The classification of 8-dimensional E-manifolds

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E-manifolds

Definition

M is an *E-manifold* if $H_{2k+1}(M;\mathbb{Z}) \cong 0$ for all k.

We will consider smooth, closed, simply-connected, oriented E-manifolds.

Examples

- *Homotopy spheres (in even dimensions)*
- *Complex and quaternionic projective spaces*
- *Bott-manifolds*
- *Complex projective complete intersections (in even complex dimensions)*

E-manifolds

Our goal is to give a method to classify 8-dimensional E-manifolds up to diffeomorphism.

Known results

- The only 2-dimensional E-manifold is S^2.
- In dimension 4 the homeomorphism classification was obtained by Freedman, the diffeomorphism classification is unknown.
- In dimension 6 classification up to diffeomorphism was given by Wall and Jupp.

Outline

The classification of E-manifolds will consist of 4 steps:

1. Define an action of $\Theta(r, w)$ on $E(r, w)$
2. Compute the group $\Theta(r, w)$
3. Classify the orbits of the action
4. Compute the stabilizers of the orbits

Outline

This talk will have 5 parts:

1. Define an action of $\Theta(r, w)$ on $E(r, w)$
2. Compute the group $\Theta(r, w)$
3. Classify the orbits of the action
4. The Q-form conjecture
5. Compute the stabilizers of the orbits

1. Definitions

Definition

A *polarized E-manifold* is a pair (M, φ), where M is an E-manifold and $\varphi : H_2(M) \to \mathbb{Z}^r$ is an isomorphism for some r.

Definition

For an $r \geq 0$ and $w : \mathbb{Z}^r \to \mathbb{Z}_2$ let

$$E(r,w) = \left\{ (M,\varphi) \,\middle|\, \begin{array}{c} M \text{ is an E-manifold} \\ \varphi : H_2(M) \cong \mathbb{Z}^r \\ w_2(M) = w \end{array} \right\} \Big/ \sim$$

where the equivalence relation $\sim$ is orientation-preserving diffeomorphism, compatible with φ.

Example

$E(0,0)$ consists of 3-connected 8-manifolds.

1. Definitions

Remark

If $w, w' : \mathbb{Z}^r \to \mathbb{Z}_2$ are both non-zero, then there is an isomorphism $I : \mathbb{Z}^r \to \mathbb{Z}^r$ such that $w = w' \circ I$.
Then $(M, \varphi) \mapsto (M, I \circ \varphi)$ is a bijection $E(r, w) \to E(r, w')$.

Definition

$\mathrm{Aut}(w) = \{f : \mathbb{Z}^r \to \mathbb{Z}^r \mid f \text{ is an isomorphism}, w \circ f = w\}$.

Proposition

$\mathrm{Aut}(w)$ acts on $E(r, w)$ by $(M, \varphi)^f = (M, f \circ \varphi)$.
The set of diffeomorphism classes of spin E-manifolds with $b_2 = r$ is $E(r, 0)/\mathrm{Aut}(\mathbb{Z}^r)$.
The set of diffeomorphism classes of non-spin E-manifolds with $b_2 = r$ is $E(r, w)/\mathrm{Aut}(w)$ for any non-zero w.

1. Definitions

Definition

For an $r \geq 0$ and $w : \mathbb{Z}^r \to \mathbb{Z}_2$ let

$$T = T(r, w) \approx D_1 \natural D_2 \natural \ldots \natural D_r$$

where

- each D_i is a D^6-bundle over S^2
- D_i is trivial if and inly if $w(e_i) = 0$, where $e_1, e_2, \ldots, e_r$ is the standard basis of $\mathbb{Z}^r$

T is well-defined up to diffeomorphism, because there is a unique nontrivial D^6-bundle over S^2 ($\pi_2(BSO_6) \cong \mathbb{Z}_2$)

1. Definitions

Definition

If $(M, \varphi) \in E(r, w)$, then φ determines an embedding $T \to M$:

- The basis elements $e_1, e_2, \dots, e_r \in \mathbb{Z}^r \cong H_2(M) \cong \pi_2(M)$ are represented by disjoint embedded spheres $S^2 \hookrightarrow M$
- The normal bundle of an S^2 is an element of $\pi_2(BSO_6) \cong \mathbb{Z}_2$, classified by the Stiefel-Whitney class w_2
- Since $w_2(M) = w$, the tubular neighbourhood of the i^{th} sphere is diffeomorphic to D_i (trivial iff $w(e_i) = 0$)
- The diffeomorphism between D_i and the neighbourhood of the i^{th} sphere is unique up to isotopy, because $\pi_2(SO_6) \cong 0$

Therefore the boundary connected sum of the tubular neighbourhoods of the spheres is an embedded copy of T in M, and the embedding is well-defined up to isotopy.

1. Definitions

Definition

There is a binary operation $\#_2$ on $E(r, w)$:

$$(M_1, \varphi_1)\#_2(M_2, \varphi_2) = (M, \varphi)$$

where

$$M = (M_1 \setminus T) \cup_{\partial T} (M_2 \setminus T)$$

and $\varphi : H_2(M) \to \mathbb{Z}^r$ is the composition of the isomorphisms

$$H_2(M) \leftarrow H_2(\partial T) \to H_2(T) \cong \mathbb{Z}^r$$

Proposition

$(E(r, w), \#_2)$ is a commutative monoid.
The zero element is $T \cup_{\partial T} T$.

1. Definitions

Definition

$$\Theta(r,w) = \{(M,\varphi) \in E(r,w) \mid H_4(M) \cong 0\} / \sim$$

Proposition

$\Theta(r,w)$ is the group of invertible elements of $E(r,w)$.

Proof.

- $H_4(M_1 \#_2 M_2) \cong H_4(M_1) \oplus H_4(M_2)$ for any $M_1, M_2 \in E(r,w)$. So if $M \in E(r,w)$ is invertible, then $H_4(M) \cong 0$.
- Elements of $\Theta(r,w)$ are invertible, because $\Theta(r,w) \cong \pi_0 \, \mathrm{SDiff}(\partial T)$.

□

1. Definitions

Definition

$$\mathrm{SDiff}(\partial T) = \{g \in \mathrm{Diff}(\partial T) \mid H_*(g) = \mathrm{id}\}$$

Proposition

If $(M, \varphi) \in \Theta(r, w)$*, then* $M \approx T \cup_g T$ *for some* $g \in \mathrm{SDiff}(\partial T)$*.*

Corollary

$\Theta(r, w) \cong \pi_0 \, \mathrm{SDiff}(\partial T)$.

Example

$\Theta(0, 0) \cong \pi_0 \, \mathrm{Diff}_+(S^7) \cong \Theta_8 \cong \mathbb{Z}_2$.

Definition

$\Theta(r, w)$ acts on $E(r, w)$ by $\#_2$.

Via the isomorphism $\Theta(r, w) \cong \pi_0 \, \mathrm{SDiff}(\partial T)$ we also get an action of $\pi_0 \, \mathrm{SDiff}(\partial T)$ on $E(r, w)$.

Proposition

The action of $g \in \mathrm{SDiff}(\partial T)$ *on* $(M, \varphi) \in E(r, w)$ *is given by*

$$(M, \varphi)^g = ((M \setminus T) \cup_g T, \varphi)$$

2. The group $\Theta(r, w)$

Theorem

- $\Theta(r, 0) \cong \mathbb{Z}^a \oplus \mathbb{Z}_2^b$, *where* $a = 6\binom{r+2}{5}$, $b = 2\binom{r+3}{4} - \binom{r-1}{2} + 2$.
- $\operatorname{rk} \Theta(r, w) = a$.
- $\Theta(1, 1) \cong \mathbb{Z}_4$.

Idea of proof: Identify $\Theta(r, w)$ with other groups that can be computed by well-known methods

2. The group $\Theta(r, w)$

Definition

- $P_3(T)$ is the third Postnikov-stage of $T \simeq \vee^r S^2$.
- ξ is a bundle over $P_3(T)$ such that $w_2(\xi) = w$.
- $\Omega_8^{String}(P_3(T), \xi)$ is a twisted String bordism group: an element is represented by a map $f : N \to P_3(T)$ together with a String structure on $\nu_N \oplus f^*(\xi)$.
- $\hat{\Omega}_8^{String}(P_3(T), \xi)$ is the subgroup of $\Omega_8^{String}(P_3(T), \xi)$ consisting of maps $f : N \to P_3(T)$ such that $\sigma(N) = 0$.

There is twisted Atiyah-Hirzebruch spectral sequence ("James spectral sequence") converging to $\Omega_8^{String}(P_3(T), \xi)$

2. The group $\Theta(r, w)$

Proposition

$\Theta(r, w) \cong \hat{\Omega}_8^{String}(P_3(T), \xi)$.

Proof.

A homomorphism $\Theta(r, w) \to \hat{\Omega}_8^{String}(P_3(T), \xi)$ is defined as follows: Let $(M, \varphi) \in \Theta(r, w)$

- The embedding $T \to M$ determines a map $P_3(T) \to P_3(M)$
- It has a homotopy inverse $P_3(M) \to P_3(T)$
- For the composition $f : M \to P_3(M) \to P_3(T)$ the bundle $\nu_M \oplus f^*(\xi)$ has a unique String structure
- $\sigma(M) = 0$

This is an isomorphism. □

3. Classification of the orbits

Theorem

The orbits of the action of $\Theta(r,w)$ on $E(r,w)$ are classified by the cohomology ring and the Pontryagin-class p_1. More precisely, the following are equivalent:

- $M \approx M' \#_2 \Sigma$ *for some* $\Sigma \in \Theta(r,w)$.
- *There is a ring isomorphism* $H^*(M) \to H^*(M')$ *that is compatible with the maps* $\varphi^* : \mathbb{Z}^r \cong H^2(M)$ *and* $(\varphi')^*$ *and preserves* p_1.

Special cases were proven by:

- Wall ('62): $r = 0$
- Schmitt ('02): any r and $w = 0$

3. Classification of the orbits

Proof.

Handlebody decomposition of $M \in E(r, w)$:

$$D^8 = W_0 \subseteq W_2 \subseteq W_4 \subseteq W_6 \subset W_8 = M$$

where $W_i = W_{i-2} \cup i$-handles

- $W_2 \approx T$
- $M \setminus W_4 \approx T$
- $M = W_4 \cup T$
- $M \approx (M')^g$ for some $g \in \mathrm{SDiff}(\partial T) \Leftrightarrow W_4 \approx W_4'$
- W_4 is determined by $H^*(M)$ and $p_1(M)$

□

Definition

Let ξ be a stable vector bundle over some space B. For a manifold M the map $f : M \to B$ is a *normal k-smoothing*, if

- $\pi_i(f)$ is an isomorphism for $i \leq k$ and $\pi_{k+1}(f)$ is surjective (that is, f is a $(k+1)$-equivalence)
- $\nu_M \cong f^*(\xi)$

Proposition

Every manifold M has a canonical k-smoothing

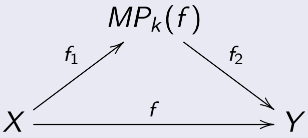
$$M \to B^k(M)$$

4. The Q-form conjecture

Definition

The k^{th} *stage of the Moore-Postnikov factorization* of a map $f : X \to Y$ is a space $MP_k(f)$ together with maps $f_1 : X \to MP_k(f)$ and $f_2 : MP_k(f) \to Y$ such that

- $f = f_2 \circ f_1$
- $\pi_i(f_1)$ is an isomorphism for $i < k$ and $\pi_k(f_1)$ is surjective
- $\pi_i(f_2)$ is an isomorphism for $i > k$ and $\pi_k(f_2)$ is injective



Examples

- *If $Y \simeq *$, then $MP_{k+1}(f) = P_k(X)$ is a Postnikov-stage of X.*
- *If $X \simeq *$, then $MP_k(f)$ is the k-connected cover of Y.*

4. The Q-form conjecture

Definition

The *normal k-type* of M is

$$B^k(M) = MP_{k+1}(\nu_M)$$

where $\nu_M : M \to BO$ is the classifying map of the stable normal bundle of M.

Proposition

- *$M \to B^k(M)$ is a normal k-smoothing.*
- *B^k is a functor.*

Example

$B^3(T) = P_3(T) \times BString$ *and* $\Omega_8(B^3(T)) = \Omega_8^{String}(P_3(T), \xi)$.

Definition

Fix some even positive integer q and a space B with a stable bundle ξ over it. If M is a $2q$-dimensional manifold and $f : M \to B$ is a $(q-1)$-smoothing, then the *Q-form* of f is the triple

$$Q(f) = (H_q(M), \lambda_M, f_*)$$

where

- $\lambda_M : H_q(M) \times H_q(M) \to \mathbb{Z}$ is the intersection pairing
- $f_* : H_q(M) \to H_q(B)$ is induced by f

4. The Q-form conjecture

Theorem (Q-form conjecture, torsion-free case)

Let M and M' be simply-connected $2q$-manifolds, and W be a cobordism between them. Let $F: W \to B$, $f = F\big|_M$, $f' = F\big|_{M'}$, and suppose that F, f and f' are all $(q-1)$-smoothings.
If

- $Q(f) \cong Q(f')$
- $H_q(B)$ *is torsion-free*

then W is cobordant to $M \times [0,1]$, therefore $M \approx M'$.

Remark

By Kreck's modified surgery theory, the existence of a bordism F between the $(q-1)$-smoothings f and f' is equivalent to the condition that M and M' are stably diffeomorphic (that is, $M \# k(S^q \times S^q) \approx M' \# l(S^q \times S^q)$ for some k, l).

5. Inertia groups

Recall that the inertia group an n-manifold M is the subgroup $\{\Sigma \in \Theta_n \mid M\#\Sigma \approx M\} \leq \Theta_n$.

Definition

The *(extended) inertia group* of an E-manifold $M \in E(r,w)$ is the subgroup

$$I(M) = \{\Sigma \in \Theta(r,w) \mid M\#_2\Sigma \approx M\} \leq \Theta(r,w)$$

We will use the Q-form conjecture to determine the inertia group of E-manifolds.

Suppose that $M \in E(r, w)$ and $\Sigma \in \Theta(r, w)$.
Let $B = B^3(M)$ and let $f : M \to B$ be the canonical map.
Let $M' = M \#_2 \Sigma$.

Lemma

a) $B^3(M') = B$

b) $Q(f) \cong Q(f')$, *where* $f' : M' \to B$ *is the canonical map.*

Proof.

a) Recall that $M = W_4 \cup T$ and $M' = W_4 \cup_g T$, where
$g \in \mathrm{SDiff}(\partial T)$ is the diffeomorphism corresponding to Σ.
The embeddings $M \leftarrow W_4 \to M'$ induce homotopy equivalences
$B^3(M) \leftarrow B^3(W_4) \to B^3(M')$.
b) $Q(f)$ is determined by W_4 and the map $W_4 \to B^3(W_4)$. □

Proposition

f and f' are bordant over B $\Longleftrightarrow \Sigma \in I(M)$

Since $M' = M \#_2 \Sigma$, in the bordism group $\Omega_8(B)$

$$[M', f'] = [M, f] + j_*([\Sigma])$$

where $[\Sigma] \in \hat{\Omega}_8^{String}(P_3(T), \xi) = \Theta(r, w)$
and $j_* : \Omega_8^{String}(P_3(T), \xi) = \Omega_8(B^3(T)) \to \Omega_8(B) = \Omega_8(B^3(M))$ is the homomorphism induced by $j : B^3(T) \to B^3(M)$

Theorem

$I(M) = \mathrm{Ker}\left(j_* : \hat{\Omega}_8^{String}(P_3(T), \xi) \to \Omega_8(B^3(M))\right)$

5. Inertia groups

Theorem

- *If* $M \in E(0,0)$*, then*

$$I(M) = \begin{cases} 0 & \text{if } p_1 \text{ is divisible by } 8 \\ \Theta(0,0) \cong \mathbb{Z}_2 & \text{otherwise} \end{cases}$$

- *If* $M \in E(1,1)$, $\pi_3(M) \cong 0$ *and* $w_4(M) \neq 0$*, then*

$$I(M) = \Theta(1,1) \cong \mathbb{Z}_4$$

- *If* $M \in E(r,w)$, $\cup_2 : H^2(M) \times H^2(M) \to H^4(M)$ *is trivial and* $p_1 = 0$*, then*

$$I(M) = 0$$

- *The subgroup* $I(M) \leq \Theta(r,w)$ *only depends on* $\cup_2$ *and* p_1*.*