{2r+1,>}. \\
d(B_{2c+1,2a+1}) &\ni \delta_2^1(W, R_5U_3) \otimes B_{2c+1,2a} + \delta_2^1(W,R_2) \otimes B_{>,>} + \delta_2^1(W,U_4) \otimes B_{2c,>} \\
&\quad + \delta_1^1(W) \otimes B_{2c+1,>} \\
&= U_3 \otimes N \otimes B_{2c+1,2a} + R_2 \otimes B_{>,2a+1} + U_4 \otimes B_{2c,2a+1} + L_5 \otimes S \otimes B_{2c+1,>} \\
&= U_3 \otimes B_{2c+1,2a} + R_2 \otimes B_{>,2a+1} + U_4 \otimes B_{2c,2a+1} + L_5 \otimes B_{2c+1,>}. \\
d(B_{>,2a+1}) &\ni \delta_2^1(W, L_2R_5U_3) \otimes B_{2c+1,2a} + \delta_2^1(W, R_5U_3) \otimes B_{>,2a} + \delta_1^1(W) \otimes B_{>,>} \\
&= L_2U_3 \otimes N \otimes B_{2c+1,2a} + U_3 \otimes N \otimes B_{>,2a} + L_5 \otimes S \otimes B_{>,>} \\
&= L_2U_3 \otimes B_{2c+1,2a} + U_3 \otimes B_{>,2a} + L_5 \otimes B_{>,>}. \\
d(B_{1,>}) &\ni \delta_2^1(S,1) \otimes A_{1,>} + \delta_2^1(S,U_2) \otimes B_{2,>} + \delta_2^1(S, C_{26}) \otimes B_{1,>} \\
&= 1 \otimes A_{1,>} + U_2 \otimes B_{2,>} + R_5U_2 \otimes W \otimes B_{1,>} \\
&= 1 \otimes A_{1,>} + U_2 \otimes B_{2,>} + R_5U_2 \otimes B_{1,2a+1}. \\
d(B_{2r,>}) &\ni \delta_3^1(S, R_5U_3, U_6) \otimes B_{2r-1,2a} + \delta_2^1(S, C_{26}) \otimes B_{2r,>} + \delta_2^1(S,U_1) \otimes B_{2r+1,>} \\
&= R_5U_3 \otimes N \otimes B_{2r-1,2a} + R_5U_2 \otimes W \otimes B_{2r,>} + U_1 \otimes B_{2r+1,>}
\end{aligned}$$

$$
\begin{aligned}
&= R_5U_3 \otimes B_{2r-1,2a} + R_5U_2 \otimes B_{2r,2a+1} + U_1 \otimes B_{2r+1,>}.\\
d(B_{2r+1,>}) &\ni \delta_2^1(S, C_{26}) \otimes B_{2r+1,>} + \delta_2^1(S, U_2) \otimes B_{2r+2,>} + \delta_2^1(S, U_4) \otimes B_{2r,>}\\
&= R_5U_2 \otimes W \otimes B_{2r+1,>} + U_2 \otimes B_{2r+2,2a+1} + U_4 \otimes B_{2r,>}\\
&= R_5U_2 \otimes B_{2r+1,2a+1} + U_2 \otimes B_{2r+2,>} + U_4 \otimes B_{2r,>}.\\
d(B_{2c+1,>}) &\ni \delta_2^1(S, C_{26}) \otimes B_{2c+1,>} + \delta_2^1(S, R_2) \otimes B_{>,>} + \delta_2^1(S, U_4) \otimes B_{2c,>}\\
&= R_5U_2 \otimes W \otimes B_{2c+1,>} + R_2 \otimes B_{>,>} + U_4 \otimes B_{2c,>}\\
&= R_5U_2 \otimes B_{2c+1,2a+1} + R_2 \otimes B_{>,>} + U_4 \otimes B_{2c,>}.\\
d(B_{>,>}) &\ni \delta_3^1(S, L_2R_5U_3, U_6) \otimes B_{2c+1,2a} + \delta_2^1(S, C_{26}) \otimes B_{>,>}\\
&= L_2R_5U_3 \otimes N \otimes B_{2c+1,2a} + R_5U_2 \otimes W \otimes B_{>,>}\\
&= L_2R_5U_3 \otimes B_{2c+1,2a} + R_5U_2 \otimes B_{>,2a+1}.
\end{aligned}
$$

This agrees with the part of the Type D structure displayed in Figure 5.15. An almost identical calculation yields the result displayed in Figure 5.16, where now the matching elements C_{14}, C_{25} and C_{36} mean that the calculation of $d(S \otimes B_{r,>})$ includes the term $\delta_2^1(S, C_{36}) = R_5U_3 \otimes W$. Because the process is so similar, it is not presented here, as the required maps are nearly identical to those taken when calculation $\mathcal{P}^5 \boxtimes \mathcal{P}^5 \boxtimes Y$ from $\mathcal{P}^5 \boxtimes Y$.

More visually, to pass from Figure 5.15 to Figure 5.16 involves:

- Swapping U_6 for U_5 with arrows $B_{r,2a} \to B_{r,2a+1}$, since $\delta_2^1(N, U_6) = U_5 \otimes N$.
- A map $B_{r,2a+1} \to B_{r,2a+2}$ with weight U_6, from $\delta_2^1(N, L_5) = U_6 \otimes W$ and $B_{r,2a+2} = W \otimes B_{r,>}$.
- A map $B_{r,2a+2} \to B_{r,2a+1}$ with weight U_2, from $\delta_2^1(W, R_5U_2) = U_2 \otimes N$.
- Maps $B_{r,>} \to B_{r,2a+2}$ with weights U_3R_5 since $\delta_2^1(S, C_{36}) = R_5U_3 \otimes W = U_3R_5 \otimes W$. Note, U_3 and R_5 commute, since 3 and 5 are sufficiently 'far' from each other.
- Maps $B_{r,2a+2} \to B_{r,>}$ with weights L_5, from $\delta_1^1(W) = L_5 \otimes S$.
- Using $\delta_3^1(S, R_5U_3, U_6) = R_5U_3 \otimes N$, and $\delta_3^1(S, L_2R_5U_3, U_6) = L_2R_5U_3 \otimes N$, one yields the maps $B_{2r,>} \to B_{2r-1,2a+1}$ and $B_{>,>} \to B_{2c+1,2a+1}$ with the corresponding weights.

Hence, since the rest of the diagram is determined, and Figure 5.16 describes a Type

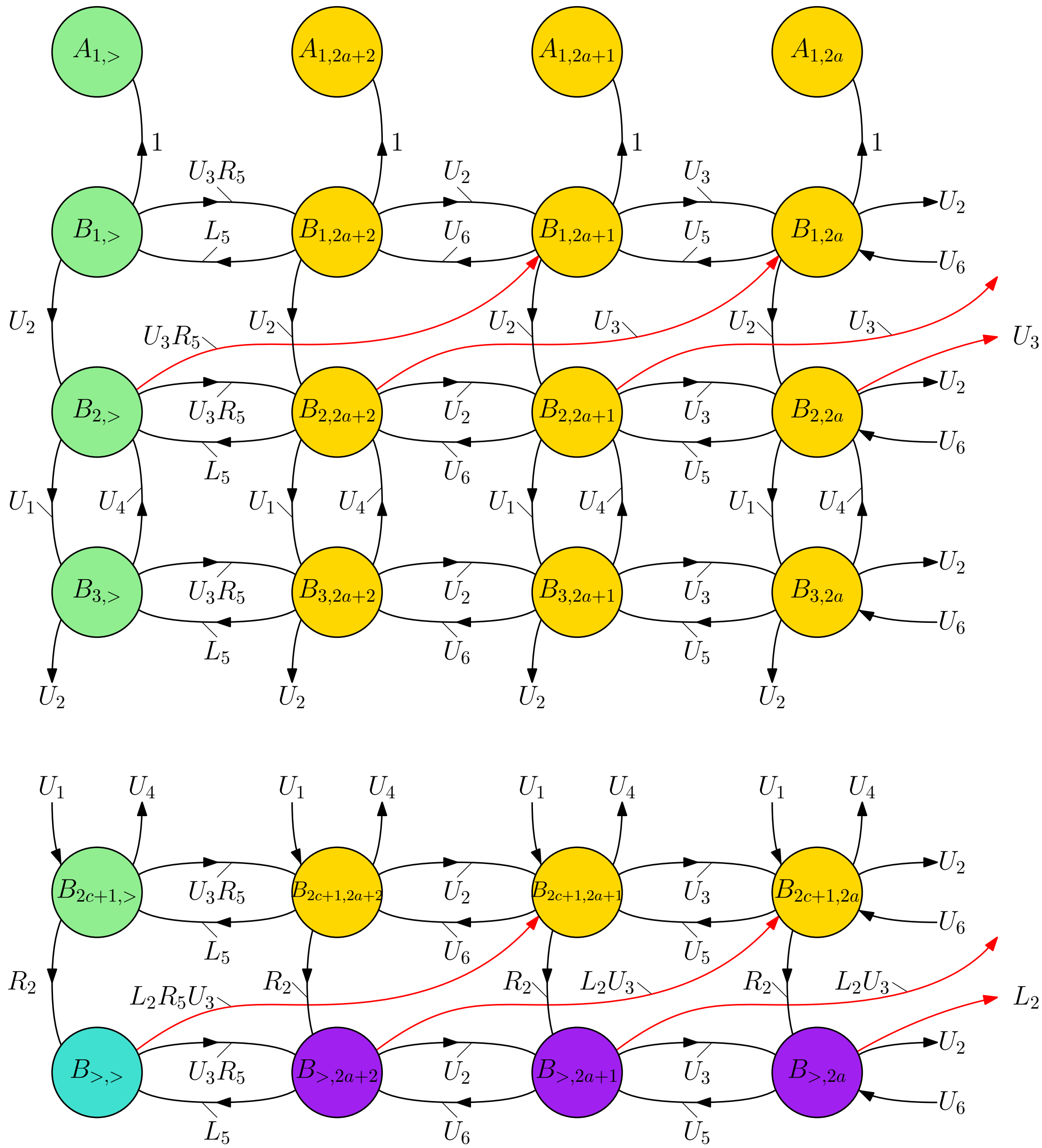


Figure 5.16: *Part of the Type D structure for $\mathcal{P}^5 \boxtimes \left((\mathcal{P}^5)^{2a+1} \boxtimes Y\right)$, represented by a weighted directed graph.*

D structure matches the statement of the lemma, Q_{2a} implies Q_{2a+2}. By induction, Lemma 5.7 then holds for all $t \in \mathbb{N}$. ■

5.5.4 The remaining maps in the Type D structure

As seen in the statement of Lemma 5.7, only some of the maps from the states B_{ik} have been determined so far. The remaining maps are maps with weight R_5 and 1, which will play a key role in the determination of $H_*(\widehat{C}(D))$ for these knots.

In the above calculation, a map $d : \left(\mathcal{P}^5\right)^{2a} \boxtimes Y \to \mathcal{A} \otimes \left(\mathcal{P}^5\right)^{2a} \boxtimes Y$ has an algebra weight of 1 arising from either an arrow in Figure 5.10 that already has 1 as a weight — for example the arrow $B_1 \to A_1$ in ${}^{\mathcal{A}}Y$ — or from a term featuring $\delta^1_2(W, R_5) = 1 \otimes N$ in $\mathcal{P}^5$. It is then useful to track where arrows with R_5 as a weight appear in $\left(\mathcal{P}^5\right)^{2a} \boxtimes Y$ in order to complete the determination of the Type D structure.

However, a complication is that R_5 is an algebraic input to a δ^1_3 term that is non-zero, namely $\delta^1_3(S, R_5, U_6) = R_5 \otimes N$. This also yields R_5 as an output algebra element of the map. To fully calculate the map d in the Type D structure one thus needs to track wherever U_6 is the weight on an outward edge at a vertex to which R_5 is the weight on an inward edge.

Remark 5.9 *From Sections 5.5.1, 5.5.2 and 5.5.3, it has been shown that the U_6 arrows are as follows in the Type D structure $\left(\mathcal{P}^5\right)^t \boxtimes Y$:*

- *When t is even:*
  - $C_{2r,\ell} \to C_{2r-1,\ell}$ *has weight U_6.*
  - $A_{\ell,2r+1} \to A_{\ell,2r+2}$ *has weight U_6.*
- *When t is odd:*
  - $C_{2r+1,\ell} \to C_{2r,\ell+2}$ *has weight U_6.*
  - $C_{\ell,2r} \to C_{\ell,2r+1}$ *has weight U_6.*
  - $C_{1,\ell} \to A_{\ell+1,1}$ *has weight U_6.*
  - $A_{\ell,2r} \to A_{\ell,2r+1}$ *has weight U_6.*
  - $A_{2r+1,1} \to A_{2r+2,1}$ *has weight U_6.*

With the positions of the U_6-weighted arrows determined, it remains to determine the R_5 arrows from the B-states.

Lemma 5.10 *In the Type D structure $\mathcal{P}^5 \boxtimes Y$, one has that*

$$d(B_{2r,>}) \ni R_5 \otimes C_{2r-1,1}$$
$$d(B_{2r+1,>}) \ni R_5 \otimes C_{2r-1,2}.$$

Proof This is relatively simple to see from Figure 5.10. Since $d(B_{2r}) \ni R_5 \otimes C_{2r,1}$ and

$d(C_{2r,1}) = U_6 \otimes C_{2r-1,1}$ in Y, one has that

$$\begin{aligned} d(S \otimes B_{2r}) &\ni \delta_3^1(S, R_5, U_6) \otimes C_{2r-1,1} \\ &= R_5 \otimes N \otimes C_{2r-1,1}. \end{aligned}$$

Furthermore, since in Y one can observe that $d(B_{2r+1}) \ni R_5 \otimes C_{2r,2}$, and $d(C_{2r,2}) = U_6 \otimes C_{2r-1,2}$, the same procedure gives

$$\begin{aligned} d(S \otimes B_{2r+1}) &\ni \delta_3^1(S, R_5, U_6) \otimes C_{2r-1,2} \\ &= R_5 \otimes N \otimes C_{2r-1,2}. \end{aligned}$$

Since all $N \otimes C_{ij}$ are relabelled as C_{ij} in $\mathcal{P}^5 \boxtimes Y$, this completes the proof of the lemma. ■

From Lemma 5.10, one can then deduce that in $\left(\mathcal{P}^5\right)^2 \boxtimes Y$ the states $W \otimes B_{2r,>}$ and $W \otimes B_{2r+1,>}$ have maps

$$\begin{aligned} d(W \otimes B_{2r,>}) &\ni \delta_2^1(W, R_5) \otimes C_{2r-1,1} \\ &= 1 \otimes N \otimes C_{2r-1,1}. \\ d(W \otimes B_{2r+1,>}) &\ni \delta_2^1(W, R_5) \otimes C_{2r-1,2} \\ &= 1 \otimes N \otimes C_{2r-1,2}. \end{aligned}$$

In this way, it is the determination of the R_5 arrows that yield the arrows with algebra weight 1 after taking the tensor product with the DA-bimodule $\mathcal{P}^5$ corresponding with the next half-twist.

Lemma 5.11 *In the Type D structure $\left(\mathcal{P}^5\right)^{2n} \boxtimes Y$, the arrows with weight R_5 from the states $B_{i,>}$ are:*

$$\begin{aligned} d(B_{2r,>}) &\ni R_5 \otimes (C_{2r-2n,2n+1} + A_{2r,2n-2r+1}) \\ d(B_{2r+1,>}) &\ni R_5 \otimes (C_{2r-2n,2n+2} + C_{2r-2n+1,2n+1} + A_{2r+1,2n-2r+1}). \end{aligned}$$

Proof Again, the proof proceeds by induction, so let P_t be the statement that the lemma holds for $n = t$.

Then, Remark 5.9 and Lemma 5.10 imply the base case $t = 1$. Lemma 5.10 proves that there is an arrow $B_{2,>} \to C_{1,1}$ with weight R_5 in $\mathcal{P}^5 \boxtimes Y$, and the remark shows that there is a U_6 weighted arrow $C_{1,1} \to A_{2,1}$. Applying these, and the observation from the remark that there is a U_6 weighted arrow $C_{2r-1,1} \to C_{2r-2,3}$ in $\mathcal{P}^5 \boxtimes Y$, one has that

$$\begin{aligned}
d(B_{2,>}) &\ni \delta_3^1(S, R_5, U_6) \otimes A_{2,1} \\
&= R_5 \otimes N \otimes A_{2,1} = R_5 \otimes A_{2,1} \\
d(B_{2r,>}) &\ni \delta_3^1(S, R_5, U_6) \otimes C_{2r-2,3} \\
&= R_5 \otimes N \otimes C_{2r-2,3} = R_5 \otimes C_{2r-2,3}.
\end{aligned}$$

Furthermore, $d(B_{3,>})$ and $d(B_{2r+1,>})$ for $r > 1$ are then

$$\begin{aligned}
d(B_{3,>}) &\ni \delta_3^1(S, R_5, U_6) \otimes A_{3,1} + \delta_3^1(S, R_5, U_6) \otimes C_{1,3} \\
&= R_5 \otimes N \otimes A_{3,1} + R_5 \otimes N \otimes C_{1,3} \\
&= R_5 \otimes A_{3,1} + R_5 \otimes C_{1,3} \\
d(B_{2r+1,>}) &\ni \delta_3^1(S, R_5, U_6) \otimes C_{2r-2,4} + \delta_3^1(SR_5, U_6) \otimes C_{2r-1,3} \\
&= R_5 \otimes N \otimes C_{2r-2,4} + R_5 \otimes N \otimes C_{2r-1,3} \\
&= R_5 \otimes C_{2r-2,4} + R_5 \otimes C_{2r-1,3}.
\end{aligned}$$

Hence P_1 is true, so the base case of the induction holds.

Assume for inductive purposes that P_t holds. For ease of calculation, divide the states $B_{i,>}$ into three categories: $B_{2r,>}$ and $B_{2r+1,>}$ with $r < t$; $B_{2t,>}$ and $B_{2t+1,>}$; and the states $B_{2r,>}$ and $B_{2r+1,>}$ with $r > t$.

When $r < t$, the inductive assumption states that $d(B_{2r,>}) = R_5 \otimes A_{2r,2t-2r+1}$ and $d(B_{2r+1,>}) = R_5 \otimes A_{2r+1,2t-2r+1}$. From Remark 5.9, since the second index in each A_{jk} term is odd, one has that there is a U_6 weighted arrow $A_{\ell,2t-2r+1} \to A_{\ell,2t-2r+2}$.

Combining $\delta_3^1(S, R_5, U_6) = R_5 \otimes N$ with this information implies that in $\left(\mathcal{P}^5\right)^{2t+1} \boxtimes Y$, one has the terms

$$\begin{aligned}
d(B_{2r,>}) &\ni R_5 \otimes A_{2r,2(t+1)-2r} \\
d(B_{2r+1,>}) &\ni R_5 \otimes A_{2r+1,2(t+1)-2r}.
\end{aligned}$$

As $t > r$ by assumption, and so $2(t+1) - 2r > 1$, using Remark 5.9 once more implies that

$$\begin{aligned}
d(S \otimes B_{2r,>}) &\ni \delta_3^1(S, R_5, U_6) \otimes A_{2r,2(t+1)-2r+1} \\
&= R_5 \otimes N \otimes A_{2r,2(t+1)-2r+1}
\end{aligned}$$

$$
\begin{aligned}
d(S \otimes B_{2r+1,>}) &\ni \delta_3^1(S, R_5, U_6) \otimes A_{2r+1,2(t+1)-2r+1} \\
&= R_5 \otimes N \otimes A_{2r+1,2(t+1)-2r+1}.
\end{aligned}
$$

This matches the form given in the statement of the lemma.

When $r = t+1$, the inductive assumption implies that in the Type D structure $\left(\mathcal{P}^5\right)^{2t} \boxtimes Y$, one has that $d(B_{2t+2,>}) \ni R_5 \otimes C_{2,2t+1}$ and $d(B_{2t+3,>}) \ni R_5 \otimes C_{2,2t+2} + R_5 \otimes C_{3,2t+1}$.

From Remark 5.9, the only U_6 arrows starting at C_{ij} states in $\left(\mathcal{P}^5\right)^{2t} \boxtimes Y$ are those from $C_{2r,\ell} \to C_{2r-1,\ell}$. Hence, combining these terms with the term $\delta_3^1(S, R_5, U_6)$ term in $\mathcal{P}^5$ yields

$$
\begin{aligned}
d(B_{2t+2,>}) &\ni R_5 \otimes C_{1,2t+1} \\
d(B_{2r+3,>}) &\ni R_5 \otimes C_{1,2t+2}.
\end{aligned}
$$

In $\left(\mathcal{P}^5\right)^{2t+1} \boxtimes Y$, the only U_6 weighted arrows from the states $C_{1,2t+1}$ and $C_{1,2t+2}$ are: $C_{1,2t+1} \to A_{2t+2,1}$; $C_{1,2t+2} \to A_{2t+3,1}$; and $C_{1,2t+2} \to C_{1,2t+3}$. Once more combining this with the $\delta_3^1(S, R_5, U_6)$ map yields

$$
\begin{aligned}
d(B_{2t+2,>}) &\ni R_5 \otimes A_{2t+2,1} \\
d(B_{2r+3,>}) &\ni R_5 \otimes A_{2t+3,1} + R_5 \otimes C_{1,2t+3}.
\end{aligned}
$$

Once more since these are now maps within the Type D structure $\left(\mathcal{P}^5\right)^{2(t+1)} \boxtimes Y$, these maps match those presented in the lemma.

When $r > t+1$, the inductive assumption states that $d(B_{2r,>}) \ni R_5 \otimes C_{2r-2t,2t+1}$ and $d(B_{2r+1,>}) \ni R_5 \otimes C_{2r-2t,2t+2} + R_5 \otimes C_{2r-2t+1,2t+1}$. From Remark 5.9, the only U_6 arrows from C_{ij} states in this Type D structure are $C_{2r,\ell} \to C_{2r-1,\ell}$. Hence, in $\left(\mathcal{P}^5\right)^{2t+1} \boxtimes Y$, one has that

$$
\begin{aligned}
d(B_{2r,>}) &\ni \delta_3^1(S, R_5, U_6) \otimes C_{2r-2t-1,2t+1} \\
&= R_5 \otimes C_{2r-2t-1,2t+1} \\
d(B_{2r+1,>}) &\ni \delta_3^1(S, R_5, U_6) \otimes C_{2r-2t-1,2t+2} \\
&= R_5 \otimes C_{2r-2t-1,2t+2}.
\end{aligned}
$$

Using Remark 5.9 once more, in $\left(\mathcal{P}^5\right)^{2(t+1)} \boxtimes Y$, one thus has:

$$
d(B_{2r,>}) \ni \delta_3^1(S, R_5, U_6) \otimes C_{2r-2t-2,2t+3}
$$

$$
\begin{aligned}
&= R_5 \otimes C_{2r-2(t+1),2(t+1)+1} \\
d(B_{2r+1,>}) &\ni \delta_3^1(S, R_5, U_6) \otimes C_{2r-2t-2,2t+4} + \delta_3^1(S, R_5, U_6) \otimes C_{2r-2t-1,2t+3}. \\
&= R_5 \otimes C_{2r-2(t+1),2(t+1)+2} + R_5 \otimes C_{2r-2(t+1)+1,2(t+1)+1}.
\end{aligned}
$$

Since $r > t+1$, all of the indices in the C_{ij} terms exist, and match the statement of the lemma. Hence, P_t being true implies that P_{t+1} is true, and so by induction Lemma 5.11 holds for all $n \in \mathbb{N}$. ■

Lemma 5.11 holds for any $n \in \mathbb{N}$, and so, using the fact that in $\mathcal{P} \boxtimes \left(\mathcal{P}^5\right)^{2n} \boxtimes Y$, the states $B_{r,2n+1} = W \otimes B_{r,>}$, one can calculate arrows with 1 as the associated algebraic weight from $\delta_2^1(W, R_5) = 1 \otimes N$.

Moreover, a quick inspection of the definition of the bimodule $\mathcal{P}^5$ shows that the only other possible maps that yield 1 as a weight are either $\delta_2^1(X, 1) = 1 \otimes X$ for X a cardinal direction, or $\delta_2^1(E, L_6) = 1 \otimes N$. Since for three strand pretzel knots in this form the idempotents are truncated as specified in Definition 4.6, and no state in the Type D structure has an associated idempotent I_x with $6 \in x$, then the only δ_2^1-maps in $\mathcal{P}^5$ that contribute an algebraic weight of 1 are the $\delta_2^1(W, R_5)$ and $\delta_2^1(X, 1)$.

Corollary 5.12 *The maps d in the type D structure $\left(\mathcal{P}^5\right)^{2a} \boxtimes Y$ with 1 as the algebraic weight are as follows.*

$$
\begin{aligned}
d(B_{2r,2n+1}) &\ni 1 \otimes C_{2r-2n,2n+1} + 1 \otimes A_{2r,2n-2r+1} \\
d(B_{2r+1,2n+1}) &\ni 1 \otimes C_{2r-2n,2n+2} + 1 \otimes C_{2r-2n+1,2n+1} + 1 \otimes A_{2r+1,2n-2r+1} \\
d(B_{2r,2n+2}) &\ni 1 \otimes C_{2r-2n-1,2n+1} + 1 \otimes A_{2r,2n-2r+2} \\
d(B_{2r+1,2n+2}) &\ni 1 \otimes C_{2r-2n-1,2n+2} + 1 \otimes A_{2r+1,2n-2r+2}.
\end{aligned}
$$

Proof From the above observation, and the fact that the Type D structure $\left(\mathcal{P}^5\right)^{2a} \boxtimes Y$ is built up by tensoring with $\mathcal{P}^5$ consecutively, when determining $\left(\mathcal{P}^5\right)^{2n+1} \boxtimes Y$ from $\left(\mathcal{P}^5\right)^{2n} \boxtimes Y$, one has the following maps:

$$
\begin{aligned}
d(B_{2r,2n+1}) &= d(W \otimes B_{2r,>}) \\
&\ni \delta_2^1(W, R_5) \otimes (C_{2r-2n,2n+1} + A_{2r,2n-2r+1}) \\
&\ni 1 \otimes (C_{2r-2n,2n+1} + A_{2r,2n-2r+1}). \\
d(B_{2r+1,2n+1}) &= d(W \otimes B_{2r+1,>}) \\
&\ni \delta_2^1(W, R_5) \otimes (C_{2r-2n,2n+2} + C_{2r-2n+1,2n+1} + A_{2r+1,2n-2r+1})
\end{aligned}
$$

$$= 1 \otimes (C_{2r-2n,2n+2} + C_{2r-2n+1,2n+1} + A_{2r+1,2n-2r+1}) .$$

Since these are maps with algebraic weight 1, and the associated idempotent to $B_{2r,2n+1}$ and $B_{2r+1,2n+1}$ in $\left(\mathcal{P}^5\right)^{2n+1} \boxtimes Y$ is I_{135}, under taking more box-tensor products with $\mathcal{P}^5$, the only non-zero tensor product is with $N \in \mathcal{P}^5$. Since $\delta_2^1(N,1) = 1 \otimes N$, these arrows are preserved under taking further box-tensor products with $\mathcal{P}^5$.

Likewise, using Remark 5.9, it is simple to see that in $\left(\mathcal{P}^5\right)^{2n+1} \boxtimes Y$, one has the R_5-weighted arrows:

$$d(B_{2r,>}) \ni R_5 \otimes (C_{2r-2n-2,2n+1} + A_{2r,2n-2r+2})$$
$$d(B_{2r+1,>}) \ni R_5 \otimes (C_{2r-2n-1,2n+2} + A_{2r+1,2n-2r+2}) .$$

Under taking the next box-tensor product with $\mathcal{P}^5$ to yield $\left(\mathcal{P}^5\right)^{2n+2} \boxtimes Y$, the same application of $\delta_2^1(W, R_5) = 1 \otimes N$ gives arrows of weight 1 to the above states from $B_{2r,2n+2}$ and $B_{2r+1,2n+2}$. Again, these arrows are preserved under further box-tensor products with $\mathcal{P}^5$ due to the only non-zero tensor products with these coordinates being $N \otimes B_{2r,2n+2}$ and $N \otimes B_{2r+1,2n+2}$, with the same weight since $\delta_2^1(N,1) = 1 \otimes N$. ■

The only remaining behaviour to determine are those maps from $B_{>,k}$ not shown in Lemma 5.7. These arise in a very similar way to the maps presented in Lemma 5.11, from the map $\delta_3^1(S, R_5, U_6) = R_5 \otimes N$ in $\mathcal{P}^5$.

Lemma 5.13 *In the Type D structure $\left(\mathcal{P}^5\right)^{2a} \boxtimes Y$, the states $B_{>,k}$ has the following maps:*

$$d(B_{>,2n}) \ni L_2 \otimes C_{2c+3-2n,2n-1}$$
$$d(B_{>,2n+1}) \ni L_2 \otimes C_{2c+2-2n,2n+1}$$
$$d(B_{>,>}) \ni L_2 R_5 \otimes C_{2c+2-2a,2a+1}.$$

Proof In $\mathcal{P}^5 \boxtimes Y$, using Figure 5.10, one has that

$$\begin{aligned} d(B_{>,>}) &= d(S \otimes B_>) \\ &\ni \delta_3^1(S, R_5, L_2 U_6) \otimes C_{2c+1,1} \\ &= L_2 R_5 \otimes C_{2c+1,1}. \end{aligned}$$

Then, using Remark 5.9, the only U_6 arrow from the state $C_{2c+1,1}$ is $C_{2c+1,1} \to C_{2c-1,1+2} = C_{2c-1,3}$. This implies that after taking the box-tensor product with $\mathcal{P}^5$ once more, the same δ_3^1 map will yield $d(B_{>,>}) \ni L_2 R_5 \otimes C_{2c-1,3} = C_{2c+1-(2a)+1,2a+1}$, where $a = 1$ here.

Using similar logic to the proof of Corollary 5.12, one yields the result, since one pairs this map with $\delta_2^1(W, L_2R_5) = L_2 \otimes N$ for the odd case, and the U_6 arrows from Remark 5.9. ■

This completes the determination of the map d acting on the B-states within the Type D structure $\left(\mathcal{P}^5\right)^{2a} \boxtimes Y$, as can be seen through a careful examination of the Type D structure for Y in Figure 5.13, and noting that all of the possible maps have been considered in the inductive methods presented in Lemma 5.7 and Corollary 5.12. Hence, combining this with the results of Sections 5.5.1 and 5.5.2 the entire Type D structure corresponding to the upper knot diagrams in Figure 5.11 is determined. In order to determine the full bordered invariant $\mathcal{C}(D)$ of Ozsváth-Szabó defined in [49], one then needs to tensor with the A_∞-module associated to the three minima, as presented in Definition 4.57.

Chapter 6

Results and the full invariant

Following Section 4.5.2, one can take the product of the Type D structure determined in Section 5.5 with the A_∞-module $Y' \boxtimes \widetilde{\mho}^2 \boxtimes \widetilde{\mho}^2_{\mathcal{A}(3)}$ to yield a chain complex over $\mathcal{R}'$. Within this chapter, the full structure of $\mathcal{C}(D)$ is determined, together with the determination of associated homology theories and associated numerical invariants.

6.1 The determination of $\mathcal{C}(D)$

As described in Section 5.5, define the Type D structure associated to the upper knot diagram of $P(2c+1, -2b-1, 2a)$ to be

$$^{\mathcal{A}(3)}\mathcal{T} := \left(\mathcal{P}^5\right)^{2a} \boxtimes \left(\mathcal{N}^3\right)^{2b+1} \boxtimes \Omega^4 \boxtimes \left(\mathcal{P}^1\right)^{2c+1} \boxtimes \Omega^2 \boxtimes \Omega^1.$$

The maps $\partial : \mathcal{T} \to \mathcal{A}(3) \otimes \mathcal{T}$ for this Type D structure were calculated in Section 5.5.1, Section 5.5.2 and Section 5.5.3, and no application of this map upon any of the generators (in correspondence with upper Kauffman states) can yield the sequence of algebra elements L_3, U_2, R_3. However, starting at some states, one can yield the the algebra elements L_5, U_4 and R_5 from applying the map ∂ thrice. Namely, such a sequence of algebra elements originates from $B_{2r+1,2a}$ for $0 \leq r \leq c$.

As a consequence, excepting at these states $B_{2r+1,2a}$, only integer weight algebra elements in $\mathcal{A}(3)$ contribute to the tensor product, since the remaining maps m_{1+j} in $Y' \boxtimes \widetilde{\mho}^2 \boxtimes \widetilde{\mho}^2$ take integer weight inputs.

What is more, since all of the generators in the A_∞-module $Y' \boxtimes \widetilde{\mho}^2 \boxtimes \widetilde{\mho}^2$ have incoming idempotent I_{135}, the chain complex $Y' \boxtimes \widetilde{\mho}^2 \boxtimes \widetilde{\mho}^2 \boxtimes \mathcal{T}$ arises from the tensor product of generators in $\mathcal{T}$ with I_{135} as the outgoing idempotent.

This is as expected, since the familiar Kauffman states for $P(2a, -2b-1, 2c+1) \cong P(2c+1, -2b-1, 2a)$ depicted in Figure 2.8 arise from the partial Kauffman states with this idempotent. The use of the DA-bimodule $\widetilde{\mho}^2$ in preference to $\mho^2$ gives a correspondence between the Kauffman states of the standard knot diagram for this family of three strand pretzel knots, as used in [5], and the generators of $Y' \boxtimes \widetilde{\mho}^2 \boxtimes \widetilde{\mho}^2 \boxtimes \mathcal{T} \cong \mathcal{C}(D)$.

Informally, maps with weights U_1 and U_5 in the Type D structure $\mathcal{T}$ contribute to increase the power of U in the chain complex, from $m_2(X, U_1^\ell) = U^\ell \otimes X$ and $m_{1+1+n}(X, U_5^n, C_{14}^{\otimes n}) = U^n \otimes X$ in $Y' \boxtimes \widetilde{\mho}^2 \boxtimes \widetilde{\mho}^2$. Similarly, maps with weights U_3 and U_6 contribute to increase the power of V in the chain complex, from the maps $m_2(X, U_6^k) = V^k \otimes X$ and $m_{1+1+r}(X, U_3^r, C_{26}^{\otimes r}) = V^r \otimes X$. By inspection, these are all maps taking only integer weight elements in Definition 4.57.

Although Ozsváth-Szabó equip the complex $\mathcal{C}(D)$ with two integer-valued gradings (Δ, A), one can use the relation $\Delta = M - A$ to recover the Maslov grading for every Kauffman state for the knot $P(2c+1, -2b-1, 2a)$. However, although an integer valued grading Δ on $\mathcal{T}$ has been described, through compatibility with the associated one-manifold with boundary, the Type D structure is currently only equipped with an Alexander multi-grading in $\left(\frac{1}{2}\mathbb{Z}\right)^6$. However, from the local contributions to Δ and A in Figure 4.1, every upper Kauffman state can be equipped with an integer-valued grading.

Moreover, this can be recovered from the multi-grading, as highlighted by [46, Sec. 1.1]. Within the computer implementation of the calculation of $\mathcal{C}(D)$ in [47] the determination of the Alexander grading associated to each upper Kauffman state is made through the formula

$$A(X) = \sum_{s \notin \mathcal{S}} w_s(X) - \sum_{s \in \mathcal{S}} w_s(X).$$

Here, $\mathcal{S}$ are the upwards oriented strands at the boundary of the one-manifold associated to a Type D structure. In order to agree with the gradings in Figure 4.1, the global minimum is assumed to be oriented right to left. As described in Section 4.6.1, if the global minimum has the reverse orientation, the roles of U and V in the $\mathcal{R}'$ are switched.

Using these gradings, the correspondence between generators and Kauffman states, and the well-defined tensor product between Type D structures and A_∞-modules outlined in

Section 4.5.2, one yields the following theorem.

Theorem 6.1 *Let D be the special knot diagram associated to the three strand pretzel knot $P(2c+1, -2b-1, 2a)$ depicted in Figure 6.1, with $a, b, c \geq 1$. The bordered invariant $\mathcal{C}(D)$ is the chain complex over $\mathcal{R}'$ generated by elements corresponding to the Kauffman states pictured in Figure 2.8, with associated Maslov and Alexander gradings as presented in Table 2.2.*

For the states A_{jk} with $1 \leq j \leq 2b+1$, $1 \leq k \leq 2a$, the differential $d : \mathcal{C}(D) \to \mathcal{C}(D)$ is given by the following.

$$
\begin{aligned}
d(A_{11}) &= U(A_{21} + C_{12}) + VA_{12} && \\
d(A_{2r,1}) &= U(A_{2r+1,1} + C_{1,2r+1}) + V(A_{2r-1,1} + A_{2r,2}) && r \in \mathbb{N} \\
d(A_{2r+1,1}) &= U(A_{2r+2,1} + C_{1,2r+2}) + VA_{2r+1,2} && r \in \mathbb{N} \\
d(A_{2b+1,1}) &= VA_{2b+1,2} && \\
d(A_{1,2p}) &= UA_{1,2p+1} && p \in \mathbb{N} \\
d(A_{2r,2p}) &= U(A_{2r+1,2p} + A_{2r,2p+1}) + VA_{2r-1,2p} && r, p \in \mathbb{N} \\
d(A_{2r+1,2p}) &= UA_{2r+1,2p+1} && r, p \in \mathbb{N} \\
d(A_{2r,2p+1}) &= UA_{2r+1,2p+1} + V(A_{2r-1,2p+1} + A_{2r,2p} + A_{2r,2p+2}) && r, p \in \mathbb{N} \\
d(A_{2r+1,2p+1}) &= V(A_{2r+1,2p} + A_{2r+1,2p+2}) && r \in \mathbb{Z}_{\geq 0}, p \in \mathbb{N} \\
d(A_{2r+1,2a}) &= 0 && r \in \mathbb{Z}_{\geq 0} \\
d(A_{2r,2a}) &= UA_{2r+1,2a} + VA_{2r-1,2a} && r \in \mathbb{N}
\end{aligned}
$$

Similarly, the differential map has the following action on the C_{ij} states, for $1 \leq i \leq 2c+1$, $1 \leq j \leq 2b+1$.

$$
\begin{aligned}
d(C_{11}) &= U(C_{12} + A_{21}) && \\
d(C_{1,2r}) &= U(C_{1,2r+1} + A_{2r+1,1}) && r \in \mathbb{N} \\
d(C_{1,2r+1}) &= U(C_{1,2r+2} + A_{2r+2,1}) + VC_{1,2r} && r \in \mathbb{Z}_{\geq 0} \\
d(C_{1,2b+1}) &= VC_{1,2b} && \\
d(C_{2s,1}) &= U(C_{2s,2} + C_{2s+1,1}) + VC_{2s-1,1} && s \in \mathbb{N} \\
d(C_{2s,2r}) &= U(C_{2s,2r+1} + C_{2s+1,2r}) + VC_{2s-1,2r} && s, r \in \mathbb{N} \\
d(C_{2s,2r+1}) &= U(C_{2s,2r+2} + C_{2s+1,2r+1}) + V(C_{2s,2r} + C_{2s-1,2r+1}) && s, r \in \mathbb{N} \\
d(C_{2s,2b+1}) &= UC_{2s+1,2b+1} + V(C_{2s,2b} + C_{2s-1,2b+1}) && s \in \mathbb{N}
\end{aligned}
$$

$$\begin{aligned}
d(C_{2s+1,1}) &= U(C_{2s+1,2} + C_{2s,3}) && s \in \mathbb{N} \\
d(C_{2s+1,2r}) &= U(C_{2s+1,2r+1} + C_{2s,2r+2}) && s, r \in \mathbb{N} \\
d(C_{2s+1,2r+1}) &= U(C_{2s+1,2r+2} + C_{2s,2r+3}) + VC_{2s+1,2r} && s, r \in \mathbb{N} \\
d(C_{2s+1,2b}) &= UC_{2s+1,2b+1} && s \in \mathbb{N} \\
d(C_{2s+1,2b+1}) &= VC_{2s+1,2b} && s \in \mathbb{N}.
\end{aligned}$$

For the states B_{ik}, with $1 \leq i \leq 2c+1$ and $1 \leq k \leq 2a$, the differential map d acts as follows.

$$\begin{aligned}
d(B_{11}) &= (A_{11} + C_{11}) + VB_{12} \\
d(B_{1,2p}) &= A_{1,2p} + UB_{1,2p+1} && p \in \mathbb{N} \\
d(B_{1,2p+1}) &= A_{1,2p+1} + VB_{1,2p+2} && p \in \mathbb{N} \\
d(B_{1,2a}) &= A_{1,2a} \\
d(B_{2s,1}) &= C_{2s,1} + UB_{2s+1,1} + VB_{2s,2} && s \in \mathbb{N} \\
d(B_{2s,2p}) &= (C_{2s-2p+1,2p-1} + A_{2s,2p-2s}) + U(B_{2s+1,2p} + B_{2s,2p+1}) + VB_{2s-1,2p-2} && s, p \in \mathbb{N} \\
d(B_{2s,2p+1}) &= (C_{2s-2p,2p+1} + A_{2s,2p-2s+1}) + UB_{2s+1,2p+1} \\
&\quad + V(B_{2s,2p} + B_{2s,2p+2} + B_{2s-1,2p-1}) && s, p \in \mathbb{N} \\
d(B_{2s,2a}) &= (C_{2s-2a+1,2a-1} + A_{2s,2a-2s}) + UB_{2s+1,2a} + VB_{2s-1,2a-2} && s \in \mathbb{N} \\
d(B_{2s+1,1}) &= (C_{2s,2} + C_{2s+1,1}) + VB_{2s+1,2} && s \in \mathbb{N} \\
d(B_{2s+1,2p}) &= (C_{2s-2p+1,2p} + A_{2s+1,2p-2s}) + UB_{2s+1,2p+1} && s, p \in \mathbb{N} \\
d(B_{2s+1,2p+1}) &= (C_{2s-2p,2p+2} + C_{2s-2p+1,2p+1} + A_{2s+1,2p-2s+1}) + V(B_{2s+1,2p+2} + B_{2s+1,2p}) && s, p \in \mathbb{N} \\
d(B_{2s+1,2a}) &= (C_{2s-2a+1,2a} + A_{2s+1,2a-2s}) + (C_{2s-2a,2a+1} + A_{2s,2a-2s+1}) && s \in \mathbb{N}
\end{aligned}$$

Proof The above are simple applications of the definition of tensor product between a Type D structure and A_∞-module as presented in Section 4.5.2. The fact that this box-tensor product is indeed a chain complex is a consequence of [25, Lem. 2.30] for general Type D structures and A_∞-modules, and more specifically for $\mathcal{C}(D)$ from [49, Sec. 8.2]. Recall, the maps in the Type D structure used within this calculation are determined in Section 5.5.

The differential map drops the Δ (and Maslov) integer gradings by the fact that the Type D structure and A_∞-modules are adapted to their respective one-manifolds. As a graded module, the map ∂ in the Type D structure drops the Maslov and Alexander gradings appropriately, as defined in Section 4.3.1.

As a special case, the states $B_{2r+1,2a}$ have a differential that is separable into two sets. The first set arises from pairing the map $\partial(B_{2r+1,2a}) \ni 1 \otimes (C_{2r-2a+1,2a} + A_{2r+1,2a-2r})$ with the map $m_2(Q, U_1^0) = U^0 \otimes Q$ for Q any generator of $Y' \boxtimes \widetilde{\mho}^2 \boxtimes \widetilde{\mho}^2$. The second arises from the only sequence of algebra elements with non-integer weights. Specifically, the maps

$$\begin{aligned}
\partial(B_{2r+1,2a}) &\ni L_5 \otimes B_{2r+1,>} \\
\partial(B_{2r+1,>}) &\ni U_4 \otimes B_{2r,>} \\
\partial(B_{2r,>}) &\ni R_5 \otimes (C_{2r-2a,2a+1} + C_{2r,2a-2s+1})
\end{aligned}$$

as determined in Lemma 5.7, Lemma 5.11 and Corollary 5.12 are paired with the map $m_4(Q, L_5, U_4, R_5) = 1 \otimes Q$ to yield the result. Note, that not all four terms can exist at the same time, due to the fact that the map only exists if the indices on A_{jk} and C_{ij} are in the required ranges.

The differential in the chain complex for the generators corresponding to other Kauffman states arise from maps in the Type D structure with integer weight algebra elements, pairing with the respective maps in Definition 4.57. ■

The fact that filtered chain homotopy type $\mathcal{C}(D)$ is an oriented knot invariant is a consequence of the construction by Ozsváth-Szabó. In [49, Thm. 1.1], they prove that for any special knot diagram D, the Type D structure associated to the upper knot diagram arising from excluding only the global minimum is invariant under bridge moves and Reidemeister moves. Although [49, Thm. 1.1] is stated only in terms of homology $H_*(\mathcal{C}(D))$, the equivalence of graded modules is proved for the Type D structures, and then there is only a single way to tensor with the A_∞-module corresponding to the global minimum.

The construction of the $\mathcal{C}(P(2c+1, -2b-1, 2a))$ within this thesis follows exactly the construction of [47, 49], and so the filtered chain-homotopy type of this bigraded chain complex is an invariant of the three strand pretzel knot.

As in classical knot Floer homology, one can construct subcomplexes and quotient complexes from $\mathcal{C}(D)$, and extract information from these. Motivated by $\widehat{HFK}(K)$ and $HFK^-(K)$, one has the following.

Definition 6.2 *Define $\widehat{\mathcal{C}}(D)$ to be the bigraded module over $\mathbb{F}_2$ resulting from setting*

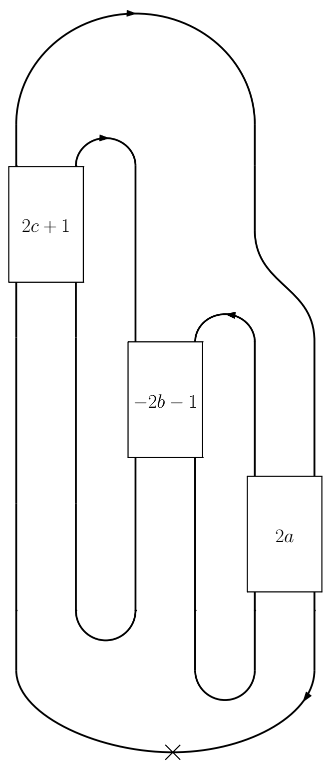


Figure 6.1: *The special knot diagram for the knot* $P(2c+1, -2b-1, 2a)$*. The complex* $\mathcal{C}(D)$ *corresponding to this knot diagram is* $Y' \boxtimes \widetilde{\mho}^2 \boxtimes \widetilde{\mho}^2 \boxtimes \mathcal{T}$*, where the Type D structure* $\mathcal{T}$ *is as defined on page 189.*

$U = 0 = V$ in $\mathcal{C}(D)$. It is thus a chain complex over $\mathbb{F}_2$ generated by Kauffman states associated to a special knot diagram, with differential $\widehat{\partial} : \widehat{\mathcal{C}}(D)_{d,s} \to \widehat{\mathcal{C}}(D)_{d-1,s}$ defined by:

$$\widehat{\partial}(x) = d(x)/(U = 0 = V),$$

where d is the differential map in $\mathcal{C}(D)$.

Likewise, define $\mathcal{C}^-(D)$ to be the bigraded module over $\mathbb{F}[U]$ resulting from setting $V = 0$ in $\mathcal{C}(D)$. Hence, as a chain complex, it is generated over $\mathbb{F}[U]$ by Kauffman states associated to a special knot diagram, with differential ∂^- defined by $\partial^-(x) = d(x)/(V = 0)$.

From both of these chain complexes, one can define the associated homology theories $\widehat{H}(D) := H_*(\widehat{C}(D))$ and $H^-(D) := H_*(\mathcal{C}^-(D))$, which are respectively bigraded $\mathbb{F}$ and $\mathbb{F}[U]$-modules recently identified with $\widehat{HFK}(D)$ and $HFK^-(D)$ in [48, Thm. 1.1].

Using Theorem 6.1, setting $U = V = 0$, one arrives at the following.

Theorem 6.3 *For the three-strand pretzel knot $P(2c+1, -2b-1, 2a)$ with oriented knot diagram as depicted in Figure 6.1, the bigraded group $\widehat{H}(D)$ decomposes as $\widehat{H}(D) =$*

$\bigoplus_{d,s} \widehat{H}(D)_{d,s}$. Here, let d be the Maslov grading $\Delta - A$, and s the Alexander grading A.

If $a \leq b \leq c$, one has that:

$$\widehat{H}(D)_{d,s} = \begin{cases} \mathbb{F}^{2b-2a+1}_{d=s+b-c} & b-c+1 \leq s \leq c-b-1 \\ \mathbb{F}^{2b-a+1}_{d=0} & s \in \{c-b, b-c\} \\ \mathbb{F}^{2b-n+1}_{d=s+b-c} & s = c-b+n, \quad 1 \leq n \leq 2b \\ \mathbb{F}^{2b-n+1}_{d=s+b-c} & s = b-c-n, \quad 1 \leq n \leq 2b. \end{cases}$$

Similarly, when $b < c$ and $a > b$, one has that:

$$\widehat{H}(D)_{d,s} = \begin{cases} \mathbb{F}^{2a-2b-1}_{d=s+b-c+1} & b-c+1 \leq s \leq c-b-1 \\ \mathbb{F}^{b}_{d=0} & s \in \{c-b, b-c\} \\ \mathbb{F}^{a-b-1}_{d=1} & s \in \{c-b, b-c\} \\ \mathbb{F}^{2b-n+1}_{d=s+b-c} & s = c-b+n, \quad 1 \leq n \leq 2b \\ \mathbb{F}^{2b-n+1}_{d=s+b-c} & s = b-c-n, \quad 1 \leq n \leq 2b. \end{cases}$$

Hence, when $a \leq b$, the complex $\widehat{H}(D)$ is homologically thin (contained in one diagonal $M - A$), and not if $a > b$.

Proof Using the map d as given in Theorem 6.1, it is simple to see that A_{jk} and C_{ij} are in $\ker(\widehat{\partial})$ for all possible values of i, j and k.

Furthermore, as proven in [49, Prop. 1.2], as a bigraded module $\mathcal{C}(D)$ is symmetric, i.e. $\widehat{H}(D)_{d,s} \cong \widehat{H}(D)_{d-2s,-s}$. This matches the symmetry for $\widehat{HFK}(D)$ in classical knot Floer homology, see [39].

As a consequence, one need only determine the groups in non-negative Alexander grading s, and use the symmetry to determine the remaining groups.

<u>Case 1: $a \leq b < c$</u>

Using Table 2.2, the states with non-negative Alexander grading are states B_{ik} with $0 \leq A(B_{ik}) \leq c - b$, and C_{ij} with $0 \leq A(C_{ij}) \leq b + c$.

Consequently, since the differential $\widehat{\partial}$ preserves the Alexander grading, every state C_{ij} with $A(C_{ij}) > c - b$ must be in $\ker(\widehat{\partial})/im(\widehat{\partial})$. There are no states B_{ik} in these Alexander gradings, so in this range no state C_{ij} can appear in $im(\widehat{\partial})$.

The Alexander grading of a state C_{ij} is given by $A(C_{ij}) = i + j - b - c - 2$. Hence, if $A(C_{ij}) = c - b + n$, then $i + j = 2c + 2 + n$. By assumption, one has that $1 \leq i \leq 2c + 1$ and $1 \leq j \leq 2b + 1$. The maximal Alexander grading of any state C_{ij} is thus $2c + 2b + 2 = A(C_{2c+1,2b+1})$. This is the unique maximal state, and corresponds to $n = 2b$.

For any n such that $1 \leq n \leq 2b$, one has that the states

$$\{C_{2c+1,n+1}, C_{2c,n+2}, \cdots, C_{2c-2b+n+1,2b+1}\}$$

have Alexander grading equal to $c - b + n$. The cardinality of this set is $2b - n + 1$, and this set provides a basis for $\widehat{H}(D)_{d,c-b+n}$. From Table 2.2, $d = s + b - c$ for all states C_{ij}, and so this is the corresponding Maslov grading for these generators.

The only states in Alexander grading $c - b$ are the states C_{ij} with $i + j = 2c + 2$ and the states B_{ik} with $i = 2c + 1$ and k odd. The set of all states in this Alexander grading are thus:

$$\{C_{2c+1,1}, C_{2c,2}, \cdots, C_{2c-2b+1,2b+1}\} \cup \{B_{2c+1,1}, B_{2c+1,3} \ldots B_{2c+1,2a-1}\}.$$

From Theorem 6.1, one can deduce that $\widehat{\partial}(B_{2c+1,2p+1}) = C_{2c-2p,2p+2} + C_{2s-2p+1,2p+1}$. Enumerating over the possible states $B_{2c+1,2p+1}$, one has that:

$$\begin{aligned}
\widehat{\partial}(B_{2c+1,1}) &= C_{2c,2} + C_{2c+1,1} \\
\widehat{\partial}(B_{2c+1,3}) &= C_{2c-2,4} + C_{2c-1,3} \\
&\vdots \\
\widehat{\partial}(B_{2c+1,2a-3}) &= C_{2c-2a+4,2a-2} + C_{2c-2a+5,2a-3} \\
\widehat{\partial}(B_{2c+1,2a-1}) &= C_{2c-2a+2,2a} + C_{2c-2a+3,2a-1}
\end{aligned}$$

By assumption, $a \leq b$, and so all of the indices for the C_{ij} terms in the image are well defined. Hence, no state B_{ij} with $i + j = 2c + 2$ lies in $\ker(\widehat{\partial})$. A basis for the homology in Alexander grading $c - b$ is then given by

$$\{C_{2c+1,1}, C_{2c-1,3}, \cdots, C_{2c-2a+3,2a-1}\} \cup \{C_{2c-2a+1,2a+1}, C_{2c-2a,2a+2}, \cdots, C_{2c-2b+1,2b+1}\},$$

which has cardinality $\frac{1}{2}(2a) + (2b - 2a + 1) = 2b - a + 1$.

In Alexander grading s for $0 \leq s \leq c - b - 1$, one has the states C_{ij} with $i + j = s + b + c + 2$, and states B_{ik} with $i = s + b + c + 2$ with k even, or $i = s + b + c + 1$ with k odd.

With $s = c - b - 1$, these states are:

$$\{C_{2c,1}, C_{2c-1,2}, \cdots, C_{2c-2b,2b+1}\} \quad \cup \quad \{B_{2c+1,2}, B_{2c+1,4} \cdots, B_{2c+1,2a}\} \\ \cup \quad \{B_{2c,1}, B_{2c,3}, \cdots, B_{2c,2a-1}\}.$$

Applying Theorem 6.1 once more, the maps $\widehat{\partial}$ in this Alexander grading are as follows.

$$\begin{aligned}
\widehat{\partial}(B_{2c+1,2}) &= C_{2c-1,2} \\
\widehat{\partial}(B_{2c+1,4}) &= C_{2c-3,4} \\
&\vdots \\
\widehat{\partial}(B_{2c+1,2a-2}) &= C_{2c-2a+3,2a-2} \\
\widehat{\partial}(B_{2c+1,2a}) &= C_{2c-2a+1,2a} + C_{2c-2a,2a+1}
\end{aligned}$$

$$\begin{aligned}
\widehat{\partial}(B_{2c,1}) &= C_{2c,1} \\
\widehat{\partial}(B_{2c,3}) &= C_{2c-2,3} \\
&\vdots \\
\widehat{\partial}(B_{2c,2a-3}) &= C_{2c-2a+4,2a-3} \\
\widehat{\partial}(B_{2c,2a-1}) &= C_{2c-2a+2,2a-1}.
\end{aligned}$$

Consequently, in Alexander grading $c - b - 1$, a basis for the homology is given by

$$\{C_{2c-2a,2a+1}, C_{2c-2a-2,2a+2}, \cdots, C_{2c-2b,2b+1}\},$$

which has cardinality $2b - 2a + 1$, as specified in the statement of the theorem.

The calculation of homology is very similar in Alexander grading $c-b-3$, where the states in this Alexander grading are now those C_{ij} with $i + j = 2c - 1$, those B_{ik} with $i = 2c - 1$ and k even, and those B_{ik} with $i = 2c - 2$, with k odd. The maps $\widehat{\partial}$ in this Alexander grading are given by:

$$\begin{aligned}
\widehat{\partial}(B_{2c-1,2}) &= C_{2c-3,2} \\
\widehat{\partial}(B_{2c-1,4}) &= C_{2c-5,4} \\
&\vdots \\
\widehat{\partial}(B_{2c-1,2a-2}) &= C_{2c-2a+1,2a-2} \\
\widehat{\partial}(B_{2c-1,2a}) &= C_{2c-2a-1,2a} + C_{2c-2a-2,2a+1}
\end{aligned}$$

$$\widehat{\partial}(B_{2c-2,1}) = C_{2c-2,1}$$
$$\widehat{\partial}(B_{2c-2,3}) = C_{2c-4,3}$$
$$\vdots$$
$$\widehat{\partial}(B_{2c-2,2a-3}) = C_{2c-2a+2,2a-3}$$
$$\widehat{\partial}(B_{2c-2,2a-1}) = C_{2c-2a,2a-1}.$$

Once more, a basis for the homology in this Alexander grading is then

$$\{C_{2c-2a-2,2a+1}, C_{2c-2a-3,2a+2}, \cdots, C_{2c-2b-2,2b+1}\}.$$

This set has cardinality $2b-2a+1$, as required. It is then simple to see that the situation is the same in Alexander grading $c-b-(2r+1)$, where this is greater than 0. To adapt the calculation, simply decrease the i index on all terms C_{ij} and B_{ik} by 2.

Now, consider the states with Alexander grading $c-b-2$, namely C_{ij} with $i+j=2c$, and B_{ik} with $i=2c$, k even, or $i=2c-1$ and k odd. Once more, all C_{ij}-states are in the kernel of $\widehat{\partial}$. The maps $\widehat{\partial}$ in this Alexander grading are as follows.

$$\widehat{\partial}(B_{2c-1,1}) = C_{2c-2,2} + C_{2c-1,1}$$
$$\widehat{\partial}(B_{2c-1,3}) = C_{2c-4,4} + C_{2c-3,3}$$
$$\vdots$$
$$\widehat{\partial}(B_{2c-1,2a-3}) = C_{2c-2a+2,2a-2} + C_{2c-2a+3,2a-3}$$
$$\widehat{\partial}(B_{2c-1,2a-1}) = C_{2c-2a,2a} + C_{2c-2a+1,2a-1}$$
$$\widehat{\partial}(B_{2c,2}) = C_{2c-1,1}$$
$$\widehat{\partial}(B_{2c,4}) = C_{2c-3,3}$$
$$\vdots$$
$$\widehat{\partial}(B_{2c,2a-2}) = C_{2c-2a+3,2a-3}$$
$$\widehat{\partial}(B_{2c,2a}) = C_{2c-2a+1,2a-1}.$$

Using a simple linear combination of the above states, one sees that the states C_{ij} with $i+j=2c$ and $j \le 2a$ all lie in $im(\widehat{\partial})$. The remaining C_{ij} states provide a basis for the homology, namely

$$\{C_{2c-2a-1,2a+1}, C_{2c-2a-2,2a+2}, \cdots, C_{2c-2b-1,2b+1}\}.$$

This is a set that has cardinality $2b-2a+1$, as required.

Likewise, to yield the states in Alexander grading $c-b-(2r)$ when this quantity is non-negative, decrease all of the i indices by 2 in the terms C_{ij} and B_{ik} above. This gives a set of the same cardinality, namely

$$\{C_{2c-2a-2r+1,2a+1}, C_{2c-2a-2r,2a+2}, \cdots, C_{2c-2b-2r+1,2b+1}\}.$$

Hence, in case 1, one has the required ranks in each Alexander grading, and since all of the generators are states C_{ij}, one has that $d-s=b-c$.

<u>Case 2: $b<c$, $b<a$</u>

In this case, the calculation is similar to that of Case 1. Since the maximal Alexander grading for any of the states B_{ik} is $c-b$, for all states C_{ij} with Alexander grading $c-b+1 \leq A(C_{ij}) \leq b+c$ the calculation remains exactly the same as presented above. Each state C_{ij} satisfying these bounds represents a generator in homology, and so the rank of the homology in Alexander grading $s=c-b+n$ is $2b-n+1$ for $1 \leq n \leq 2b$.

The possible states with Alexander grading equal to $c-b$ are B_{ik} with $i=2c+1$ and k odd, and C_{ij} with $i+j=2c+2$:

$$\{C_{2c+1,1}, C_{2c,2}, \cdots, C_{2c-2b+1,2b+1}\} \cup \{B_{2c+1,1}, B_{2c+1,3} \ldots B_{2c+1,2a-1}\}.$$

Enumerating over the possible states $B_{2c+1,2p+1}$ one has the following calculation for $\widehat{\partial}$.

$$\begin{aligned}
\widehat{\partial}(B_{2c+1,1}) &= C_{2c,2} + C_{2c+1,1} \\
\widehat{\partial}(B_{2c+1,3}) &= C_{2c-2,4} + C_{2c-1,3} \\
&\vdots \\
\widehat{\partial}(B_{2c+1,2b-1}) &= C_{2c-2b+2,2b} + C_{2c-2b+3,2b-1} \\
\widehat{\partial}(B_{2c+1,2b+1}) &= C_{2c-2b+1,2b+1} \\
\widehat{\partial}(B_{2c+1,2b+3}) &= 0 \\
&\vdots \\
\widehat{\partial}(B_{2c+1,2a-1}) &= 0
\end{aligned}$$

A basis for homology in this Alexander grading is then given by the set

$$\{C_{2c,2}, C_{2c-2,4}, \cdots, C_{2c-2b+2,2b}\} \quad \cup \quad \{B_{2c+1,2b+3}, B_{2c+1,2b+5}, \cdots, B_{2c+1,2a-1}\}.$$

The subset of the above with only C_{ij} states has cardinality b, and Maslov grading equal to 0, while the subset with only B_{ik} states has cardinality $a-b-1$, and Maslov grading 1. This is precisely as required.

The calculation for states with Alexander grading $c-b-1$ is similar to Case 1. The states in this Alexander grading are as enumerated on page 197, with the differential $\widehat{\partial}$ as follows.

$$\begin{aligned}
\widehat{\partial}(B_{2c+1,2}) &= C_{2c-1,2}\\
\widehat{\partial}(B_{2c+1,4}) &= C_{2c-3,4}\\
&\vdots\\
\widehat{\partial}(B_{2c+1,2b}) &= C_{2c-2b+1,2b}\\
\widehat{\partial}(B_{2c+1,2b+2}) &= 0\\
&\vdots\\
\widehat{\partial}(B_{2c+1,2a}) &= 0\\
\widehat{\partial}(B_{2c,1}) &= C_{2c,1}\\
\widehat{\partial}(B_{2c,3}) &= C_{2c-2,3}\\
&\vdots\\
\widehat{\partial}(B_{2c,2b-1}) &= C_{2c-2b+2,2b-1}\\
\widehat{\partial}(B_{2c,2b+1}) &= C_{2c-2b,2b+1}\\
\widehat{\partial}(B_{2c,2b+3}) &= 0\\
&\vdots\\
\widehat{\partial}(B_{2c,2a-1}) &= 0.
\end{aligned}$$

A basis for the homology in this Alexander grading is then given by

$$\{B_{2c+1,2b+2}, B_{2c,2b+3}, B_{2c+1,2b+4}, \cdots, B_{2c,2a-1}, B_{2c+1,2a}\},$$

which has cardinality $2a-2b-1$, as required.

Likewise, similarly to Case 1, the calculation of $\widehat{\partial}$ for those states in Alexander grading $c-b-(2r+1)$ proceeds in exactly the same fashion, with all of the i indices in C_{ij} and B_{ik} decreased by 2. In particular, a basis for the homology is given by the following set, with cardinality $2a-2b-1$:

$$\{B_{2c-2r+1,2b+2}, B_{2c-2r,2b+3}, B_{2c-2r+1,2b+4}, \cdots, B_{2c-2r,2a-1}, B_{2c-2r+1,2a}\}.$$

Similarly, consider those states with Alexander grading $c-b-2$. These are once more as stated in Case 1. The calculation of the map $\widehat{\partial}$ is as follows:

$$\widehat{\partial}(B_{2c-1,1}) = C_{2c-2,2} + C_{2c-1,1}$$

$$\begin{aligned}
\widehat{\partial}(B_{2c-1,3}) &= C_{2c-4,4} + C_{2c-3,3} \\
&\vdots \\
\widehat{\partial}(B_{2c-1,2b-1}) &= C_{2c-2b,2b} + C_{2c-2b+1,2b-1} \\
\widehat{\partial}(B_{2c-1,2b+1}) &= C_{2c-2b-1,2b+1} \\
\widehat{\partial}(B_{2c-1,2b+3}) &= 0 \\
&\vdots \\
\widehat{\partial}(B_{2c-1,2a-1}) &= 0 \\
\widehat{\partial}(B_{2c,2}) &= C_{2c-1,1} \\
\widehat{\partial}(B_{2c,4}) &= C_{2c-3,3} \\
&\vdots \\
\widehat{\partial}(B_{2c,2b-2}) &= C_{2c-2b+3,2b-3} \\
\widehat{\partial}(B_{2c,2b}) &= C_{2c-2b+1,2b-1} \\
\widehat{\partial}(B_{2c,2b+2}) &= C_{2c-2b-1,2b+1} \\
\widehat{\partial}(B_{2c,2b+4}) &= 0 \\
&\vdots \\
\widehat{\partial}(B_{2c,2a}) &= 0.
\end{aligned}$$

A basis for the homology is then given by the set

$$\{(B_{2c-1,2b+1} + B_{2c,2b+2}), B_{2c-1,2b+3}, B_{2c,2b+4}, \cdots, B_{2c,2a}\}.$$

Note, this set includes a term that is the sum of two B_{ik} states, but as a generating set this has cardinality $2a-2b-1$. Similarly, one can adapt the calculation to states with Alexander grading $c-b-(2r)$ by the same method as before. The generating set for the homology in this Alexander grading has cardinality $2a-2b-1$, namely:

$$\{(B_{2c-2r+1,2b+1} + B_{2c-2r+2,2b+2}), B_{2c-2r+1,2b+3}, B_{2c-2r+2,2b+4}, \cdots, B_{2c-2r+2,2a}\}.$$

Hence, in Cases 1 and 2, generating sets for the homology have been given, and they match the statement in the theorem. ■

For specific values of a, b, c, it is easy to verify the above using the computer implementation of the construction [47], and the wrapper for this written by the author [58]. The output of the program are the ranks of the homology groups in each Maslov and Alexander grading.

Remark 6.4 *As noted in Section 1.1, in [5] Eftekhary calculated the hat version of the knot Floer homology* $\widehat{HFK}(K)$ *for the above family of pretzel knots. In [5, Thm. 2], it is stated that the hat version of the knot Floer homology is contained in precisely two diagonals:* $d - s = b - c$ *and* $d - s = b - c + 1$*, and for each value of* s *is non-zero in at most one of these diagonals.*

Since the conjectural equivalence between the theory of the bordered invariant $\mathcal{C}(D)$ *and classical knot Floer homology holds, as recently proven in [48, Thm. 1.1], then there is a contradiction with this result. In particular, Theorem 6.3 states that in Alexander grading* 2*, the homology group* $\widehat{H}(P(7,-3,6))_{d,2}$ *is* $\mathbb{F}_{d=1} \oplus \mathbb{F}_{d=0}$*. For this three strand pretzel knot* $P(2c+1, -2b-1, 2a)$*, the coefficients describing the knot are* $c = 3 = a$ *and* $b = 1$*.*

So $a - b - 1 = 1 = b$*, and in Alexander grading* $c - b = 2$*, one has a direct summand* $\mathbb{F}$ *with Maslov grading equal to* 0*, and a direct summand* $\mathbb{F}$ *with Maslov grading equal to* 1*. Using the bimodule* $\mho^2$ *as defined by Ozsváth and Szabó in [49], the computer implementation [47] verifies this calculation, with the homology having the same ranks as described in Theorem 6.3.*

Using Theorem 6.1, one can also determine the ranks of the homology groups $\widehat{H}(D)$ for D a special knot diagram of the three strand pretzel knot $P(2c+1, -2b-1, 2a)$ with $b \geq c$. However, this is a less interesting case, since it has already been demonstrated that this knot has $\tau(D) = \nu(D) = b - c = g_4(D)$ from Lemma 3.19. However, for completeness, the ranks of the homology groups $\widehat{H}(D)_{d,s}$ are presented in Corollary 6.5.

Corollary 6.5 *Let* D *be a special knot diagram of the three strand pretzel knot* $P(2c+1, -2b-1, 2a)$ *with* $b \geq c$*, as presented in Figure 6.1. Then, the bigraded homology groups* $\widehat{H}(D) = \oplus_{d,s\in\mathbb{Z}} \widehat{H}(D)_{d,s}$ *are given as follows. When* $b > c$*, one has that:*

$$\widehat{H}(D)_{d,s} = \begin{cases} \mathbb{F}^{2a+2c+1}_{d=s+b-c} & c-b+1 \leq s \leq b-c-1 \\ \mathbb{F}^{a+2c+1}_{d=0} & s \in \{c-b, b-c\} \\ \mathbb{F}^{2c-n+1}_{d=s+b-c} & s = b-c+n, \quad 1 \leq n \leq 2c \\ \mathbb{F}^{2c-n+1}_{d=s+b-c} & s = c-b-n, \quad 1 \leq n \leq 2c. \end{cases}$$

When $b = c$, one has that the homology groups are given by:

$$\widehat{H}(D)_{d,s} = \begin{cases} \mathbb{F}^{2b+1}_{d=0} & s = 0 \\ \mathbb{F}^{2c-n+1}_{d=s+b-c} & s = n, \quad 1 \leq n \leq 2c \\ \mathbb{F}^{2c-n+1}_{d=s+b-c} & s = -n, \quad 1 \leq n \leq 2c. \end{cases}$$

Proof As in the proof of Theorem 6.3, one can exploit the symmetry of $\mathcal{C}(D)$ proven in [49, Prop. 1.2], and examine only non-negative Alexander gradings.

Furthermore, all of the states C_{ij} and A_{jk} are in $\ker(\widehat{\partial})$. From Table 2.2, any state C_{ij} or A_{jk} with Alexander grading greater than the maximum Alexander grading of some state B_{ik} cannot appear in $im(\widehat{\partial})$, and hence provide generators of homology.

Consequently, it is simple to see that for any n in the range $1 \leq n \leq 2c$, the states

$$\{C_{n+1,2b+1}, C_{n+2,2b}, \cdots, C_{2c+1,2b-2c+n+1}\}$$

have Alexander grading equal to $b - c + n$, and provide a basis for $\widehat{H}(D)_{d,b-c+n} \cong \mathbb{F}^{2c-n+1}_{d=2b-2c+n}$.

<u>Case 1: $b > c$</u>

When $b > c$, there are no states B_{ik} with non-negative Alexander grading. Hence, all of the states C_{ij} and A_{jk} with non-negative Alexander are generators of homology.

In Alexander grading $b - c$, the generators of the homology group $\widehat{H}(D_{d,s})$ are then

$$\{C_{1,2b+1}, C_{2,2b}, \cdots, C_{2c+1,2b-2c+1}\} \cup \{A_{2b+1,1}, A_{2b+1,3}, \cdots, A_{2b+1,2a-1}\},$$

which is a set of cardinality $2c + a + 1$.

For Alexander grading $0 \leq k \leq b - c - 1$, the homology group has rank $2a + 2c + 1$, as it is generated by the set

$$\{C_{1,b+c+k+1}, C_{2,b+c+k}, \cdots, C_{2c+1,b-c+k+1}\} \cup \{A_{b+c+k+1,2r+1}\}_{r=0}^{a-1} \cup \{A_{b+c+k+2,2r}\}_{r=1}^{2a}.$$

Since these are all of the states in the non-negative Alexander gradings, this completes the calculation of the homology groups when $b > c$.

<u>Case 2: $b = c$</u>

In this case, one has that $b - c = 0 = c - b$. The only states with positive Alexander grading are the states C_{ij} with $i + j > b + c + 2$. All of these states are generators of homology, and the calculation of the ranks is the same as in Case 1.

In Alexander grading 0, the generating sets for the homology depend on whether $a < b$, $a = b$ or $a > b$, however the rank doesn't change.

Using the fact that $\widehat{\partial}(B_{2c+1,2p+1}) = C_{2c-2p,2p+2} + C_{2c-2p+1,2p+1} + A_{2c+1,2p-2s+1}$, one can easily check that a generating set is given by

$$\begin{aligned} \{A_{2b+1,2r+1}\}_{r=0}^{a-1} \cup \{C_{2c-2r+1,2r+1}\}_{r=0}^{a-1} \cup \{C_{2c-2a+1,2a+1}, C_{2c-2a,2a+2}, \cdots, C_{1,2b+1}\} &\quad a < c = b \\ \{C_{2c+1,1}, C_{2c-1,3}, \cdots, C_{2c-2a+1,2a+1}\} \cup \{A_{2b+1,1}, A_{2b+1,3}, \cdots, A_{2b+1,2a-1}\} &\quad a = c = b \\ \{C_{2c-2r+1,2r+1}\}_{r=0}^{b} \cup \{A_{2b+1,2a-2b+1}, A_{2b+1,2a-2b+3}, \cdots, A_{2b+1,2a-1}\} &\quad a > c = b, \end{aligned}$$

all of which are sets of cardinality $2b + 1 = 2c + 1$. Note, the only states B_{ik} in Alexander grading 0 are $B_{2c+1,2p+1}$ for $0 \leq p \leq a - 1$, none of which lie in $\ker(\widehat{\partial})$. ■

For the family of three strand pretzel knots given by $P(2c + 1, -2b - 1, 2a)$, the hat version of the homology $\widehat{H}(P(2c + 1, -2b - 1, 2a))$ is thin (contained in a single diagonal $M - A$) when $b \geq c$, and when $c > b \geq a$. Using the equivalence between $\widehat{HFK}(D)$ and $\widehat{H}(D)$ from [48, 49], in these cases the classical knot Floer complex $CFK^{\infty}(D)$ is completely determined by the concordance invariant τ and the Alexander polynomial, as proven by [52]. Applying Proposition 3.18 and Lemma 3.19 allows for the determination of τ, which is equal to $-\frac{\sigma}{2}$, where σ is signature of these knots. This signature is calculable using the techniques of [7]. These knots are called σ-thin, following the terminology of [52].

However, in remaining case when $c > b$ and $a > b$, the homology groups $\widehat{H}(D)$ are not thin, and so $HFK^{-}(D)$ and $CFK^{\infty}(D)$ are not directly calculable simply from τ and the Alexander polynomial.

6.2 Calculation of $H^{-}(D)$

The determination of $H^{-}(D)$, when D is a special knot diagram of $P(2c+1, -2b-1, 2a)$ with $min(a, c) > b$ follows as a corollary from the determination of $\mathcal{C}(D)$ in Theorem 6.1. For comparison, in [49], the bigraded homology group $H^{-}(D)$ for a special knot diagram D is denoted $\mathcal{J}^{U}(D) = H(\mathcal{C}(D)/V = 0)$.

Theorem 6.6 *Let D be a special knot diagram of $P(2c+1, -2b-1, 2a)$ with $min(a, c) >$ b, with a diagram of the form described in Figure 6.1. Denote by $H^{-}(D, s)$ the group $\oplus_{d} H^{-}(D)_{d,s}$, where s is the Alexander grading and d the Maslov grading. The homology* *groups $H^-(D)$ as described in Definition 6.2 are then as follows:*

$$\begin{aligned}
H^-(D, 2\ell - b - c - 1) &= \mathbb{F}^{\ell}_{d=2\ell-2c-1} && 1 \leq \ell \leq b \\
H^-(D, 2\ell - b - c) &= \mathbb{F}^{\ell}_{d=2\ell-2c} && 1 \leq \ell \leq b-1 \\
H^-(D, b - c) &= \mathbb{F}^{b+1}_{d=2b-2c} && \\
H^-(D, b - c + 1) &= \mathbb{F}[U]_{d=2b-2c+2} \oplus \mathbb{F}^{a-b-1}_{d=2b-2c+2} && \\
H^-(D, 2\ell - b - c) &= \mathbb{F}^{a-b-1}_{d=2\ell-2c+1} && b+1 \leq \ell \leq c \\
H^-(D, 2\ell - b - c - 1) &= \mathbb{F}^{a-b}_{d=2\ell-2c} && b+1 < \ell \leq c \\
H^-(D, c - b + 2\ell + 1) &= \mathbb{F}^{b-\ell}_{d=2\ell+1} && 0 \leq \ell \leq b-1 \\
H^-(D, c - b + 2\ell) &= \mathbb{F}^{b-\ell+1}_{d=2\ell} && 1 \leq \ell \leq b.
\end{aligned}$$

Proof From Table 2.2, the minimal Alexander grading of any state A_{jk}, B_{ik} or C_{ij} is $-b-c-1$, which is achieved for states

$$\{A_{1,2r}\}_{r=1}^{a} \cup \{B_{1,2r}\}_{r=1}^{2a}.$$

Using Theorem 6.1, one has that

$$\begin{aligned}
\partial^-(A_{1,2r}) &= UA_{1,2r+1} \\
\partial^-(B_{1,2r}) &= A_{1,2r} + UB_{1,2r+1}.
\end{aligned}$$

Hence, since no cancellation can occur, one has that none of the elements above live in $\ker(\partial^-)$, and so the homology in Alexander grading $-b-c-1$ is trivial. It is also simple to check that $(\partial^-)^2 = 0$ when applied to either of these states, as can be expected from a chain complex.

Likewise, the states in Alexander grading $-b-c$ are

$$\{A_{1,2r+1}, B_{1,2r+1}\}_{r=0}^{a-1} \cup \{A_{2,2r}, B_{2,2r}\}_{r=1}^{a} \cup \{C_{11}\}.$$

Applying Theorem 6.1 once more, the action of ∂^- on these states is as follows:

$$\begin{aligned}
\partial^-(A_{11}) &= U(A_{21} + C_{12}) \\
\partial^-(A_{1,2r+1}) &= 0 \\
\partial^-(A_{2,2r}) &= U(A_{3,2r} + A_{2,2r+1}) \\
\partial^-(B_{11}) &= A_{11} + C_{11} \\
\partial^-(B_{1,2r+1}) &= A_{1,2r+1}
\end{aligned}$$

$$\partial^-(B_{2,2r}) = (C_{2-2r+1,2r-1} + A_{2,2r-2}) + U(B_{3,2r} + B_{2,2r+1})$$
$$\partial^-(C_{11}) = U(A_{21} + C_{12}).$$

Using this, it is clear that $\ker(\partial^-)$ in this Alexander grading is spanned by

$$\{A_{11} + C_{11}\} \cup \{A_{1,2r+1}\}_{r=0}^{a-1},$$

which are also terms that feature in $im(\partial^-)$. Hence, the homology group is also trivial in Alexander grading $-b-c$.

Alexander grading $2\ell - b - c - 1$

The states with Alexander grading $2\ell - b - c - 1$ for $1 \leq \ell \leq b$ are:

$$\{A_{2\ell,2r+1}, B_{2\ell,2r+1}\}_{r=0}^{a-1} \cup \{A_{2\ell+1,2r}, B_{2\ell+1,2r}\}_{r=1}^{2a} \cup \{C_{ij}\}_{i+j=2\ell+1}.$$

Fixing ℓ in the range $1 \leq \ell \leq b$, applying Theorem 6.1 the differential ∂^- acts as follows upon these states:

$$\begin{aligned}
\partial^-(A_{2\ell,2r+1}) &= UA_{2\ell+1,2r+1} && \\
\partial^-(A_{2\ell+1,2r}) &= UA_{2\ell+1,2r+1} && \\
\partial^-(C_{2p+1,2\ell-2p}) &= U(C_{2p+1,2\ell-2p+1} + C_{2p,2\ell-2p+2}) && 0 \leq p \leq \ell - 1 \\
\partial^-(C_{2p,2\ell-2p+1}) &= U(C_{2p+1,2\ell-2p+1} + C_{2p,2\ell-2p+2}) && 0 \leq p \leq \ell \\
\partial^-(B_{2\ell,2r+1}) &= C_{2\ell-2r,2r+1} + UB_{2\ell+1,2r+1} && 0 \leq r \leq \ell - 1 \\
\partial^-(B_{2\ell,2r+1}) &= A_{2\ell,2r-2\ell+1} + UB_{2\ell+1,2r+1} && \ell \leq r \leq a - 1 \\
\partial^-(B_{2\ell+1,2r}) &= C_{2\ell-2r+1,2r} + UB_{2\ell+1,2r+1} && 1 \leq r \leq \ell \\
\partial^-(B_{2\ell+1,2r}) &= A_{2\ell+1,2r-2\ell} + UB_{2\ell+1,2r+1} && \ell + 1 \leq r \leq a - 1 \\
\partial^-(B_{2\ell+1,2a}) &= A_{2\ell+1,2a-2\ell} + A_{2\ell,2a-2\ell+1}. &&
\end{aligned}$$

Using the above, one has that in this Alexander grading, $\ker(\partial^-)$ contains the terms

$$\{A_{2\ell+1,2r} + A_{2\ell,2r+1}\}_{r=1}^{a-1} \cup \{A_{2\ell+1,2a}\} \cup \{C_{2p+1,2\ell-2p} + C_{2p,2\ell-2p+1}\}_{p=0}^{\ell-1}.$$

Note that the sums of the C_{ij} terms appear in the image of ∂^-, since for $1 \leq r \leq \ell - 1$, one has that:

$$\partial^-(B_{2\ell+1,2} + B_{2\ell,3}) = C_{2\ell-1,2} + C_{2\ell-2,3}$$
$$\partial^-(B_{2\ell+1,4} + B_{2\ell,5}) = C_{2\ell-3,4} + C_{2\ell-4,5}$$

$$\vdots$$

$$\partial^-(B_{2\ell+1,2\ell-2}+B_{2\ell,2\ell-1})=C_{3,2\ell-2}+C_{2,2\ell-1}.$$

Similarly, for $l+1\leq r\leq a-1$, the sums of A_{jk} terms also appear in $im(\partial^-)$, since

$$\begin{aligned}\partial^-(B_{2\ell+1,2\ell+2}+B_{2\ell,2\ell+3})&=A_{2\ell+1,2}+A_{2\ell,3}\\ \partial^-(B_{2\ell+1,2\ell+4}+B_{2\ell,2\ell+5})&=A_{2\ell+1,4}+A_{2\ell,5}\\ &\vdots\\ \partial^-(B_{2\ell+1,2a-2}+B_{2\ell,2a-1})&=A_{2\ell+1,2a-2\ell-2}+A_{2\ell,2a-2\ell-1}\\ \partial^-(B_{2\ell+1,2a})&=A_{2\ell+1,2a-2\ell}+A_{2\ell,2a-2\ell+1}.\end{aligned}$$

Furthermore, since $\partial^-(A_{2\ell,2r})=U(A_{2\ell+1,2r}+A_{2\ell,2r+1})$, for every $1\leq r\leq a-1$, the homology in Alexander grading $2\ell-b-c-1$ is given by

$$\left(\{A_{2\ell+1,2t}+A_{2\ell,2t+1}\}_{t=a-\ell+1}^{a-1}\cup\{A_{2\ell+1,2a}\}\right)/\left(\{U(A_{2\ell+1,2t}+A_{2\ell,2t+1})\}_{t=a-\ell+1}^{a-1}\cup\{UA_{2\ell+1,2a}\}\right),$$

which gives a group isomorphic to $\mathbb{F}^\ell$.

Alexander grading $2\ell-b-c$

The states with Alexander grading $2\ell-b-c$ for $1\leq\ell\leq b-1$ are

$$\{A_{2\ell+1,2r+1},B_{2\ell+1,2r+1}\}_{r=0}^{a-1}\cup\{A_{2\ell+2,2r},B_{2\ell+2,2r}\}_{r=1}^{a}\cup\{C_{ij}\}_{i+j=2\ell+2}.$$

Applying the calculation in Theorem 6.1 and setting $V=0$ for the map $d:\mathcal{C}(D)\to\mathcal{C}(D)$, one has that:

$$\begin{aligned}\partial^-(A_{2\ell+1,1})&=U(A_{2\ell+2,1}+C_{1,2\ell+2})&&\\ \partial^-(A_{2\ell+1,2r+1})&=0&&1\leq r\leq a-1\\ \partial^-(A_{2\ell+2,2r})&=UA_{2\ell+3,2a}&&\\ \partial^-(C_{1,2\ell+1})&=U(A_{2\ell+2,1}+C_{1,2\ell+2})&&\\ \partial^-(C_{2p+1,2\ell-2p+1})&=U(C_{2p+1,2\ell-2p+2}+C_{2p,2\ell-2p+3})&&1\leq p\leq\ell\\ \partial^-(C_{2p,2\ell-2p+2})&=U(C_{2p,2\ell-2p+3}+C_{2p+1,2\ell-2p+2})&&1\leq p\leq\ell\\ \partial^-(B_{2\ell+1,2r+1})&=C_{2\ell-2r,2r+2}+C_{2\ell-2r+1,2r+1}&&0\leq r\leq\ell-1\\ \partial^-(B_{2\ell+1,2\ell+1})&=C_{1,2\ell+1}+A_{2\ell+1,1}&&\\ \partial^-(B_{2\ell+1,2r+1})&=A_{2\ell+1,2r-2\ell+1}&&\ell+1\leq r\leq a-1\end{aligned}$$

$$\partial^-(B_{2\ell+2,2r}) = C_{2\ell-2r+3,2r-1} + U(B_{2\ell+3,2r} + B_{2\ell+2,2r+1}) \qquad 1 \le r \le \ell$$
$$\partial^-(B_{2\ell+2,2r}) = A_{2\ell+2,2r-2\ell-2} + U(B_{2\ell+3,2r} + B_{2\ell+2,2r+1}) \qquad \ell+1 \le r \le a.$$

As a consequence, it is easy to see that the terms $A_{2\ell+1,1} + C_{1,2\ell+1}$, $\{A_{2\ell+1,2r+1}\}_{r=0}^{a-1}$ and $\{C_{2p+1,2\ell-2p+1} + C_{2p,2\ell-2p+2}\}_{p=1}^{\ell}$ are all of the generators in $\ker(\partial^-)$ in Alexander grading $2\ell - b - c$ for $1 \le \ell \le b-1$.

The term $A_{2\ell+1,1} + C_{1,2\ell+1}$ lies in $im(\partial^-)$, as it is equal to $\partial^-(B_{2\ell+1,2\ell+1})$. Also, each term in the set $\{C_{2p+1,2\ell-2p+1} + C_{2p,2\ell-2p+2}\}_{p=1}^{\ell}$ lies in $im(\partial^-)$ using the calculation of $\partial^-(B_{2\ell+1,2r+1})$ above. Furthermore, $\{A_{2\ell+1,2r+1}\}_{r=1}^{a-\ell-1}$ also lie in $im(\partial^-)$ from the same term with $\ell+1 \le r \le a-1$.

What is more, since $\partial^-(A_{2\ell+1,2r}) = UA_{2\ell+1,2r+1}$ when $1 \le \ell \le b-1$, $H^-(D, 2\ell - b - c)$ is isomorphic to

$$\{A_{2\ell+1,2r+1}\}_{r=a-\ell}^{a-1} / \{UA_{2\ell+1,2r+1}\}_{r=a-\ell}^{a-1} \cong \mathbb{F}^{\ell}.$$

Alexander grading $b - c$

In Alexander grading $2b - b - c = b - c$, the terms in $\ker(\partial^-)$ are:

$$\{C_{1,2b+1}\} \cup \{A_{2b+1,2r+1}\}_{r=0}^{a-1} \cup \{C_{2p+1,2b-2p+1} + C_{2p,2b-2p+2}\}_{p=1}^{b}.$$

Likewise, as in the calculation above, one has that

$$im(\partial^-) \ni \{A_{2b+1,1} + C_{1,2b+1}\} \cup \{A_{2b+1,2r+1}\}_{r=1}^{a-b-1} \cup \{C_{2p+1,2b-2p+1} + C_{2p,2b-2p+2}\}_{p=1}^{b}.$$

Once more, employing the fact that $\partial^-(A_{2b+1,2r}) = UA_{2b+1,2r+1}$, the group $H^-(D, b-c)$ is isomorphic to

$$\{A_{2\ell+1,2r+1}\}_{r=a-\ell}^{a-1} / \{UA_{2\ell+1,2r+1}\}_{r=a-\ell}^{a-1} \oplus \{A_{2b+1,1}\} / \{UA_{2b+1,1}\} \cong \mathbb{F}^{b+1}.$$

Alexander grading $b - c + 1$

The possible Kauffman states in Alexander grading $b - c + 1$ are

$$\{B_{2b+2,2r+1}\}_{r=0}^{a-1} \cup \{B_{2b+3,2r}\}_{r=1}^{a} \cup \{C_{ij}\}_{i+j=2b+3}.$$

Recall, from Table 2.2, there are no A_{jk} states in this Alexander grading or in any greater Alexander grading.

Note here, that the only assumption in this corollary is that $min(a,c) > b$, hence one might have that $c = b+1$. Thus, $2b+3 = 2(b+1)+1 = 2c+1$, so each B_{ik} term is a generator corresponding to an appropriate Kauffman state.

Applying once more the calculation of ∂^- from adapting Theorem 6.1, the action of the differential on these states is as follows.

$$\begin{aligned}
\partial^-(C_{2,2b+1}) &= UC_{3,2b+1} && \\
\partial^-(C_{2p,2b+3-2p}) &= U(C_{2p,2b+4-2p} + C_{2p+1,2b+3-2p}) && 2 \leq p \leq b \\
\partial^-(C_{2b+2,1}) &= U(C_{2b+2,2} + C_{2b+3,1}) && \\
\partial^-(C_{3,2b}) &= UC_{3,2b+1} && \\
\partial^-(C_{2p+1,2b+2-2p}) &= U(C_{2p,2b+4-2p} + C_{2p+1,2b+3-2p}) && 2 \leq p \leq b \\
\partial^-(B_{2b+2,2r+1}) &= C_{2b+2-2r,2r+1} + UB_{2b+3,2r+1} && 0 \leq r \leq b \\
\partial^-(B_{2b+2,2r+1}) &= UB_{2b+3,2r+1} && b+1 \leq r \leq a-1 \\
\partial^-(B_{2b+3,2r}) &= C_{2b+3-2r,2r} + UB_{2b+3,2r+1} && 1 \leq r \leq b \\
\partial^-(B_{2b+3,2r}) &= UB_{2b+3,2r+1} && b+1 \leq r \leq a-1 \\
\partial^-(B_{2b+3,2a}) &= 0. &&
\end{aligned}$$

Hence, in this Alexander grading, one has that $ker(\partial^-)$ is spanned by

$$\{C_{2p,2b+3-2p} + C_{2p+1,2b+2-2p}\}_{p=1}^{b} \cup \{B_{2b+2,2r+1} + B_{2b+3,2r}\}_{r=b+1}^{a-1} \cup \{B_{2b+3,2a}\}.$$

Each of the terms $C_{2p,2b+3-2p}+C_{2p+1,2b+2-2p}$ appear in $im(\partial^-)$, from the terms $\partial^-(B_{2b+2,2r+1}+B_{2b+3,2r})$ with $1 \leq r \leq b$.

Since $\partial^-(B_{2b+2,2r}) = U(B_{2b+3,2r} + B_{2b+2,2r+1})$ when $b+2 \leq r \leq a-1$, and $\partial^-(B_{2b+2,2a}) = UB_{2b+3,2a}$, the group $H^-(D, b-c+1)$ contains a subgroup $\mathbb{F}^{a-b-2} \oplus \mathbb{F} \cong \mathbb{F}^{a-b-1}$.

However, the term $B_{2b+2,2b+3} + B_{2b+3,2b+2}$ and any U^k-multiple of this does not appear in $im(\partial^-)$. This is because the term $\partial^-(B_{2b+2,2b+2}) = C_{1,2b+1} + U(B_{2b+3,2b+2} + B_{2b+2,2b+3})$. An examination of Theorem 6.1 demonstrates that there is no way to cancel the $C_{1,2b+1}$ term in this sum. As a consequence, $B_{2b+2,2b+3} + B_{2b+3,2b+2} \in \ker(\partial^-)$, but since $U^k(B_{2b+2,2b+3} + B_{2b+3,2b+2})$ for any $k \in \mathbb{N}$, this generates a subgroup $\mathbb{F}[U]$ in homology. Hence, the group $H^-(D, b-c+1)$ is only non-zero in Maslov grading $2b-2c+2$, and is isomorphic to

$$H^-(D, b-c+1) \cong \mathbb{F}[U] \oplus \mathbb{F}^{a-b-1}.$$

As a remark, there is a special case when $a = b+1$, since then there is no term $B_{2b+2,2b+3} = B_{2b+2,2a+1}$. However, similar logic holds, since $B_{2b+3,2a} \in \ker(\partial^-)$, yet because

$$\partial^-(B_{2b+2,2b+2}) = \partial^-(B_{2b+2,2a}) = C_{1,2b+1} + UB_{2b+3,2a} \neq 0,$$

no U-multiples of $B_{2b+3,2a}$ ever appear in $im(\partial^-)$. Hence, this generates $\mathbb{F}[U]$, and this is the only homology in this Alexander grading when $a = b + 1$.

Alexander grading $2\ell - b - c$, $\ell > b$

The states in Alexander grading $2\ell - b - c$ with $\ell > b$ are

$$\{B_{2\ell+2,2p}\}_{p=1}^{a} \cup \{B_{2\ell+1,2p+1}\}_{p=0}^{a-1} \cup \{C_{ij}\}_{i+j=2\ell+2},$$

noting once more that there are no states A_{jk} with Alexander grading greater than $b-c < 0$. The action of ∂^- on these states is then given by the following:

$$\begin{aligned}
\partial^-(C_{2p,2\ell+2-2p}) &= U(C_{2p,2\ell+3-2p} + C_{2p+1,2\ell+2-2p}) && \ell - b + 1 \leq p \leq \ell \\
\partial^-(C_{2p+1,2\ell+1-2p}) &= U(C_{2p,2\ell+3-2p} + C_{2p+1,2\ell+2-2p}) && \ell - b + 1 \leq p \leq \ell \\
\partial^-(C_{2\ell-2b+1,2b+1}) &= 0 && \\
\partial^-(B_{2\ell+2,2p}) &= C_{2\ell+3-2p,2p-1} + U(B_{2\ell+3,2p} + B_{2\ell+2,2p+1}) && 1 \leq p \leq b+1 \\
\partial^-(B_{2\ell+2,2p}) &= U(B_{2\ell+3,2p} + B_{2\ell+2,2p+1}) && b+2 \leq p \leq a-1 \\
\partial^-(B_{2\ell+2,2a}) &= UB_{2\ell+3,2a} && \\
\partial^-(B_{2\ell+1,2p+1}) &= C_{2\ell-2p,2p+2} + C_{2\ell-2p+1,2p+1} && 0 \leq p \leq b-1 \\
\partial^-(B_{2\ell+1,2b+1}) &= C_{2\ell-2b+1,2b+1} && \\
\partial^-(B_{2\ell+1,2p+1}) &= 0 && b+1 \leq p \leq a-1.
\end{aligned}$$

As a consequence, the generators in $\ker(\partial^-)$ in this Alexander grading are

$$\{B_{2\ell+1,2p+1}\}_{p=b+1}^{a-1} \cup \{C_{2p,2\ell+2-2p} + C_{2p+1,2\ell+1-2p}\}_{p=\ell-b+1}^{\ell} \cup \{C_{2\ell-2b+1,2b+1}\}.$$

Note that $C_{2\ell-2b+1,2b+1} = \partial^-(B_{2\ell+1,2b+1})$, and $C_{2p,2\ell+2-2p}+C_{2p+1,2\ell+1-2p} = \partial^-(B_{2\ell+1,2\ell-2p+1})$. Furthermore, since $\partial^-(B_{2\ell+1,2p}) = UB_{2\ell+1,2p+1}$ when $p \geq b+1$, one deduces that

$$H^-(D, 2\ell - b - c) \cong \mathbb{F}_{d=2\ell-2c+1}^{a-b-1} \qquad \text{for } l > b.$$

Alexander grading $2\ell - b - c - 1$, $\ell > b + 1$

The only states in this Alexander grading are

$$\{B_{2\ell+1,2p}\}_{p=1}^{a} \cup \{B_{2\ell,2p+1}\}_{p=0}^{a-1} \cup \{C_{ij}\}_{i+j=2\ell+1}.$$

The differential ∂^- acts on these states as follows:

$$\begin{aligned}
\partial^-(C_{2\ell-2b,2b+1}) &= UC_{2\ell-2b+1,2b+1} && \\
\partial^-(C_{2p,2\ell+1-2p}) &= U(C_{2p,2\ell+2-2p} + C_{2p+1,2\ell+1-2p}) && \ell-b+1 \le p \le \ell \\
\partial^-(C_{2\ell-2b+1,2b}) &= UC_{2\ell-2b+1,2b+1} && \\
\partial^-(C_{2p+1,2\ell-2p}) &= U(C_{2p,2\ell+2-2p} + C_{2p+1,2\ell+1-2p}) && \ell-b+1 \le p \le \ell-1. \\
\partial^-(B_{2\ell+1,2p}) &= C_{2\ell-2p+1,2p} + UB_{2\ell+1,2p+1} && 1 \le p \le b \\
\partial^-(B_{2\ell+1,2p}) &= UB_{2\ell+1,2p+1} && b+1 \le p \le a-1 \\
\partial^-(B_{2\ell+1,2a}) &= 0 && \\
\partial^-(B_{2\ell,2p+1}) &= C_{2\ell-2p,2p+1} + UB_{2\ell+1,2p+1} && 0 \le p \le b \\
\partial^-(B_{2\ell,2p+1}) &= UB_{2\ell+1,2p+1} && b+1 \le p \le a-1.
\end{aligned}$$

From above, $\partial^-(B_{2\ell,2p+1} + B_{2\ell+1,2p}) = C_{2\ell-2p,2p+1} + C_{2\ell-2p+1,2p}$ for $1 \le p \le b$. Hence, despite the fact that $C_{2\ell-2p,2p+1} + C_{2\ell-2p+1,2p} \in \ker(\partial^-)$, these terms do not contribute to the homology.

Furthermore, $\partial^-(B_{2\ell,2p}) = U(B_{2\ell,2p+1} + B_{2\ell+1,2p})$ for $p \ge b+2$. What is more, using Theorem 6.1, one can check that $\partial^-(B_{2\ell,2b+2} + B_{2\ell-1,2b+1}) = U(B_{2\ell+1,2b+2} + B_{2\ell,2b+3})$. Hence, one has that $\{UB_{2\ell,2p+1}+UB_{2\ell+2,2p}\}_{p=b+1}^{a-1} \in im(\partial^-)$. Similarly, since $\partial^-(B_{2\ell,2a}) = UB_{2\ell+1,2a}$, one has that $im(\partial^-) \ni UB_{2\ell+1,2a}$

Then, since one has that $\{UB_{2\ell,2p+1}+UB_{2\ell+2,2p}\}_{p=b+1}^{a-1} \cup \{B_{2\ell+1,2a}\} \in \ker(\partial^-)$ in Alexander grading $2\ell-b-c-1$, the group $H^-(D, 2\ell-b-c-1)$ is thus

$$H^-(D, 2\ell-b-c-1) \cong \mathbb{F}_{d=2\ell-2c}^{a-b}.$$

<u>Alexander grading $c-b+2\ell+1$, $0 \le \ell \le b-1$</u>

As described before in the calculation of $\widehat{H}(D)$, the only states with Alexander grading greater than $c-b$ are the states C_{ij} with $i+j > 2c+2$.

Specifically, in Alexander grading $c-b+2\ell+1$, with ℓ in the range $0 \le \ell \le b-1$, one has that states

$$\{C_{2p,2c+2\ell+3-2p}\}_{p=c+\ell+1-b}^{c} \cup \{C_{2p+1,2c+2\ell+2-2p}\}_{p=c+\ell+1-b}^{c}.$$

The differential ∂^- acts as follows on these states:

$$\partial^-(C_{2(c+\ell+1-b),2b+1}) = UC_{2(c+\ell+1-b)+1,2b+1}$$
$$\partial^-(C_{2p,2c+2\ell+3-2p}) = U(C_{2p,2c+2\ell+4-2p} + C_{2p+1,2c+2\ell+3-2p}) \quad c+\ell+2-b \leq p \leq c$$
$$\partial^-(C_{2(c+\ell+1-b)+1,2b}) = UC_{2(c+\ell+1-b)+1,2b+1}$$
$$\partial^-(C_{2p+1,2c+2\ell+2-2p}) = U(C_{2p,2c+2\ell+4-2p} + C_{2p+1,2c+2\ell+3-2p}) \quad c+\ell+2-b \leq p \leq c.$$

Consequently, one has that $\{C_{2p,2c+2\ell+3-2p} + C_{2p+1,2c+2\ell+2-2p}\}_{p=c+\ell+1-b}^{c}\} \in \ker(\partial^-)$. Since one also has that $\partial^-(C_{2p,2c+2\ell+2-2p}) = U(C_{2p,2c+2\ell+3-2p} + C_{2p+1,2c+2\ell+2-2p})$ from Theorem 6.1, it is clear that each of these terms generates a subgroup $\mathbb{F}$ in $H^-(D, c-b+2\ell+1)$. Hence,

$$H^-(D, c-b+2\ell+1) \cong \mathbb{F}^{b-\ell}.$$

<u>Alexander grading $c-b+2\ell$, $1 \leq \ell \leq b$</u>

Like in the previous case, one has that the states in this Alexander grading are C_{ij} such that $i+j = 2c+2\ell+2$. These states are

$$\{C_{2p,2c+2\ell+2-2p}\}_{p=c+\ell+1-b}^{c} \cup \{C_{2p+1,2c+2\ell+1-2p}\}_{p=c+\ell-b}^{c}$$

The differential acts on these states by:

$$\partial^-(C_{2p,2c+2\ell+2-2p}) = U(C_{2p,2c+2\ell+3-2p} + C_{2p+1,2c+2\ell+2-2p}) \quad c+\ell+1-b \leq p \leq c$$
$$\partial^-(C_{2(c+\ell-b)+1,2b+1}) = 0$$
$$\partial^-(C_{2p+1,2c+2\ell+1-2p}) = U(C_{2p,2c+2\ell+3-2p} + C_{2p+1,2c+2\ell+2-2p}) \quad c+\ell+1-b \leq p \leq c.$$

Clearly, fixing ℓ, the generators of the $\ker(\partial^-)$ in this Alexander grading are

$$\{C_{2p,2c+2\ell+2-2p} + C_{2p+1,2c+2\ell+1-2p}\}_{p=c+\ell+1-b}^{c} \cup \{C_{2(c+\ell-b)+1,2b+1}\}.$$

Then, since $\partial^-(C_{2(c+\ell-b),2b+1}) = UC_{2(c+\ell-b)+1,2b+1}$, and

$$\partial^-(C_{2p,2c+2\ell+1-2p}) = U(C_{2p,2c+2\ell+2-2p} + C_{2p+1,2c+2\ell+1-2p})$$

for $c+\ell+1-b \leq p \leq c$, all of these terms generate the subgroup $\mathbb{F}$ in homology. Hence

$$H^-(D, c-b+2\ell) \cong \mathbb{F}^{b-\ell+1}. \quad \blacksquare$$

Using Theorem 6.6, one can extract numerical invariants that have recently been proven to be equivalent to the classical concordance invariants ν, τ and ϵ as defined in Chapter 3.

6.2.1 Numerical invariants

As introduced by Ozsváth-Szabó in [49, Sec. 1.1], the module $H(\mathcal{C}(D))$ — denoted by them $\mathcal{J}(D)$ — is a (Δ, A)-graded module over $\mathcal{R}'$ with multiplication actions by both U and V. These actions are inherited from the actions of U and V on the complex $\mathcal{C}(D)$, as defined in Definition 4.2.

Using [49, Prop. 1.4], which states that $H(\mathcal{C}(D))\otimes_{\mathbb{F}[U]}\mathbb{F}[U,U^{-1}] \cong \mathbb{F}[U,U^{-1}]$ as $\mathcal{R}'$-modules with multiplication by V acting as multiplication by 0 on $\mathbb{F}[U,U^{-1}]$, for a special knot diagram D one can define the numerical invariants $\underline{\tau}$ and $\underline{\nu}$. See [49, Def. 1.5,1.6].

Definition 6.7 *With the decomposition*

$$H(\mathcal{C}(D)) \cong \bigoplus_{s\in\mathbb{Z}} H(\mathcal{C}(D), s),$$
$$H^-(D) \cong \bigoplus_{s\in\mathbb{Z}} H^-(D, s),$$

where s is the Alexander grading, define the knot invariant $\underline{\tau}(D) \in \mathbb{Z}$ as

$$\underline{\tau}(D) = -\max_{s\in\mathbb{Z}} \left\{ \theta \in H^-(D,s) \;\middle|\; U^d \cdot \theta \neq 0 \;\; \forall d \in \mathbb{N} \right\}.$$

Similarly, define the invariant $\underline{\nu}(D) \in \mathbb{Z}$ as

$$\underline{\nu}(D) = -\max_{s\in\mathbb{Z}} \left\{ \theta \in H(D,s) \;\middle|\; U^d \cdot \theta \neq 0 \;\; \forall d \in \mathbb{N} \right\}.$$

With the recently proven equivalence equivalence between $\mathcal{C}(D)$ and $CFK_{\mathcal{R}'}(K)$, as demonstrated in [48], these invariants $\underline{\tau}$ and $\underline{\nu}$ are equal to their counterparts τ and ν from classical knot Floer homology.

Independent of this equivalence, in [49] it was demonstrated that these numerical invariants extracted from $\mathcal{C}(D)$ satisfy the same crossing change inequalities as their classical counterparts, as described on page 45.

Proposition 6.8 *[49, Prop. 1.7] For D_+ a special knot diagram with specified (oriented) positive crossing, define D_- as the special knot diagram with this crossing switched to a negative crossing. Then*

$$\underline{\tau}(D_+) - 1 \leq \underline{\tau}(D_-) \leq \underline{\tau}(D_+)$$
$$\underline{\nu}(D_+) - 1 \leq \underline{\nu}(D_-) \leq \underline{\nu}(D_+).$$

Furthermore, for any special knot diagram D,

$$\underline{\tau}(D) \leq \underline{\nu}(D).$$

These inequalities are proven in [49, Sec. 11.2]. In particular, the fact that $\underline{\tau}(D) \leq \underline{\nu}(D)$ follows from the fact that any U-nontorsion element in $H(\mathcal{C}(D))$ is also U-nontorsion in $H^-(D)$. In the language of [4], the U-nontorsion element in $H(\mathcal{C}(D))$ is the U-tower for the complex $\mathcal{C}(D)$. The fact that $\mathcal{C}(D)$ is a knot-like complex comes from the recently proven equivalence of $\mathcal{C}(D)$ with classical knot Floer homology, and the work of [4].

Theorem 6.6 then gives the determination of $\underline{\tau}(P(2c+1, -2b-1, 2a))$ with $\min\{a, c\} > b$.

Corollary 6.9 *For D a special knot diagram of the three strand pretzel knot $P(2c + 1, -2b-1, 2a)$ with $\min\{a, c\} > b$, the quantity $\underline{\tau}$ is given by*

$$\underline{\tau}(D) = c - b - 1.$$

Proof From the proof of Theorem 6.6, in Alexander grading $b-c+1$, the term $B_{2b+2,2b+3} + B_{2b+3,2b+2}$ generates a subgroup $\mathbb{F}[U]$ in $H^-(D)$ when $a > b+1$, else the term $B_{2b+3,2a}$ generates $\mathbb{F}[U]$ when $a = b+1$.

Hence, from Definition 6.7, as this is the only Alexander grading with a non-torsion element,

$$\underline{\tau}(D) = -(b - c + 1) = c - b - 1.$$

■

Note, it was determined in Proposition 3.18 that the classical τ invariant defined by Ozsváth-Szabó lies in the range

$$c - b - 1 \leq \tau(P(2c + 1, -2b - 1, 2a)) \leq g_4(P(2c + 1, -2b - 1, 2a)) \leq c - b.$$

Gratifyingly, using the equivalence between the classical and bordered invariants, Corollary 6.9 does not contradict this restriction. Furthermore, since $\nu(K) \in \{\tau(K), \tau(K) + 1\}$ for every oriented knot K, for these special knot diagrams one has $\underline{\nu}(D) \in \{c - b - 1, c - b\}$.

If $\underline{\nu}\,(P(2c + 1, -2b - 1, 2a)) = c - b - 1$, one would require that the maximal Alexander grading s with a U-nontorsion element in $H(\mathcal{C}(D, s))$ is $s = b - c + 1$.

Lemma 6.10 *For D a special knot diagram of $P(2c+1, -2b-1, 2a)$ with $\min\{a, c\} > b$, one has that*

$$\underline{\nu}(D) = c - b - 1$$

Proof Consider the element $B_{2b+3,2a}$. From Theorem 6.1, one has that $d(B_{2b+3,2a}) = 0$. In the study of ∂^- within Theorem 6.6, this appears in $im(\partial^-)$ as

$$\partial^-(B_{2b+2,2a}) = UB_{2b+3,2a}.$$

However, when $V \neq 0$, one has that

$$d(B_{2b+2,2a}) = UB_{2b+3,2a} + VB_{2b+1,2a-2}.$$

One can check that no other term can cancel this $VB_{2b+1,2a-2}$ term in the sum – i.e. $VB_{2b+1,2a-2} \notin im(d)$. Hence, $UB_{2b+3,2a} \notin im(d)$, and so $B_{2b+3,2a}$ is a U-nontorsion state because $U^d \cdot B_{2b+3,2a} \neq 0$ in $H(\mathcal{C}(D))$ for every $d \in \mathbb{N}$.

The state $B_{2b+3,2a}$ has Alexander grading $b - c + 1$. If there were some terms with strictly larger Alexander grading that are also U-nontorsion, one would thus have that

$$\underline{\nu}(D) < -(b - c + 1) = \underline{\tau}(D).$$

However, this cannot be true, as Proposition 6.8 states that $\underline{\tau}(D) \leq \underline{\nu}(D)$. Hence, this must be the maximal Alexander grading with such a U-nontorsion element, and so $\underline{\nu}(D) = c - b - 1$. ■

6.2.2 Other invariants of $\mathcal{R}'$-modules

The information within $H(\mathcal{C}(D))$ and $H^-(D)$ can be used to place bounds upon other numerical invariants, such as the concordance invariants $\{\varphi_j\}_{j\in\mathbb{N}}$ introduced by Dai et al in [4].

As proven in [4, Thm. 1.1], for each value $j \in \mathbb{N}$, one can use the methods outlined by Dai-et-al in [4] to construct a surjective homomorphism from $\mathcal{C}$ (the concordance group of knots) to $\mathbb{Z}$. These homomorphisms φ_j are in fact defined as invariants of local equivalence classes of bigraded chain complexes over $\mathcal{R}'$. By [4, Thm. 2.5] and [62], two concordant knots K_1 and K_2 have locally equivalent complexes $CFK_*(K_1)$ and $CFK_*(K_2)$, where $* \in \{\mathcal{R}', \mathcal{R}\}$. These complexes are as defined in Section 1.3.

Hence, even without using the recently proven equivalence between $CFK_{\mathcal{R}'}(D)$ and $\mathcal{C}(D)$, for a special knot diagram D one can determine numerical invariants φ_j associated to local equivalence classes of complexes $\mathcal{C}(D)$.

For the sake of brevity, a full definition of these invariants φ_j will be omitted, however what follows is a rough outline of their construction from [4].

As outlined in Section 1.4.1, every knot-like complex over $\mathcal{R}'$ is locally equivalent to a reduced knot-like complex. Recall, this is a complex such that the differential acting upon any element of the complex is strictly increasing in either the power of U, power of V, or both. The invariants $\{\varphi_j\}$ are defined from *standard complexes*, a subset of reduced, knot-like complexes.

Definition 6.11 *[4, Def. 4.3] For n an even natural number, let $(b_1, b_2, \cdots, b_n)$ be a sequence of non-zero integers. A standard complex of type $(b_1, \cdots, b_n)$, denoted $C(b_1, \cdots, b_n)$, is the knot-like complex freely generated over $\mathcal{R}'$ by $\{x_0, x_1, \cdots, x_n\}$, such that:*

- *Every pair of generators x_{2k}, x_{2k+1} are connected by a $U^{|b_{2k+1}|}$-arrow. If b_{2k+1} is positive, then the arrow goes from x_{2k+1} to x_{2k}, and the reverse otherwise.*
- *Every pair of generators x_{2k+1}, x_{2k+2} are connected by a $V^{|2k+2|}$-arrow.*

A U^ℓ-arrow (respectively V^ℓ-arrow) is a map that when applied to some generator x yields generator $U^\ell y$ (respectively $V^\ell y$). In the diagrammatic representation of the complex, following Figure 1.5, these would be horizontal (respectively vertical) arrows of length ℓ.

The above should be sufficient to define the complex $C(b_1, b_2, \cdots, b_{2k})$ and the associated differential, however full detail is presented in [4]. The state x_0 is the V-tower, and the state x_n is the U-tower, as defined by Definition 1.15. Hence, $gr_U(x_0) = 0$, and $gr_V(x_n) = 0$, since this is a knot-like complex. From the following result of [4], every knot-like complex is locally equivalent to some standard complex.

Theorem 6.12 *[4, Thm 6.1] Every knot-like complex $\mathcal{C}$ is locally equivalent to a standard complex.*

Idea of proof Using the fact that one can place a total order upon the set of standard complexes, for any knot-like complex C one can define integers $\{a_i(C)\}_{i\in\mathbb{N}}$ such that there is a standard complex with standard sequence $\{a_i\}_{i\in\mathbb{N}}$ that is locally equivalent to C. Furthermore, it is proven in [4, Prop. 6.3] that for every knot-like complex there is some

$N \in \mathbb{N}$ such that $a_i(C) = 0$ for $i > N$. These invariants a_i are defined in [4, Sec. 5].

Consequently, for each knot-like complex C there is some standard complex $C(a_1, \cdots, a_{2k})$ to which it is locally equivalent to it. The invariants φ_j of any knot-like complex are defined as follows.

Definition 6.13 *[4, Def. 7.1] For C a knot-like complex, let the standard complex that is locally equivalent to this be denoted $C(a_1, a_2, \cdots a_{2k})$. Then, for each $j \in \mathbb{N}$, define*

$$\varphi_j(C) = \#\{a_i | a_i = j, i \text{ odd}\} - \#\{a_i | a_i = -j, i \text{ odd}\}.$$

The invariants φ_j are thus a signed count of the number of horizontal arrows of length j in the standard complex that is locally equivalent to then knot-like complex.

These invariants $\{\varphi_j\}$ are recoverable from any knot-like complex, and only use the purely vertical and purely horizontal information in $CFK_{\mathcal{R}}(K)$. Properties of these invariants are presented in [4]; by construction they provide homomorphisms from the concordance group to $\mathbb{Z}$ for each $j \in \mathbb{N}$. Furthermore, they are linearly independent, and can be used to bound the concordance genus and concordance unknotting number of knots, see [4, Sec. 1.3].

But, for three-strand pretzel knots, the local equivalence class of the bigraded invariant $\mathcal{C}(D)$ can be used to show that the invariants $\varphi_j(\mathcal{C}(D))$ are only possibly non-zero for one value of j.

Lemma 6.14 *For D a special knot diagram of a three-strand pretzel knot $P(2c+1, -2b-1, 2a)$, and $\mathcal{C}(D)$ the bigraded chain complex defined by [49], then*

$$\varphi_j(\mathcal{C}(D)) = 0 \qquad j > 1.$$

Proof Excepting the case where $\min\{a, c\} > b$, the homology theory $\widehat{H}(D)$ of a special knot diagram D isotopic to $P(2c+1, -2b-1, 2a)$ is contained in a single diagonal $\Delta = M - A$, where M and A are the Alexander gradings. Hence, the knots are homologically thin. Following [4, Prop. 1.4], the invariants $\varphi_j(K)$ for homologically thin knots K are determined by their τ-invariants (equivalently their $\underline{\tau}$-invariants). Namely:

$$\varphi_j(K) = \begin{cases} \tau(K) & \text{if } j = 1 \\ 0 & \text{otherwise.} \end{cases}$$

The fact that $CFK_{\mathcal{R}'}(D)$ is a knot-like complex is demonstrated in [4, Sec. 2]. The complex $CFK_{\mathcal{R}'}(D)$ has a single V-tower with Maslov grading (equivalent to the gr_U-

grading) equal to 0. This is the generator of $H(C\{i=0\})$ as introduced in Definition 3.2. From the symmetry of knot Floer homology, the V-tower in $CFK_{\mathcal{R}'}(\overline{D})$ corresponds under the symmetry of the complex $CFK_{\mathcal{R}'}(D)$ and $CFK_{\mathcal{R}'}(\overline{D})$ that interchanges the values of gr_U and gr_V to the U-tower of $CFK_{\mathcal{R}'}(D)$.

Consequently, using the recent equivalence between $\mathcal{C}(D)$ and $CFK_{\mathcal{R}'}(D)$, $\mathcal{C}(D)$ is a knot-like complex. Hence, one can determine the invariants φ_j for the complexes defined using the method of [49].

Denote by $Tors_U(H^-(D))$ the U-torsion submodule of the $\mathbb{F}[U]$-module $H^-(D)$. From Theorem 6.6, this torsion submodule is generated by all elements that are not the U-tower (the generator of the $\mathbb{F}[U]$-term).

Hence, since all other terms in the $H^-(D)$ are $\mathbb{F}$, one has that $U^j \cdot Tors_U(H^-(D)) = 0$ for all $j \geq 1$. Employing [4, Prop. 1.15], $\varphi_j(D) = 0$ for all $j > 1$. ■

From [4, Prop. 1.2], the classical concordance invariant τ can be determined from the family of invariants $\{\varphi_j(K)\}$. Namely, one has that

$$\tau(K) = \sum_{j \in \mathbb{N}} j \cdot \varphi_j(K).$$

This provides the following easy corollary, determining $\varphi_1(K)$ in all cases.

Corollary 6.15 *For any three-strand pretzel knot $K = P(2c+1, -2b-1, 2a)$, one has that $\varphi_1(K) = \tau(K)$.*

Proof Combining the calculation from Lemma 6.14, and [4, Prop. 1.2], one has that

$$\sum_{j \in \mathbb{N}} j \cdot \varphi_j(K) = \varphi_1(K) = \tau(K). \quad ■$$

6.2.3 Numerical invariants as concordance invariants

The classical invariants ν and τ extracted from $CFK^\infty(K)$ or appropriate sub- or quotient complexes of this are concordance invariants, as discussed in Chapter 3. However, without using the equivalence between the theories, the invariants $\underline{\nu}$ and $\underline{\tau}$ are only numerical invariants of the complex $\mathcal{C}(D)$.

More strictly, as described in [49, Sec. 11.2], if C^1 and C^2 are quasi-isomorphic as bigraded chain complexes over $\mathcal{R}'$, then $\underline{\nu}(C^1) = \underline{\nu}(C^2)$ and $\underline{\tau}(C^1) = \underline{\tau}(C^2)$. This would make $\underline{\nu}$ and $\underline{\tau}$ invariants of the quasi-isomorphism class of $H_*(\mathcal{C}(D))$. But, since the quasi-isomorphism

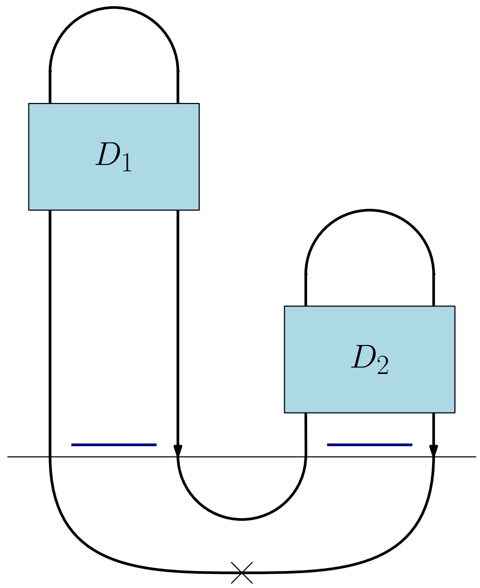


Figure 6.2: *Above the horizontal line, one has the special knot diagram associated to the disjoint uinion of $\overline{D_1} \sqcup \overline{D_2}$. The idempotent of all upper Kauffman states for this upper knot diagram is I_{13}, displayed in blue. Below the horizontal line, one has the two minima that form the connect sum $D_1\#D_2$.*

class of $H_*(\mathcal{C}(D))$ is a knot invariant [49, Thm. 1.1], so are $\underline{\nu}$ and $\underline{\tau}$. Yet without using the recently proven equivalence between the theories, $\underline{\tau}$ and $\underline{\nu}$ are invariants associated to knots, not concordance classes.

In [49, Sec. 9.1], Ozsváth-Szabó examined $\mathcal{C}(D_1\#D_2)$, where D_1 is an oriented special knot diagram of knot K_1, and D_2 an oriented special knot diagram of K_2. Let $\overline{D_1}$ and $\overline{D_2}$ be the upper knot diagrams for D_1 and D_2 that result from excising their global minima. Then, because one can place all maxima, minima and crossings of $\overline{D_2}$ at y-values below those of $\overline{D_1}$, the generators of the Type D structure associated to $\overline{D_1} \sqcup \overline{D_2}$ are tensor products of upper Kauffman states of $\overline{D_1}$ and $\overline{D_2}$. This is demonstrated in Figure 6.2.

Due to the fact that the upper Kauffman states of the upper knot diagrams $\overline{D_1}$ and $\overline{D_2}$ have associated idempotents I_1 and I_3, as only the global minimum has been excised, the upper Kauffman states corresponding to generators of $\overline{D_1} \sqcup \overline{D_2}$ have associated idempotent I_{13}. More specifically, the upper Kauffman states of $\overline{D_1} \sqcup \overline{D_2}$ are

$$\sum_{X_i \text{ K.S. for } \overline{D_i}} I_1 \cdot X_1 \otimes I_3 \cdot X_2.$$

Proposition 6.16 *For D_1 and D_2 special knot diagrams, with global minima oriented* *right to left, the chain complex* $\mathcal{C}(D_1 \# D_2)$ *satisfies the Künneth relation, that is*

$$\mathcal{C}(D_1 \# D_2) \cong \mathcal{C}(D_1) \otimes \mathcal{C}(D_2).$$

Proof To yield $D_1 \# D_2$ from $\overline{D_1} \sqcup \overline{D_2}$, as pictured in Figure 6.2, one attaches the lower knot diagram corresponding to $t\mho^1 \boxtimes \widetilde{\mho}^2$. The DA-bimodule $\widetilde{\mho}^2$ is as defined in Definition 4.45, with a single generator $I_1 \cdot Q \cdot I_{13}$.

Since by construction the Kauffman states for the D_1 and D_2 components do not include 2 in their idempotents, the non-zero maps in $\widetilde{\mho}^2$ are:

$$\begin{aligned}
\delta_2^1(Q, U_1^k) &= U_1^k \otimes Q \\
\delta_2^1(Q, U_4^k) &= U_2^k \otimes Q \\
\delta_{1+1+k}^1(Q, U_2^k, C_{34}^{\otimes k}) &= U_2^k \otimes Q \\
\delta_{1+1+k}^1(Q, U_3^k, C_{12}^{\otimes k}) &= U_1^k \otimes Q \\
\delta_3^1(Q, C_{12}, C_{34}) &= C_{12} \otimes Q.
\end{aligned}$$

Pairing this with the A_∞-module $Y'_{\mathcal{A}(1)}$ as defined in Definition 4.43, the A_∞-module corresponding to the minima in Figure 6.2 is $Y' \boxtimes \widetilde{\mho}^2_{\mathcal{A}(2)}$, with a single generator $H \cdot I_{13}$, and maps

$$m_{1+j} : \left(Y' \boxtimes \widetilde{\mho}^2\right) \otimes \mathcal{A}(2)^{\otimes j} \to Y' \boxtimes \widetilde{\mho}^2,$$

defined by

$$\begin{aligned}
m_2(H, U_1^k) &= U^k \otimes H \\
m_2(H, U_4^k) &= V^k \otimes H \\
m_{1+1+k}(H, U_2^k, C_{34}^{\otimes k}) &= V^k \otimes H \\
m_{1+1+k}(H, U_3^k, C_{12}^{\otimes k}) &= U^k \otimes H.
\end{aligned}$$

Roughly, this means that elements U_1 and U_3 correspond with U in $\mathcal{R}'$, and the elements U_2 and U_4 with V in $\mathcal{R}'$.

Abusing notation slightly, and letting $\overline{D_i}$ denote the Type D structure associated with the upper knot diagram $\overline{D_i}$, in $\overline{D_1}$ one would have that U_1 would pair with $m_2(H \cdot I_1, U_1) = U \otimes H$ in Y', and U_2 with $m_2(H \cdot I_1, U_2) = V \otimes H$. Similarly for U_1 and U_2 in $\overline{D_2}$. In the disjoint union $\overline{D_1} \sqcup \overline{D_2}$, the term U_1 in $\overline{D_2}$ would correspond to U_3 in $\overline{D_1} \sqcup \overline{D_2}$, and U_2 with U_4.

If in $\overline{D_1}$ and $\overline{D_2}$ there are elements x^1 and x^2 respectively, such that

$$d(x^1) = \sum a_i \otimes y_i^1 \qquad d(x^2) = \sum b_i \otimes y_i^2$$

for $a_i, b_i \in \{U_1^k, U_2^\ell\}$, then in $\overline{D_1} \sqcup \overline{D_2}$ one thus has that

$$d(I_{13} \cdot (x^1 \otimes x^2)) = a_i \otimes (y_i^1 \otimes x^2) + \phi(b_i) \otimes (x^1 \otimes y_i^2),$$

where the map ϕ is such that $\phi(U_j) = U_{j+2}$. This is perhaps easiest to visualise using the interpretation of these maps as corresponding to partial domains between intersection points in upper Heegaard diagrams. Because the Maslov and Alexander gradings are determined by local contributions, and the two components are disconnected, partial domains between two upper Kauffman states with Maslov grading differing by one are either contained solely in the upper Heegaard diagram for $\overline{D_1}$ or the upper Heegaard diagram for $\overline{D_2}$.

Hence, pairing this with the A_∞-module defined above, and using the familiar box-tensor product between Type D structures and A_∞-modules (see Section 4.5.2) one has that

$$\partial_{D_1\#D_2}(H \cdot I_{13} \otimes (x^1 \otimes x^2)) = \partial_{D_1}(H \cdot I_1 \otimes x^1) \otimes x^2 + x^1 \otimes \partial_{D_2}(H \cdot I_1 \otimes x^2).$$

This is because a_i and $\phi(b_i)$ (together with appropriate matching terms) yield U and V terms in the chain complex as in the disjoint union. This shows the differential acts as it would under the tensor product of two chain complexes, as required. ■

Corollary 6.17 *The knot invariants $\underline{\nu}$ and $\underline{\tau}$ are additive under the connect sum operation.*

Proof Since the Alexander grading is determined by the local contributions at Kauffman states at each crossing, and that any element in the kernel of $\partial_{D_1\#D_2}$ (or quotient with $V = 0$) needs to have tensor-coordinate components that are in the kernel of ∂_{D_1} and ∂_{D_2}, then the non-torsion element in $H(\mathcal{C}(D_1\#D_2))$ has to be the tensor product of the non-torsion elements in D_1 and D_2. The local contributions of the grading then imply that the Alexander gradings of each tensor coordinate are summed together, so making the invariants additive. ■

6.3 Further directions for study

As remarked within this thesis, the divide-and-conquer construction of $\mathcal{C}(D)$ is a useful tool for providing a combinatorial method for the construction of a bigraded complex equivalent to $CFK_{\mathcal{R}'}(D)$. From $\mathcal{C}(D)$, one can then extract homology theories equivalent to $\widehat{HFK}(D)$ and $HFK^-(D)$, and concordance invariants equivalent to τ and ν. As evinced by the speed of the C++ program [47] published by Ozsváth-Szabó, this combinatorial method of determining a complex equivalent to knot Floer homology $\widehat{HFK}(K)$ has the advantage of using fewer generators than the grid homology of [30, 35], allowing the swift determination of some concordance invariants for specific examples of knots.

Although this thesis determines the invariant $\mathcal{C}(D)$ for one family of pretzel knots, one could extend this construction to pretzel knots with four, five, or more strands. As well as having well-structured Kauffman states — which are in bijection with the generators of $\mathcal{C}(D)$ — all pretzel knots admit special knot diagrams with width three: that is they admit isotopic special knot diagrams with at most six intersection points between some generic line $y = \ell$ and the special knot diagram.

For a knot with width n, the differential graded algebra $\mathcal{A}(n)$ as defined in Definition 4.12 is associated to the horizontal level intersecting the knot at $2n$ points. Hence, at the widest point of the special knot diagram, the algebra used is simpler: i.e. has fewer permitted idempotents than knots with greater width. For this reason, three-strand pretzel knots (and by extension, all pretzel knots) are particularly amenable to study using this combinatorial construction, since the algebras and number of possible idempotents does not grow too computationally complicated. One can verify that specific examples of pretzel knots D have homology theories $\widehat{H}(D)$ that can be determined quickly by the program [47], even for those pretzel knots with high numbers of strands and crossings.

It is however slightly beyond the scope of this thesis to study families of pretzel knots with more than three-strands. This is because the proofs within this chapter and within Chapter 5 rely upon good knowledge of the Kauffman states in order to use induction. This is aided by the simplicity of diagrams representing Type D structures, yet because pretzel knots with more strands have more complicated Kauffman states, determining the structure and using induction becomes more difficult. However, with patience, the author does believe that one could determine $\mathcal{C}(D)$ for D a representative of a family of pretzel knots with more than three strands. It would be particularly advantageous for such constructions to determine explicitly the DA-bimodules associated to any number of half twists, so for any k and n the bimodules

$$\left(\mathcal{P}^k\right)^{\boxtimes n} \text{ and } \left(\mathcal{N}^k\right)^{\boxtimes n}.$$

6.3.1 Concordance invariants and slice-genus of pretzel knots

This thesis determines the algebraic invariant of Ozsváth-Szabó for an infinite family of knots — the three-strand pretzel knots $P(2a,-2b-1,2c+1)$. Since $\mathcal{C}(D)$ is equivalent to the bigraded complex $CFK_{\mathcal{R}'}(D)$ defined in [4], and within [4] the authors extract an infinite family of concordance invariants from the complex, one could then hope that the determination of these concordance invariants may answer open questions as to the slice-genus of representatives of this family.

In [20], Lecuona studies obstructions to sliceness for many examples of three-strand pretzel knots. The techniques used include examining the Alexander polynomial and Casson-Gordon invariants of families of pretzel knots whose slice genus is not known, and examining double-branched covers of the knots. The Fox-Milnor theorem [6, Thm. 2], states that if a knot K is slice, then its Alexander polynomial is of the form

$$\Delta_K(t) = f(t)\cdot f(t^{-1}),$$

where f is a polynomial with integer coefficients. In [20, Thm. 4.5], Lecuona uses the Fox-Milnor theorem to obstruct the sliceness of many infinite families of pretzel knot, by demonstrating that their Alexander polynomial does not have the required form.

Three strand pretzel knots of the form

$$P\left(a, -a-2, \frac{-(a+1)^2}{2}\right),$$

with $a \equiv 1, 11, 37, 47, 49, 59 \mod (60)$, are the only family of three-strand pretzel knots whose slice genus is not yet determined using the methods of [20] or otherwise. However, in [20, Conj. 1.3] Lecuona conjectures that this family is not slice. Although this family of pretzel knots is of the form considered in this thesis — i.e. $P(2c+1,-2b-1,2a')$ for $a', b, c \in \mathbb{N}$ and $c = b+1$, for $a \geq 3$ — this is a knot where $a > b$ and $c > b$. Hence, applying Corollary 6.9 and Lemma 6.10, one has that

$$\underline{\tau}\left(P(a,-a-2,\frac{-(a+1)^2}{2})\right) = \underline{\nu}\left(P(a,-a-2,\frac{-(a+1)^2}{2})\right) = 0.$$

So, these two invariants do not obstruct sliceness in the cases where the slice-genus is not known.

The infinite family of concordance invariants $\{\varphi_j\}_{j\in\mathbb{N}}$ also do not obstruct being slice in this case. From [4, Thm. 1.1], for each $j \in \mathbb{N}$, $\varphi_j : \mathcal{C} \to \mathbb{Z}$ is a surjective homomorphism. However, from Lemma 6.14 and Corollary 6.15, one has that $\varphi_j\left(P\left(a, -a-2, \frac{-(a+1)^2}{2}\right)\right) = 0$ for all $j \in \mathbb{N}$.

But, Theorem 6.1 determines $\mathcal{C}(P(2c+1, -2b-1, 2a))$ for any $a, b, c \in \mathbb{N}$, not simply the homology theory $HFK^-(P(2c+1, -2b-1, 2a))$ or $\widehat{HFK}(P(2c+1, -2b-1, 2a))$. Although it is beyond the scope of this thesis, it is hoped that the information within this bigraded complex could be used to try to obstruct sliceness for the remaining family of knots given by Lecuona, as they are conjectured to have slice genus equal to one.

6.3.2 Information about domains in Heegaard diagrams

Using the recently proven equivalence between algebraic invariant $\mathcal{C}(D)$ and classical knot Floer homology, it would be interesting to consider whether one could recover information about domains within a Heegaard diagram from the knowledge of counts provided by the differential within $\mathcal{C}(D)$.

One of the difficulties in computing the knot Floer homology of pretzel knots directly from the Heeagaard diagrams produced from thickened up knot projections is that domains can arise whose counts are not known. In Eftekhary's examination of the hat version of knot Floer homology for pretzel knots, [5], Eftekhary examines domains that have a known count — see [5, Fig. 7]. These domains are known as arborescent punctured polygons, introduced by Greene in [9] following the work of Ozsváth-Szabó in [34].

Arborescent punctured polygons, as defined in [9, Def. 6.5], are an extension of the 'disky differentials' as considered in [53]. That is, all disky differentials are arborescent punctured polygons, but not all arborescent punctured polygons are disky differentials. In particular, punctured polygons admit boundary components that are solely α or β curves, in addition to the polygonal boundary composed of alternating α and β curves with only internal corners.

Without loss of generality, let the internal boundary components in a punctured polygon be β-curves. These β-curves can be connected to each other, or the boundary, by α-curves that have degenerate corners on the β-curve boundary components. A punctured polygon domain D is then arborescent if the complement of these curves in D is connected. Greene then proved in [9, Lem. 6.6] that if D is an arborescent punctured polygon that is a domain representing a Whitney disk $\phi \in \pi_2(x, y)$, then $\mu(\phi) = 1$ and $\#\widehat{\mathcal{M}}(\phi) = \pm 1$.

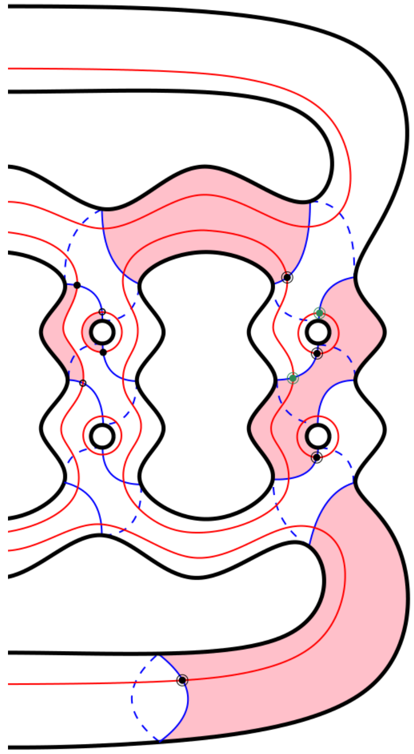


Figure 6.3: *A domain between intersection points on a subsection of a Heegaard diagram associated to a three-strand pretzel knot. Using the black states on the topmost two crossings on the right hand strand, the domain pictured is an arborescent punctured polygon. However, uing the green states at these two crossings yields a non-arborescent punctured polygon, but with the same interior.*

Examining the domains between generators in said Heegaard diagrams for three-strand pretzel knots, one can find examples of domains that are non-arborescent punctured polygons, but with other domains that have the same interior (but different corners) and are arborescent punctured polygons. An example is provided in Figure 6.3. Although one cannot apply [9, Lem. 6.6] to determine the count of pseudo-holomorphic representatives in the case of non-arborescent punctured polygons, one might be able to use the known differentials within $\mathcal{C}(D)$ to determine information about the counts of the corresponding Whitney disks. This utilises the equivalence between $\mathcal{C}(D)$ and $CFK_{\mathcal{R}'}(D)$ recently proven in [48], and the correspondence between differentials in $\mathcal{C}(D)$ and domains within Heegaard diagrams. The author hypothesises that the pseudo-holomorphic counts for Whitney disks admitting non-arborescent punctured polygons as featured in Figure 6.3 would match the counts on the differentials within $\mathcal{C}(D)$, and this would be an interesting direction for future work.

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