Vector algebra - revision

1.1 Determinants

Recall the idea of a *determinant* as follows,

2×2 determinant

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc, \quad \text{e.g.} \begin{vmatrix} 4 & 5 \\ 6 & 9 \end{vmatrix} = 36 - 30 = 6.$$

3×3 determinant

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} == a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

For example,

$$\begin{vmatrix} 4 & 2 & 3 \\ 1 & 5 & 4 \\ -2 & 7 & 6 \end{vmatrix} = 4(5 \cdot 6 - 4 \cdot 7) - 2(1 \cdot 6 - (-2) \cdot 4) + 3(1 \cdot 7 - (-2) \cdot 5) = 31.$$

1.2 Scalar product in $\mathbb{R}^3$

Recall the scalar product ($\mathbf{a} \cdot \mathbf{b}$) *is a scalar* defined by:

1. $\mathbf{a} \cdot \mathbf{b}$=$|\mathbf{a}||\mathbf{b}| \cos\theta$,
   where θ is the angle between the vectors.
   or equivalently by,

2. $\mathbf{a} \cdot \mathbf{b}$=$a_1 b_1 + a_2 b_2 + a_3 b_3$
   in component form.

Some useful identities come from 1 and 2 above:

- $\mathbf{a} \cdot \mathbf{b} = 0 \iff \mathbf{a}$ and $\mathbf{b}$ are perpendicular.
- The *length of* $\mathbf{a} = (a_1, a_2, a_3)$ is $|\mathbf{a}|$ and is

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

Example 1.1 Let $\mathbf{a} = (7, 2, 1)$ and $\mathbf{b} = (5, -3, 4)$. Calculate $\mathbf{a} \cdot \mathbf{b}$ and $|\mathbf{a}|$ and $|\mathbf{b}|$.

Solution :
$\mathbf{a}\cdot\mathbf{b} = 7.5 + -3.2 = 1.4 = 33$, $|\mathbf{a}| = \sqrt{7^2+2^2+1^2} = \sqrt{54}$ and $|\mathbf{b}| = \sqrt{5^2+(-3)^2+4^2} = \sqrt{50}$. □

1.3 Vector product in $\mathbb{R}^3$

Recall the *vector product* $(\mathbf{a}\times\mathbf{b})$ is *a vector* defined by,

1. $\mathbf{a}\times\mathbf{b}$=$|\mathbf{a}||\mathbf{b}|\sin\theta\mathbf{c}$,
   where θ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$ and $\mathbf{c}$ is a unit vector perpendicular to $\mathbf{a}$ and $\mathbf{b}$ (right handed screw law).

   or equivalently by giving the formula in determinant form,

2. $$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

Example 1.2 Find $\mathbf{a}\times\mathbf{b}$ where $\mathbf{a} = (2,5,3)$ and $\mathbf{b} = (-1,4,8)$

Solution :
$$\mathbf{a}\times\mathbf{b} = (5.8 - 3.4)\mathbf{i} - (2.8 - 3.(-1))\mathbf{j} + (2.4 - 5.(-1))\mathbf{k} = (28,-19,13).$$

□

Some useful observations about vector products,

- $\mathbf{b}\times\mathbf{a} = -(\mathbf{a}\times\mathbf{b})$
- $\mathbf{a}\times\mathbf{b} = 0 \iff \mathbf{b}$ is a multiple of $\mathbf{a}$.
- In particular, $\mathbf{a}\times\mathbf{a} = \mathbf{0}$.

1.4 Triple scalar product

Definition

$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$ is the *triple scalar product* of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ and is denoted by $[\mathbf{a},\mathbf{b},\mathbf{c}]$. (It is a **number** not a vector).

Remark We begin by noting some properties of triple scalar products.

1. From the definitions on the previous pages, we see that the easiest way to calculate the triple scalar product is as
$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = [\mathbf{a},\mathbf{b},\mathbf{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix},$$
i.e. the 3×3 determinant of the components.
2. From the properties of interchanging rows of a determinant, we can see immediately that
$$[\mathbf{a},\mathbf{b},\mathbf{c}] = [\mathbf{b},\mathbf{c},\mathbf{a}] = [\mathbf{c},\mathbf{a},\mathbf{b}] = -[\mathbf{a},\mathbf{c},\mathbf{b}] = -[\mathbf{c},\mathbf{b},\mathbf{a}] = -[\mathbf{b},\mathbf{a},\mathbf{c}].$$
3. The triple scalar $[\mathbf{a},\mathbf{b},\mathbf{c}]$ is a *number* that measures the volume of the parallelepiped with sides $\mathbf{a},\mathbf{b},\mathbf{c}$
4. In particular, $[\mathbf{a},\mathbf{a},\mathbf{c}] = 0$, since the parallelepiped then has volume 0.

Example 1.3 Find $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$ for $\mathbf{a} = (2,1,5)$, $\mathbf{b} = (0,0,3)$ and $\mathbf{c} = (7,5,-6)$.

Solution :
$$\begin{vmatrix} 2 & 1 & 5 \\ 0 & 0 & 3 \\ 7 & 5 & -6 \end{vmatrix} = -51.$$
□

1.5 The triple vector product

The following is a key identity that you should try an learn:
$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}.$$

Note the RHS is a vector. The proof of this rule is quite hard to construct from nothing and is not covered in this course.

Example 1.4 Show that if $\mathbf{a}$ and $\mathbf{b}$ are perpendicular, then $\mathbf{a}\times(\mathbf{b}\times\mathbf{r})$ is a multiple of $\mathbf{b}$.

Solution : $\mathbf{a}\times(\mathbf{b}\times\mathbf{r}) = (\mathbf{a}\cdot\mathbf{r})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{r}$. But, $\mathbf{a}\cdot\mathbf{b} = 0$ since $\mathbf{a},\mathbf{b}$ are perpendicular. So, $\mathbf{a}\times(\mathbf{b}\times\mathbf{r}) = (\mathbf{a}\cdot\mathbf{r})\mathbf{b}$, i.e. a muliple of $\mathbf{b}$. □