Chapter 2

Extreme values

As an application of partial derivatives, we consider the problem of finding extreme, either minimum or maximum, values of functions of two or more variables.

2.1 Maxima and minima for functions of two variables

Definition 2.1 If

$$f(x,y) - f(a,b) > 0,$$

for all $(x,y) \neq (a,b)$ in the domain of f then we say that f has a *global minimum* at (a,b). If this inequality holds for $(x,y) \neq (a,b)$ *sufficiently close to* (a,b) then we say that f has a *local minimum* (or simply a *minimum*) at (a,b).

Similarly, if

$$f(x,y) - f(a,b) < 0,$$

for all $(x,y) \neq (a,b)$ in the domain of f then we say that f has a *global maximum* at (a,b) and if this holds for $(x,y) \neq (a,b)$ *sufficiently close to* (a,b) then we say that f has a *local maximum* (or simply a *maximum*) at (a,b).

A maximum or a minimum value is called an *extremum*. The word *extrema* is the plural of *extremum*.

Example 2.1 Show that $(0,0)$ is a global maximum point for the function $f(x,y) = 1 - x^2 - y^2$ and find this maximum value.

Solution :

Answer: f has a global maximum at $(0,0)$. The maximum value is $f(0,0) = 1$. □

Critical points (Stationary points)

Definition 2.2 Suppose that the partial derivatives f_x and f_y exist at (a, b). If $f_x = f_y = 0$ at (a, b) then it is said to be a *critical point (stationary point).*

As for functions of one variable, there is a close connection between (local) extrema and critical points.

Theorem *Let f have an extremum at (a, b). Then (a, b) is a critical point.*

Remark This states that every local extreme point is be a critical point. However, a critical point is *not necessarily* an extreme point.

Example 2.2 Show that $(0, 0)$ is a critical point for each of the functions

$$\text{(a) } f(x, y) = x^2 + y^2, \quad \text{(b) } g(x, y) = x^2 - y^2.$$

In each determine whether or not $(0, 0)$ is an extreme point.

Solution :

□

Remark In general, global extrema may be located at critical points, points on the boundary of the domain or points where derivatives are not defined. For example, consider the cone

$$z = 1 - \sqrt{x^2 + y^2}, \quad \text{where } x^2 + y^2 \leq 1,$$

shown in Figure **??**. Clearly, the global maximum is at $(0, 0)$, and has value 1, and every point on the circle $x^2 + y^2 = 1$ is a global minimum point giving value 0. Since

$$\frac{\partial z}{\partial x} = \frac{-x}{\sqrt{x^2 + y^2}}, \quad \frac{\partial z}{\partial y} = \frac{-y}{\sqrt{x^2 + y^2}},$$

are not defined at $(0, 0)$, there are no critical points.

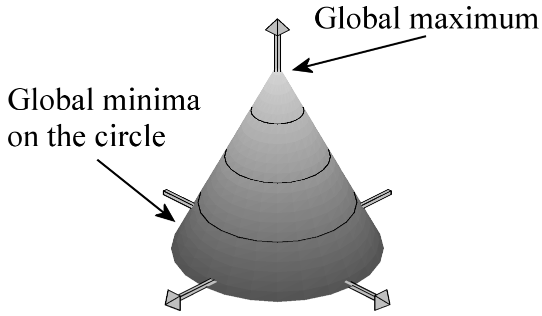

Figure 2.1: Global extrema on a cone

Definition 2.3 A critical point which is not an extreme point is called a *saddle point*.

Typical examples of critical points, showing both the surfaces and their level curves are shown in Figure **??**.

Classifying critical points

Let (a, b) be a critical point of f. By considering the difference

$$\Delta(h, k) = f(a + h, b + k) - f(a, b),$$

for h, k sufficiently small but not both zero, we can determine the nature of this critical point. If

$$\Delta(h,k) = \begin{cases} > 0 & (a,b) \text{ is a local minimum,} \\ < 0 & (a,b) \text{ is a local maximum,} \\ \text{otherwise} & (a,b) \text{ is a saddle point.} \end{cases}$$

Remark Using this criterion directly can be difficult (see later for an example) but fortunately the next theorem gives a simple test to determine the nature of a critical point. However, it is not always applicable. In the cases where it cannot be applied, a first principles method using this criterion must be used instead.

Theorem (Second Derivative Test) *Let (a, b) be a critical point of f and let $A = f_{xx}$ and $D = f_{xx}f_{yy} - f_{xy}^2$, where all derivatives are evaluated at (a, b). Then*

1. *If $A > 0$ and $D > 0$ then (a, b) is a minimum point,*
2. *If $A < 0$ and $D > 0$ then (a, b) is a maximum point,*
3. *If $D < 0$ then (a, b) is a saddle point.*

If $D = 0$ then no conclusion about the nature of (a, b) is made.

The key idea in the proof of this theorem is to use Taylor's theorem to approximate the difference $\Delta(h, k)$ as a quadratic.

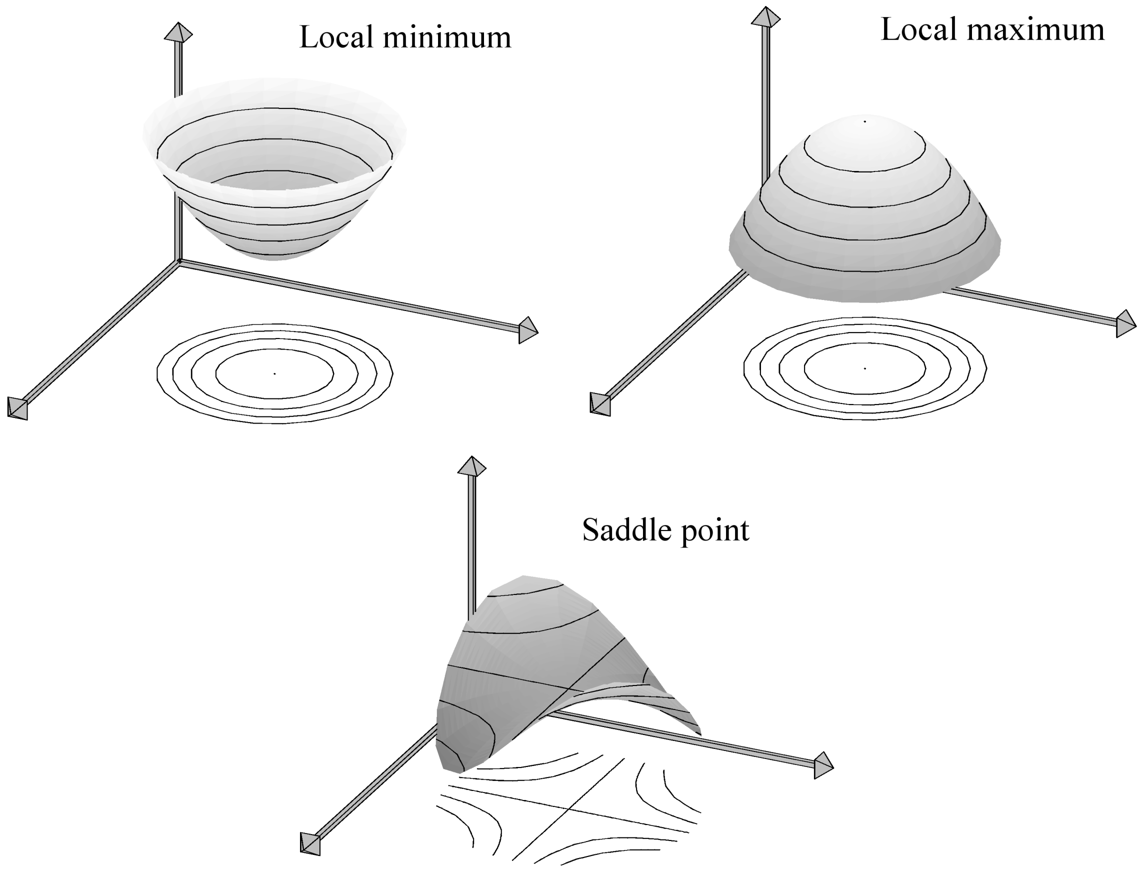

Figure 2.2: Minimum point, maximum point and saddle point

Remark D is the determinant of the Hessian Matrix H, where $H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$, the matrix of second derivatives of f.

Example 2.3 Find, and classify, the stationary points of

$$\text{(a) } u(x,y) = 3x^2 + 6xy - y^3, \quad \text{(b) } v(x,y) = x^2 + 2x + y^4, \quad \text{(c) } w(x,y) = 3x^2 + 3xy^2 + y^3.$$

Solution : We will use the second derivative test whenever possible, and otherwise use the first principles method.

□

Remark In some cases the "sufficiently close" part of the first principles definition is vital. As an example of this most difficult type of first principles argument, suppose that

$$\Delta(h, k) = h^2 + k^2 - hk^2.$$

At first sight, one would suspect that this could be positive and negative for different values of (h, k) and so that the critical point under consideration is a saddle point. However, by writing it as

$$\Delta(h, k) = h^2 + k^2(1 - h),$$

we see that it is positive for all $(h, k) \neq (0, 0)$ for $h < 1$. That is, it is positive for all (h, k) *sufficiently close* to $(0, 0)$ and so the critical point is a minimum point.

Example 2.4 Find, and classify, the stationary points of $f(x, y) = 4x^2y - y^2 - 8x^2 - 2x^4$.

Solution :

Answer: $(0, 0)$ is a maximum. $(\pm\sqrt{2}, 4)$ are both saddle points. □

Example 2.5 Find the shortest distance of the point $(3, 4, 0)$ to the cone $z = \sqrt{x^2 + y^2}$.

Solution : Answer: The required shortest distance is therefore

$$\sqrt{f(\tfrac{3}{2},2)} = \sqrt{(\tfrac{3}{2}-3)^2+(2-4)^2+(\tfrac{3}{2})^2+2^2} = \sqrt{\frac{25}{2}} = \frac{5\sqrt{2}}{2}.$$

□

2.2 Constrained extrema and Lagrange multipliers

Problems involving finding extreme values often also include *constraints*. For example,

- find the minimum surface area of a cuboid *having a particular volume*,
- find the minimum distance from the origin of a point *lying on a given curve*.

Suppose, for example, that we wish to find the maximum height on a footpath as it crosses a mountain. The height is given by $z = f(x,y)$ and the path by $g(x,y) = 0$. The contours and the constraint curve for a typical situation are illustrated in Figure **??**.

To solve the problem we must to find the highest contour ($f(x,y) = C$ for the largest C) and point (a,b) lying on this contour and the path.

We see from Figure **??** that these conditions are satisfied when the contour $f(x,y) = C$ and $g(x,y) = 0$ *touch* at (a,b), so that they have a common normal.

Theorem *Let (a,b) be an extremum for $f(x,y)$ subject to the constraint $g(x,y) = 0$. Then, (provided g_x and g_y are not both zero at (a,b)),*

$$f_x(a,b) + \lambda g_x(a,b) = 0,$$
$$f_y(a,b) + \lambda g_y(a,b) = 0,$$

for some constant λ. The constant λ is called a Lagrange multiplier*.*

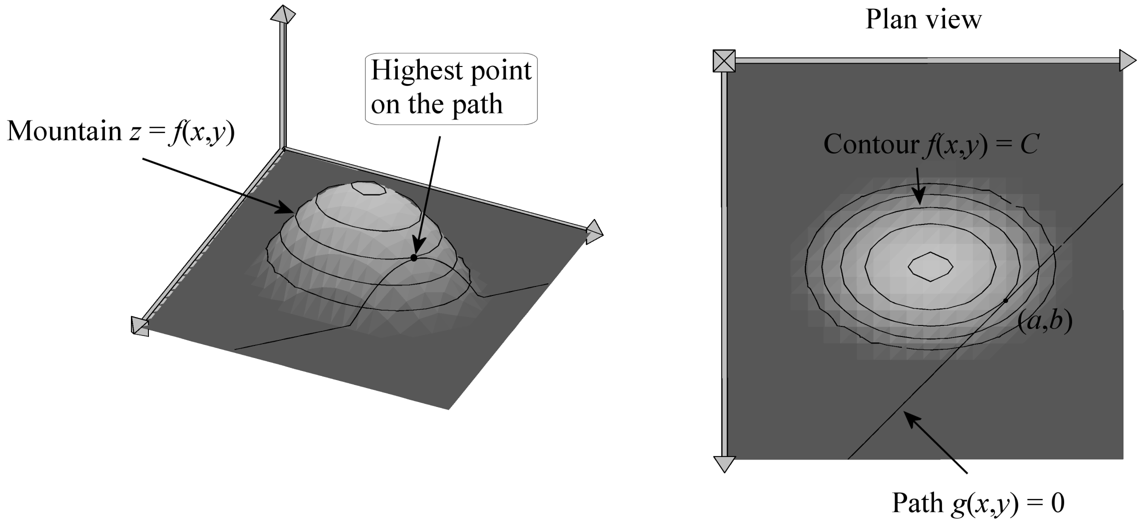

Figure 2.3: Contours and constraint curve

Remarks

1. The theorem tells us that if we know that a solution to a constrained problem exists, it may be found by finding all points (x, y) where

$$\begin{aligned} f_x + \lambda g_x &= 0, \\ f_y + \lambda g_y &= 0, \\ g &= 0. \end{aligned}$$

   Note that there are 3 equations in 3 unknowns (x, y and λ). The theorem *does not* prove that such a solution exists; this must be done independently. Often proving this existence use techniques beyond the scope of this course and existence will either be assumed or justified by heuristic arguments.

2. The theorem extends in a natural way to a function of n variables $f(x_1, \ldots, x_n)$ with m constraints $g_1(x_1, \ldots, x_n) = 0$, $\ldots$,$g_m(x_1, \ldots, x_n) = 0$. Here an extreme point satisfies

$$\begin{aligned} \nabla f + \lambda_1 \nabla g_1 + \cdots + \lambda_m \nabla g_m &= 0, \\ g_1(x_1, \ldots, x_n) &= 0, \\ &\vdots \\ g_m(x_1, \ldots, x_n) &= 0, \end{aligned}$$

   where $\nabla f = (f_{x_1}, \ldots, f_{x_n})$ and $\lambda_1, \ldots, \lambda_m$ are m Lagrange multipliers ($m + n$ equations in $m + n$ unknowns). Example **??** uses this result.

Example 2.6 Find the minimum distance of the curve $xy = 1$ from the origin.

Solution : The distance of a general point (x, y) to the origin is $\sqrt{x^2 + y^2}$. For simplicity, we consider the square of the distance and so we must minimise $f(x, y) = x^2+y^2$ subject to the constraint $g(x, y) = xy-1 = 0$.

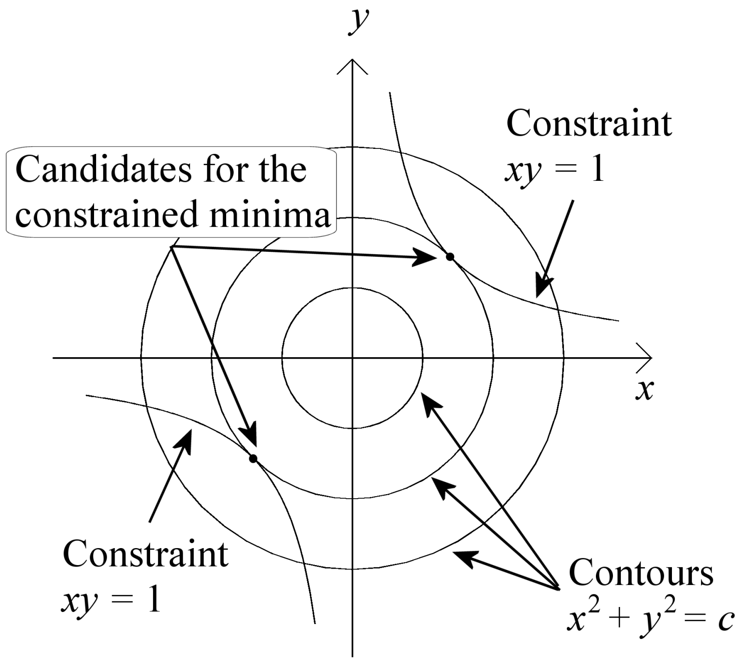

Figure 2.4: Candidates for the constrained minima

Answer: The minimum distance is $\sqrt{2}$. □

Example 2.7 Find the maximum area of an isosceles triangle with perimeter of length 2. What are the lengths of the sides which give this maximum?

Solution :

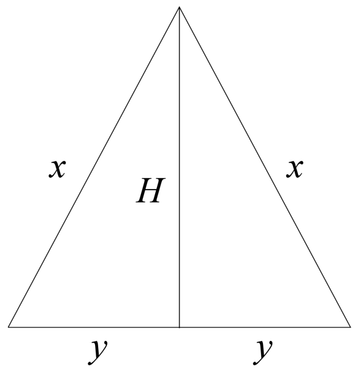

Figure 2.5: Isosceles triangle

Answer: The maximum area for an isosceles triangle of perimeter 2 is $A(\frac{2}{3}, \frac{1}{3}) = \frac{1}{3}\sqrt{\frac{1}{3}} = 3^{-3/2}$, given by sides of length $x = \frac{2}{3}$ and $2y = \frac{2}{3}$ (i.e., an equilateral triangle with perimeter 2.) □

Example 2.8 Find the minimum surface area of a cuboid of fixed volume. (You may assume that such a minimum exists—establishing this fact is beyond the scope of this course.)

Solution :

Answer: The minimum surface area is $S = 6V^{\frac{2}{3}}$. □

Remark It is not entirely clear from this working that the surface area is a minimum rather than a maximum. Conider a very long thin cuboid (e.g. a wafer). The length of the cuboid is 10^n units and the width 10^{-n} units and height V units. The volume is V as required. The surface area of one side of box is $10^n V$, by increasing n this can be made as large as we like. This proves the surface area (for a given volume V) is unbounded, So the surface area in the question above must actually be the minimum.

Example 2.9 Find the maximum and minimum values of xy at points on the ellipse $4x^2 + 2xy + y^2 = 36$.

Solution :

Answer: The maximum value of xy is 6, the mimimum is -9.

$\square$