Summary of Lecture 16- -Conservative vector fields

Question:

Are *all* vector fields the grad of some scalar field?

Answer:

No, only vector fields that are defined everywhere and satisfy curl $\mathbf{F} = \mathbf{0}$ can be expressed as

$$\mathbf{F} = \text{grad}\,\phi$$

for some scalar field ϕ. The vector field is then called conservative.

Practicalities

Given a conservative vector field $\mathbf{F} = (F_1, F_2, F_3)$ we find the scalar field ϕ such that $\mathbf{F} = \text{grad}\,\phi$ by solving:

$$\frac{\partial \phi}{\partial x} = F_1, \quad \frac{\partial \phi}{\partial y} = F_2, \quad \frac{\partial \phi}{\partial z} = F_3.$$

Steps:

1. Integrate the first equation w.r.t. x to give an expression for ϕ. The constant of integration, $A(y,z)$, will involve y and z.

2. Substitute ϕ into the next equation. You will be left with an equation involving $\partial A/\partial y$. Integrate this w.r.t. y to give A. The integration generates a constant of integration, $B(z)$, which depends only on z.

3. Substitute the expression for $A(y,z)$ into ϕ, so it remains to find $B(z)$. Substitute ϕ into the last equation, $\frac{\partial \phi}{\partial z} = F_3$, and use this to find $B(z)$.