Homotopy equivalence and simple homotopy equivalence of manifolds

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Simple homotopy equivalence

Definition

Let X be a finite CW complex. An *elementary expansion* is an inclusion

$$X \to X \cup_g D^n$$

where $\partial D^n = S^{n-1} = D_+^{n-1} \cup D_-^{n-1}$ and the gluing map is $g : D_-^{n-1} \to X$.
The deformation retraction $X \cup_g D^n \to X$ is called an *elementary collapse*.

Definition

Let X and Y be finite CW complexes. A map $f : X \to Y$ is a *simple homotopy equivalence* if it is homotopic to a composition

$$X = X_0 \to X_1 \to \ldots \to X_k = Y$$

of elementary expansions and collapses.

Questions

Definition

Let X and Y be finite CW complexes.
X and Y are *simple homotopy equivalent* if there is a simple homotopy equivalence $f : X \to Y$.

Is simple homotopy equivalence strictly stronger than homotopy equivalence?

- In general yes.
- It depends on the fundamental group. For example, if X and Y are simply-connected, then every homotopy equivalence $X \to Y$ is simple.

Questions

Question

Are there (closed, connected, orientable) n-manifolds M and N that are homotopy equivalent but not simple homotopy equivalent?

If $n = 1$ or 2, then homotopy equivalent manifolds are homeomorphic.
If $n \geq 3$ is odd, then there are examples of lens spaces that are homotopy equivalent but not simple homotopy equivalent.

Question

Are there n-manifolds that are homotopy equivalent but not simple homotopy equivalent if $n \geq 4$ is even?

Motivation: classifying manifolds

Definition

Let M and N be closed n-dimensional manifolds, and let W be a cobordism between M and N, ie. $\partial W = M \sqcup N$.

- $(W; M, N)$ is an *h-cobordism* if the inclusions $M \to W$ and $N \to W$ are homotopy equivalences.
- $(W; M, N)$ is an *s-cobordism* if the inclusions $M \to W$ and $N \to W$ are simple homotopy equivalences.

Theorem (s-cobordism theorem)

If $n \geq 5$ and $(W; M, N)$ is an s-cobordism, then $W \approx M \times I$, therefore $M \approx N$.

The s-cobordism theorem also holds for $n = 4$ in the topological category (Freedman).

The Whitehead torsion

Theorem (Whitehead)

Let $f : X \to Y$ be a homotopy equivalence between finite CW complexes. Then there is an invariant, the Whitehead torsion

$$\tau(f) \in \mathrm{Wh}(\pi_1(Y))$$

such that

$$f \text{ is simple} \iff \tau(f) = 0.$$

The Whitehead group

Definition

Let G be a group. Its Whitehead group $\mathrm{Wh}(G)$ is defined as follows:

- $\mathbb{Z}G$ denotes the group ring
- $GL_n(\mathbb{Z}G) = \{\text{invertible } n \times n \text{ matrices over } \mathbb{Z}G\}$
- $GL_n(\mathbb{Z}G) \to GL_{n+1}(\mathbb{Z}G)$, $A \mapsto \left(\begin{smallmatrix} A & 0 \\ 0 & 1 \end{smallmatrix}\right)$
- $GL(\mathbb{Z}G) = \mathrm{dirlim}_n \, GL_n(\mathbb{Z}G)$.
- $\mathrm{Wh}(G) = GL(\mathbb{Z}G)/(E(\mathbb{Z}G), \pm G)$

where $E(\mathbb{Z}G)$ is the set of elementary matrices (ie. the identity matrix with one non-zero off-diagonal entry) and $\pm G \subset GL_1(\mathbb{Z}G)$.

Proposition

$\mathrm{Wh} : Grp \to Ab$ *is a (covariant) functor.*

The Whitehead torsion

Definition (Special case)

Let $f : X \to Y$ be a homotopy equivalence between finite CW complexes with $\pi_1(X) = \pi_1(Y) = G$.

- Let $C_*(X;\mathbb{Z}G)$ and $C_*(Y;\mathbb{Z}G)$ denote the cellular chain complexes of X and Y with $\mathbb{Z}G$ coefficients. The cells of X and Y determine bases in $C_*(X;\mathbb{Z}G)$ and $C_*(Y;\mathbb{Z}G)$.
- f induces a chain homotopy equivalence $f_* : C_*(X;\mathbb{Z}G) \to C_*(Y;\mathbb{Z}G)$.
- Assume that $f_i : C_i(X;\mathbb{Z}G) \to C_i(Y;\mathbb{Z}G)$ is an isomorphism for every i. Then it is given by an invertible matrix $f_i \in GL(\mathbb{Z}G)$.
- In this special case

$$\tau(f) = \sum_i (-1)^i [f_i] \in \mathrm{Wh}(G)$$

The Whitehead group - examples

Example

$\mathrm{Wh}(\{e\}) \cong 0$.

Every homotopy equivalence between simply-connected CW complexes is simple.

Example

- $\mathrm{Wh}(\mathbb{Z}^n) \cong 0$.
- $\mathrm{Wh}(F_n) \cong 0$.
- $\mathrm{Wh}(\mathbb{Z}_m) \cong \mathbb{Z}^{\lfloor m/2 \rfloor + 1 - \delta(m)}$, *where* $\delta(m)$ *is the number of positive integers dividing* m.

Lens spaces

Definition

- Let $2k-1 \geq 3$, $m \geq 2$ and $q_1, \ldots, q_k$ be positive integers such that $gcd(m, q_j) = 1$.
- $S^{2k-1} = \{(z_1, \ldots, z_k) \mid \sum_{j=1}^{k} |z_j|^2 = 1\} \subset \mathbb{C}^k$.
- The cyclic group $\mathbb{Z}_m$ acts freely on S^{2k-1}, the generator acts by $(z_1, \ldots, z_k) \mapsto (\zeta^{q_1} z_1, \ldots, \zeta^{q_k} z_k)$, where $\zeta = e^{2\pi i/m}$.
- $L_{2k-1}(m; q_1, \ldots, q_k) = S^{2k-1}/\mathbb{Z}_m$.

$\pi_1(L_{2k-1}(m; q_1, \ldots, q_k)) \cong \mathbb{Z}_m$.

Proposition

$L_3(7; 1, 1)$ and $L_3(7; 2, 1)$ are homotopy equivalent but not simple homotopy equivalent.

Even-dimensional manifolds

Proposition

$S^1 \times L_3(7;1,1)$ and $S^1 \times L_3(7;2,1)$ are simple homotopy equivalent.

Proposition

If M, M', N, N' are odd-dimensional manifolds such that $M \simeq M'$ and $N \simeq N'$, then $M \times N$ is simple homotopy equivalent to $M' \times N'$.

Proof.

Let $f : M \to M'$ and $g : N \to N'$ be homotopy equivalences. Take $f \times g : M \times N \to M' \times N'$. Then

$$\tau(f \times g) = \chi(N') \cdot i_*(\tau(f)) + \chi(M') \cdot j_*(\tau(g)) = 0$$

where $i : \pi_1(M') \to \pi_1(M' \times N')$ and $j : \pi_1(N') \to \pi_1(M' \times N')$ are the inclusions. □

Idea of the construction

Goal: find n-manifolds M, N that are homotopy equivalent but not simple homotopy equivalent.

- Fix a specially chosen M.
- Construct a homotopy equivalence $f : N \to M$ which is not simple. This depends on $\mathrm{Wh}(\pi_1(M))$.
- Show that no other homotopy equivalence $g : N \to M$ can be simple. This depends on $\mathrm{hAut}(M)$.

If $g : N \to M$ is a simple homotopy equivalence, then $f \circ g^{-1} : M \to M$ is a homotopy automorphism of M with $\tau(f \circ g^{-1}) = \tau(f)$.

The s-cobordism theorem

Theorem (s-cobordism theorem, strong version)

If $n \geq 5$ and M is an n-manifold with $\pi_1(M) = G$, then for every $x \in \mathrm{Wh}(G)$ there is a (unique) h-cobordism $(W; M, N)$ with $\tau(M \to W) = x$.

Proposition

The homotopy equivalences $M \to W \leftarrow N$ determine a homotopy equivalence $N \to M$, and we have

$$\tau(N \to M) = (-1)^n \bar{x} - x$$

where $x \mapsto \bar{x}$ is a naturally defined involution on $\mathrm{Wh}(G)$.

Corollary

If $n \geq 5$ and M is an n-manifold with $\pi_1(M) = G$, then for every element of the form $(-1)^n \bar{x} - x \in \mathrm{Wh}(G)$ there is a homotopy equivalence $N \to M$ with $\tau(N \to M) = (-1)^n \bar{x} - x$.

Examples from lens spaces

Let $M = S^1 \times L$ for some lens space $L = L_{2k-1}(m; q_1, \dots, q_k)$.

Theorem (N-Nicholson-Powell)

If $m \notin \{2, 3, 5, 6, 7, 10, 11, 13, 14, 15, 17, 19, 29\}$, then $\{\bar{x} - x \mid x \in \mathrm{Wh}(\mathbb{Z} \oplus \mathbb{Z}_m)\} \neq 0$.

Theorem (N-Nicholson-Powell)

Every homotopy automorphism of $S^1 \times L$ is simple.

Theorem (N-Nicholson-Powell)

If $m \notin \{2, 3, 5, 6, 7, 10, 11, 13, 14, 15, 17, 19, 29\}$, then there is a $2k$-manifold N that is homotopy equivalent (h-cobordant) but not simple homotopy equivalent to $S^1 \times L$.

Examples with other fundamental groups

Proposition

Let M and N be n-manifolds with fundamental group G. Suppose that $\{(-1)^n\bar{x} - x \mid x \in \mathrm{Wh}(G)\} = 0$. If M and N are h-cobordant, then they are simple homotopy equivalent.

Proof.

An h-cobordism $(W; M, N)$ determines a homotopy equivalence $f : N \to M$ with

$$\tau(f) = (-1)^n\bar{x} - x$$

where $x = \tau(M \to W)$. □

Examples with other fundamental groups

Theorem (N-Nicholson-Powell)

Let $S^j \to M \to K$ be an orientable sphere bundle, where

- *K is a k-manifold*
- *$j > k$ and j is odd*

Then every homotopy automorphism of M is simple.

If $k \geq 4$, then every group can be realised as $\pi_1(M) = \pi_1(K)$.

Theorem (N-Nicholson-Powell)

Let $n \geq 11$ or $n = 9$. Let G be a finitely presented group. Then the following are equivalent:

- *There is a pair of n-manifolds with fundamental group G that are h-cobordant but not simple homotopy equivalent.*
- $\{(-1)^n \bar{x} - x \mid x \in \mathrm{Wh}(G)\} \neq 0$.