INERTIA GROUPS OF 3-CONNECTED 8-MANIFOLDS

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ABSTRACT. We announce the determination of the inertia groups of closed 3-connected 8-manifolds. This result completes the smooth classification of 3-connected 8-manifolds, begun by Wall in 1962. Full proofs rely on the second author's PhD thesis and will appear in later work.

1. Statement of results

The purpose of this note is to announce the determination of the inertia groups of 3-connected 8-manifolds. All manifolds M are closed, smooth and oriented, all maps preserve orientation and all coefficients for (co)homology groups are integral.

Definition 1.1. For $n \geq 5$, the group of n-dimensional homotopy spheres is denoted by Θ_n, and in dimension 8 we have $\Theta_8 \cong \mathbb{Z}_2$; see [4]. We denote the non-trivial element of Θ_8 by Σ^8_{ex}.

Definition 1.2. Let M be an n-manifold. The *inertia group* of M is the subgroup

$$I(M) := \{\Sigma \in \Theta_n \mid M\#\Sigma \approx M\} \subseteq \Theta_n,$$

where $M\#\Sigma$ denotes connected sum and $\approx$ denotes diffeomorphism.

In 1962 Wall classified 3-connected 8-manifolds (and more generally $(q-1)$-connected $2q$-manifolds $(q \geq 3)$) up to connected sum with homotopy spheres [10]. This reduced the classification of 3-connected 8-manifolds M to the determination of the inertia group $I(M)$. This is a subtle problem and despite work on special cases [5] and on related inertia groups [3], the problem remained open. The following result settles this problem, proving a conjecture of the first author [1, Conjecture 2.4].

Theorem 1.3. *Let M be a 3-connected 8-manifold and let $d_M \in \mathbb{Z}_{\geq 0}$ denote the divisibility of its first Pontryagin-class $p_1(M)$ (ie. $p_1(M) = d_M x$ for a primitive element $x \in H^4(M)$). Then*

$$I(M) = \begin{cases} 0 & \text{if } 8 \mid d_M, \\ \Theta_8 & \text{if } 8 \nmid d_M. \end{cases}$$

We note that every 3-connected manifold is spin and therefore d_M is even (see [7, Lemma 2.2]).

Now let $n_+(M)$ denote the number of diffeomorphism classes of smooth manifolds which are homeomorphic to M. The corollaries below follow immediately from [10, p. 170] and Theorem 1.3.

Corollary 1.4. *Let M be a 3-connected 8-manifold. Then $n_+(M) = 2$ if $8 \mid d_M$ and $n_+(M) = 1$ if $8 \nmid d_M$.*

Corollary 1.5. *Let M_0 and M_1 be 3-connected 8-manifolds and $A\colon H^4(M_1) \to H^4(M_0)$ an isomorphism preserving the intersection form and the first Pontryagin class, and suppose that $8 \nmid d_{M_0} = d_{M_1}$. Then M_0 and M_1 are diffeomorphic.*

When M_0 and M_1 are diffeomophic, we have the following result, which is proven using Wall's results [10, p. 170] and modified surgery over the normal 4-type, specifically [6, Theorem 4].

Theorem 1.6. *Let M_0 and M_1 be diffeomorphic 3-connected 8-manifolds and $A\colon H^4(M_1) \to H^4(M_0)$ an isomorphism preserving intersection form and the first Pontryagin class. Then A is realised by a diffeomorphism $f\colon M_0 \to M_1$.*

2. Discussion of the proof of Theorem 1.3

The proof of Theorem 1.3 relies on a special case of the so called *Q-form conjecture* of the first author. In order to state this conjecture first we introduce some definitions.

Definition 2.1. Suppose that $B \to BSO$ is a fibration. A map $M \to B$ from a smooth manifold M is a *normal k-smoothing*, if it is $(k+1)$-connected and the composition $M \to B \to BSO$ is the classifying map of the stable normal bundle of M (up to homotopy).

Definition 2.2. Let M be a simply-connected $2q$-manifold with q even. The *Q-form* of a normal $(q-1)$-smoothing $f : M \to B$ is the triple

$$Q_q(f) = (H_q(M), \lambda_M, f_*)\,,$$

where $\lambda_M : H_q(M) \times H_q(M) \to \mathbb{Z}$ is the intersection form of M and $f_* : H_q(M) \to H_q(B)$ is the homomorphism induced by f.

The following result has been proven as part of the second author's PhD thesis [8].

Theorem 2.3 (Q-form conjecture, special case)**.** *Suppose that q is even and $B \to BSO$ is a fibration with $\pi_1(B) = 0$ and $H_q(B)$ torsion-free. Let W be a cobordism between the $2q$-manifolds M_0 and M_1, and let $F : W \to B$, $f_0 = F\big|_{M_0} : M_0 \to B$ and $f_1 = F\big|_{M_1} : M_1 \to B$ be normal $(q-1)$-smoothings. If the Q-forms of f_0 and f_1 are isomorphic, then W is cobordant to an h-cobordism.*

We now briefly sketch the proof: It begins with the definition of an algebraic monoid $l_{2q+1}(B)$ and also a surgery obstruction $\theta_W \in l_{2q+1}(B)$, which generalise the monoid $l_{2q+1}(\mathbb{Z})$ and corresponding surgery obstruction of [3, Theorem 4] in the simply-connected case. The surgery obstruction θ_W is an invariant of the cobordism class of the $(q-1)$-smoothing $W \to B$ and it is elementary (an algebraically defined notion) if and only if W is cobordant, relative to its boundary, to an h-cobordism over B. Finally, if $Q_q(f_0)$ is isomorphic to $Q_q(f_1)$, then θ_W is elementary.

Definition 2.4. Let $BSpin_a$ denote the central space in the 4^{th} Moore-Postnikov stage of a map $S^4 \to BSpin$ representing $a \in \mathbb{Z} \cong \pi_4(BSpin)$; ie. $BSpin_a$ is a space admitting maps $S^4 \xrightarrow{f_a} BSpin_a \xrightarrow{g_a} BSpin$ such that f_a is 4-connected, g_a is 4-co-connected and $[g_a \circ f_a] = a$.

For example $BSpin_1 = BSpin$ and $BSpin_0 = BString$.

We will apply the Q-form conjecture in the following setting: Let $M_0 = M$ be a 3-connected 8-manifold, and $M_1 = M \# \Sigma^8_{\text{ex}}$. If the divisibility of $p_1(M)$ is $2a$, then let $B = BSpin_a$. The natural map $BSpin_a \to BSO$ defines the normal 3-type (as defined in [3, §2]) of both M_0 and M_1. Hence there are canonical normal 3-smoothings $f_i : M_i \to B$. For these maps $Q_4(f_0) \cong Q_4(f_1)$, so we get the following:

Proposition 2.5. *The following are equivalent:*
1. *M is diffeomorphic to $M \# \Sigma^8_{\text{ex}}$.*
2. *f_0 and f_1 are bordant over B.*
3. *The map $i : \Theta_8 \to \Omega_8^{Spin_a}$ is trivial.*

The proof of Theorem 1.3 is then completed by the following result.

Theorem 2.6. *The map $i : \Theta_8 \to \Omega_8^{Spin_a}$ is trivial if and only if a is not divisible by 4.*

Theorem 2.6 is proven using the long exact sequence $\ldots \to \Omega_*^{Spin_a} \to \Omega_*^{Spin} \to \Omega_*^{Spin,Spin_a} \to \ldots$ and a relative version of the James spectral sequence from [9, Theorem 3.1.1].

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