Chapter 1

Revision of some vector algebra

1.1 Determinants

Recall the idea of a *determinant* as follows,

2×2 determinant

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc, \quad \text{e.g.} \begin{vmatrix} 4 & 5 \\ 6 & 9 \end{vmatrix} = 36 - 30 = 6.$$

3×3 determinant

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} =$$

For example,

$$\begin{vmatrix} 4 & 2 & 3 \\ 1 & 5 & 4 \\ -2 & 7 & 6 \end{vmatrix} =$$

1.2 Scalar product in $\mathbb{R}^3$

Recall the scalar product ($\mathbf{a} \cdot \mathbf{b}$) *is a scalar* defined by:

1. $\mathbf{a} \cdot \mathbf{b}$=$|\mathbf{a}||\mathbf{b}| \cos\theta$,
   where θ is the angle between the vectors.
   or equivalently by,

2. $\mathbf{a} \cdot \mathbf{b}$=$a_1b_1 + a_2b_2 + a_3b_3$
   in component form.

Some useful identities come from 1 and 2 above:

- $\mathbf{a} \cdot \mathbf{b} = 0 \iff \mathbf{a}$ and $\mathbf{b}$ are perpendicular.
- The *length of* $\mathbf{a} = (a_1, a_2, a_3)$ is $|\mathbf{a}|$ and is

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

Example 1.1

Solution : □

1.3 Vector product in $\mathbb{R}^3$

Recall the *vector product* ($\mathbf{a} \times \mathbf{b}$) is *a vector* defined by,

1. $\mathbf{a} \times \mathbf{b}$=$|\mathbf{a}||\mathbf{b}| \sin\theta \mathbf{c}$,
   where θ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$ and $\mathbf{c}$ is a unit vector perpendicular to $\mathbf{a}$ and $\mathbf{b}$ (right handed screw law.

   or equivalently by,

2. $$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} =$$

Example 1.2 Find $\mathbf{a} \times \mathbf{b}$ where $\mathbf{a} = (2, 5, 3)$ and $\mathbf{b} = (-1, 4, 8)$

Solution : □

Some useful observations about vector products,

- $\mathbf{b} \times \mathbf{a} = -\mathbf{a} \times \mathbf{b}$
- $\mathbf{a} \times \mathbf{b} = 0 \iff \mathbf{b}$ is a multiple of $\mathbf{a}$.
- In particular, $\mathbf{a} \times \mathbf{a} = \mathbf{0}$.

1.4 Triple scalar product

Definition

$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ is the *triple scalar product* of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ and is denoted by $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$. (It is a **number** not a vector).

Properties of triple scalar products,

- From the definitions on the previous pages, we see that the easiest way to calculate the triple scalar product is as

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = [\mathbf{a}, \mathbf{b}, \mathbf{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix},$$

  i.e. the 3 determinant of the components.

- From the properties of interchanging rows of a determinant, we can see immediately that

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = [\mathbf{b}, \mathbf{c}, \mathbf{a}] = [\mathbf{c}, \mathbf{a}, \mathbf{b}] = -[\mathbf{a}, \mathbf{c}, \mathbf{b}] - [\mathbf{c}, \mathbf{b}, \mathbf{a}] = -[\mathbf{b}, \mathbf{a}, \mathbf{c}].$$

- The triple scalar $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ is a number that measures the volume of the parallelepiped with sides $\mathbf{a}, \mathbf{b}, \mathbf{c}$

- In particular, $[\mathbf{a}, \mathbf{a}, \mathbf{c}] = 0$, since the parallelepiped then has volume 0.

Example 1.3 Find $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ for $\mathbf{a} = (2, 1, 5)$, $\mathbf{b} = (0, 0, 3)$ and $\mathbf{c} = (7, 5, -6)$.

Solution :

□

1.5 The triple vector product

The following is a key identity:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}.$$

Note the RHS is a vector.

Example 1.4 Show that if $\mathbf{a}$ and $\mathbf{b}$ are perpendicular, then $\mathbf{a} \times (\mathbf{b} \times \mathbf{r})$ is a multiple of $\mathbf{b}$.

Solution :

□