Cobordism groups of branched coverings

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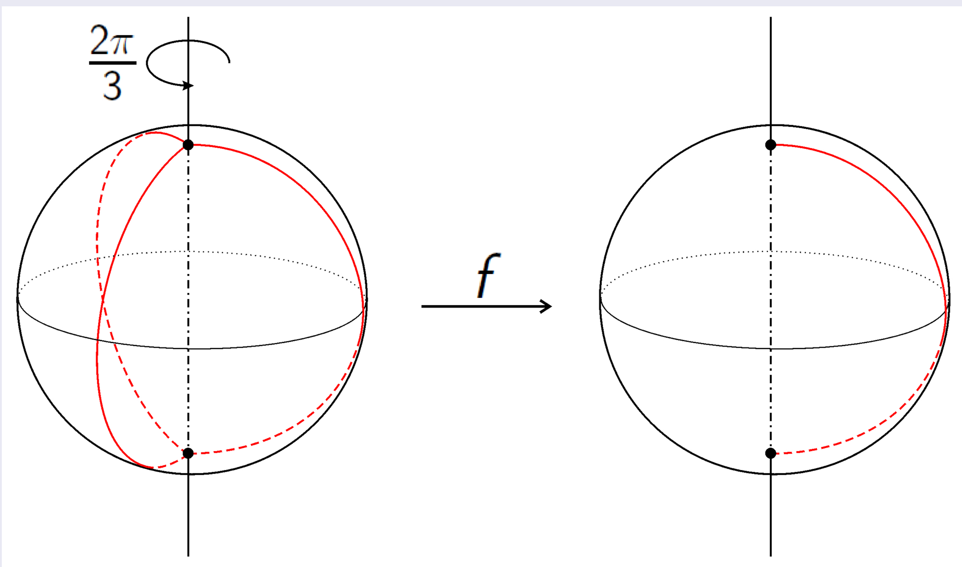
Definition

Let M' and M be closed oriented smooth n-manifolds. Let z^j denote the complex j^{th} power function.
The smooth map $f : M' \to M$ is a *k-fold branched covering*, if it has degree k and all of its singularities are of type $z^j \times \mathrm{id}$ for some j. That is, we can choose smooth charts around any $x \in M'$ and $f(x) \in M$ such that between these charts f has the form $\mathrm{id}_{\mathbb{R}^n}$ or $z^j \times \mathrm{id}_{\mathbb{R}^{n-2}}$.

Example

The cyclic group C_k acts on the sphere S^2: the homeomorphism corresponding to a generator of the group is rotation by $\frac{2\pi}{k}$ around an axis of the sphere. The factor space is homeomorphic to S^2. If an appropriate differentiable structure is chosen on it, then the factor map $f : S^2 \to S^2$ is a k-fold branched covering with two singularities of type z^k.



Example

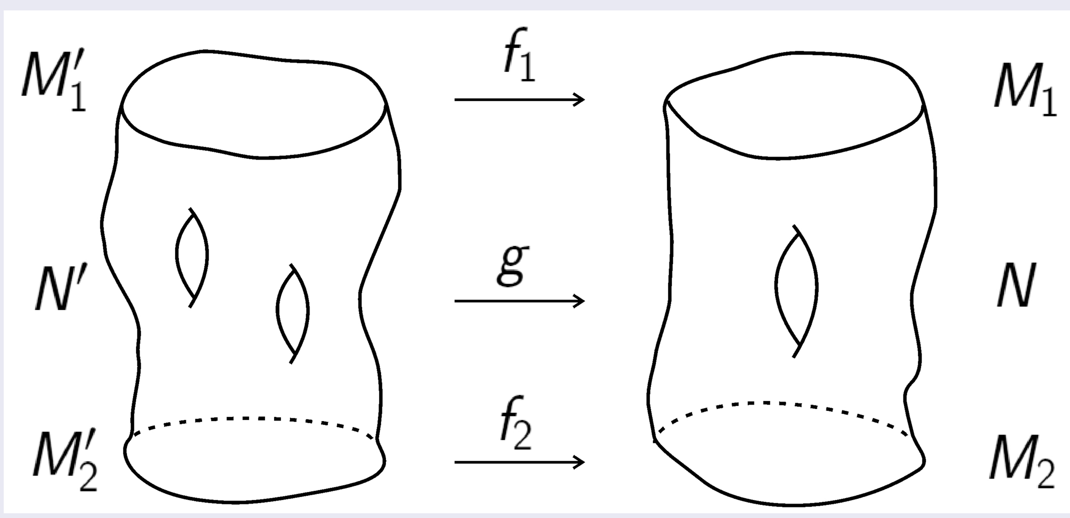
Any non-constant holomorphic map between Riemann surfaces is a branched covering.

Definition

Branched coverings $f_1 : M_1' \to M_1$, $f_2 : M_2' \to M_2$ are *cobordant* if there is

- a cobordism N' between M_1' and M_2',
- a cobordism N between M_1 and M_2, and
- a branched covering $g : N' \to N$

such that the restriction of g to the boundary is $f_1 \bigsqcup f_2$.
If $M_1 = M_2 = M$ and $N = M \times [0, 1]$, then f_1 and f_2 are *concordant*.



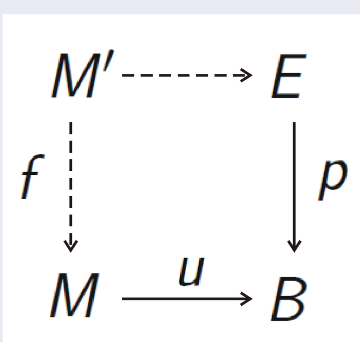
Definition

$\mathrm{Cob}(n, k)$ is the set of cobordism classes of n-dimensional k-fold branched coverings. This is an Abelian group (the operation is disjoint union).
$\mathrm{Con}(S^n, k)$ is the set of concordance classes of k-fold branched coverings over S^n, this is also an Abelian group (with an appropriate operation).

Definition

If $p : E \to B$ is a branched covering, and $u : M \to B$ is a "generic" map, then u *induces* a branched covering $f : M' \to M$ by a pullback diagram:

- $M' = \{(x, y) \in M \times E \mid u(x) = p(y)\}$
- $f : M' \to M$, $f(x, y) = x$



Definition

$p : E \to B$ is a *universal branched covering* (and B is a *classifying space* of branched coverings), if

- for any $f : M' \to M$ branched covering there is a u inducing f, and
- this u is unique up to homotopy.

Theorem

For each k there is a universal k-fold branched covering $p_k : E(k) \to B(k)$.

Idea of proof

First proof: Brown's representability theorem.
Second proof: Explicit construction. For each $m \geq 0$ we consider k-fold branched coverings $f : M' \to M$ with the property that any point of M has at most m singular preimiges. There is a universal branched covering of this type, $p_k^m : E^m(k) \to B^m(k)$. We can construct these recursively:
$p_k^0 : E^0(k) \to B^0(k)$ is the universal k-fold covering. $E^m(k)$ and $B^m(k)$ are obtained from $E^{m-1}(k)$ and $B^{m-1}(k)$ by gluing blocks to them; one new block is needed for each possible type of a point in M with exactly m singular preimiges. $E(k) = \bigcup_m E^m(k)$ and $B(k) = \bigcup_m B^m(k)$.

Theorem

$\mathrm{Cob}(n, k) = \Omega_n(B(k))$ and $\mathrm{Con}(S^n, k) = \pi_n(B(k))$.
($\Omega_n(X)$ denotes the n-dimensional oriented bordism group of a space X, and $\pi_n(X)$ is its n^{th} homotopy group.)

Theorem

$$\mathrm{rk}\,\mathrm{Cob}^m(n, k) = \sum_{p=0}^{\lfloor \frac{n}{4} \rfloor} \pi(p) R(n - 4p, k, m)$$

where $R(0, k, m) = 1$, $R(j, k, m) = 0$ for j odd, and for j even, $j > 0$:

$$R(j, k, m) = \mathrm{rk}\, H_j(B^m(k)) = \sum_{\alpha} \sum_{p_1 + \ldots + p_r = \frac{j}{2} - m_\alpha} \prod_{i=1}^{r} \pi_{\leq m_i}(p_i)$$

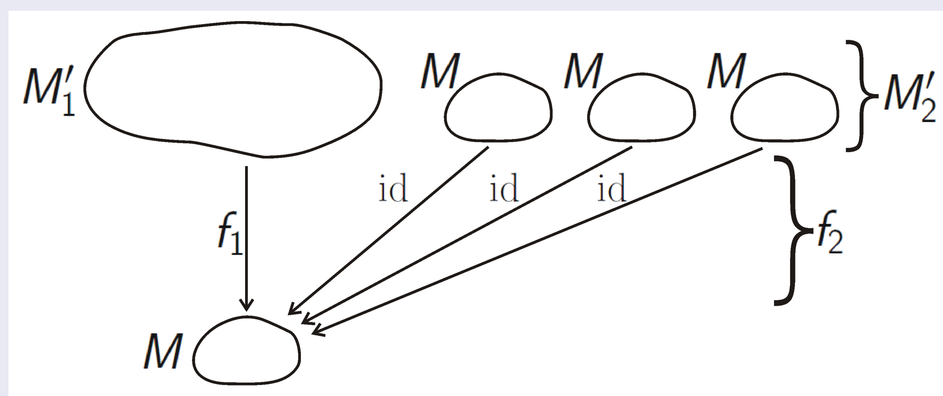
where the summation goes over all $\alpha = (k_1, \ldots, k_r, m_1, \ldots, m_r)$ with

- $2 \leq k_1 < \ldots < k_r$
- $m_\alpha = m_1 + \ldots + m_r \leq \min\left(m, \frac{j}{2}\right)$
- $k_1 m_1 + \ldots + k_r m_r \leq k$

(α is a possible type for points in M. A point has type α if it has m_i preimages with singularity of the type $z^{k_i} \times \mathrm{id}$.)
$\pi(x)$ denotes the number of partitions of x, while $\pi_{\leq y}(x)$ denotes the number of partitions that contain at most y summands.

Definition

Branched coverings $f_1 : M_1' \to M$ and $f_2 : M_2' \to M$ are *stably equivalent*, if $M_2' = M_1 \bigsqcup d \cdot M$ and $f_2 = f_1 \bigsqcup d \cdot \mathrm{id}_M$ for some d.
The equivalence classes are called *stable branched coverings*.



Definition

Cobordism and concordance of stable branched coverings can be defined using representatives. The cobordism and concordance classes form groups, these are denoted by $\mathrm{Cob}(n, \infty)$ and $\mathrm{Con}(S^n, \infty)$.

Theorem

$B(\infty) = \mathrm{dir\,lim}_k\, B(k)$ is the classifying space of stable branched coverings.
$\mathrm{Cob}(n, \infty) = \Omega_n(B(\infty))$ and $\mathrm{Con}(S^n, \infty) = \pi_n(B(\infty))$.

Definition

$B_t(\infty)$ is the classifying space of stable branched coverings that have singularities of type $z^j \times \mathrm{id}$ only if $j \leq t$.

Theorem

$B_t(\infty)$ is rational homotopy equivalent to $\Omega^\infty S^\infty B_t^1(\infty)$.
(ΩX is the loop space of X, SX is its suspension, $\Omega^\infty S^\infty X = \mathrm{dir\,lim}_n\, \Omega^n S^n X$.)

Theorem

$$\mathrm{rk}\,\pi_n(B_t(\infty)) = \mathrm{rk}\, H_n(B_t^1(\infty)) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ t - 1 & \text{if } n \text{ is even} \end{cases}$$