The Sullivan-conjecture in complex dimension 4

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Outline

1. Introduction to complete intersections
2. Statement of the Sullivan-conjecture on the classification of complete intersections
3. Proof of the Sullivan-conjecture in complex dimension 4

Complete intersections - definition

Definition

A *complete intersection* of (complex) dimension n is the transverse intersection of k algebraic hypersurfaces in $\mathbb{C}P^{n+k}$ (for some k). In particular, it is a smooth complex projective variety.

Definition

If a complete intersection is defined by hypersurfaces of degrees $d_1, d_2, \ldots, d_k$, then

- its *multidegree* is $\underline{d} = (d_1, d_2, \ldots, d_k)$
- its *total degree* is $d = d_1 d_2 \ldots d_k$

Complete intersections - examples

Example

Any nonsingular algebraic hypersurface is a complete intersection. For example, the equation

$$z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0$$

determines a smooth degree-4 hypersurface in $\mathbb{C}P^3$ *(a K3 surface).*

Counterexample

The set of points $[z_0 : z_1 : z_2 : z_3] \in \mathbb{C}P^3$ *such that*

$$z_1^2 - z_0 z_2 = 0$$
$$z_2^2 - z_1 z_3 = 0$$
$$z_0 z_3 - z_1 z_2 = 0$$

is a smooth projective variety (diffeomorphic to $\mathbb{C}P^1 = S^2$*), but not a complete intersection.*

Complete intersections - Thom's theorem

Theorem (Thom)

The diffeomorphism class of a complete intersection depends only on its multidegree.

Proof.

The space of tuples of polynomials $(f_1, \ldots, f_k)$ of degrees $(d_1, \ldots, d_k)$ that define complete intersections is connected. □

In fact, two complete intersections with the same multidegree are deformation equivalent.

Definition

The n-dimensional complete intersection with multidegree $\underline{d}$ is denoted by $X_n(\underline{d})$.

Complete intersections - examples

Example

$X_n(1) = \mathbb{C}P^n$.

Example

$X_1(d_1, d_2, \ldots, d_k)$ *is an oriented surface of genus*

$$\frac{2 - d(2 + k - \sum_{i=1}^{k} d_i)}{2}$$

Example

$X_2(4)$, $X_2(3,2)$ *and* $X_2(2,2,2)$ *are K3 surfaces.*

Complete intersections - cohomology

Proposition

The inclusion $j : X_n(\underline{d}) \to \mathbb{C}P^{n+k}$ is n-connected.
That is, $\pi_i(j) : \pi_i(X_n(\underline{d})) \to \pi_i(\mathbb{C}P^{n+k})$ is an isomorphism if $i < n$, and $\pi_n(j) : \pi_n(X_n(\underline{d})) \to \pi_n(\mathbb{C}P^{n+k})$ is surjective.

Proof.

Application of the Lefschetz hyperplane theorem. □

Corollaries

- *If $n \geq 2$, then $X_n(\underline{d})$ is simply-connected.*
- $$H^i(X_n(\underline{d})) \cong \begin{cases} \mathbb{Z} & \text{if } i \neq n \text{ is even} \\ 0 & \text{if } i \neq n \text{ is odd} \\ \text{free Abelian} & \text{if } i = n \end{cases}$$

Complete intersections - cohomology

Definition

Let $y \in H^2(\mathbb{C}P^{n+k})$ be a generator and $x = j^*(y) \in H^2(X_n(\underline{d}))$.

Proposition

- *If* $2i < n$, *then* x^i *is a generator of* $H^{2i}(X_n(\underline{d})) \cong \mathbb{Z}$.
- $x^n = d \cdot \mathit{generator} \in H^{2n}(X_n(\underline{d})) \cong \mathbb{Z}$.

Proof.

A degree-d_i hypersurface in $\mathbb{C}P^{n+k}$ represents the Poincaré-dual of $d_i y$, so $X_n(\underline{d})$ represents the dual of dy^k. □

Corollary

If $2 < n$, *then* d *is a homotopy invariant of* $X_n(\underline{d})$.

Complete intersections - normal bundle

Definition

Let $H \to \mathbb{C}P^{n+k}$ denote the complex conjugate of the tautological line bundle ($c_1(H) = y$).

Proposition

The stable normal bundle of $\mathbb{C}P^{n+k}$ *is* $\nu_{\mathbb{C}P^{n+k}} \cong -(n+k+1)H$*. The normal bundle of a degree-*d_i *hypersurface in* $\mathbb{C}P^{n+k}$ *is* $H^{\otimes d_i}$*.*

Proposition

The stable normal bundle of $X_n(\underline{d})$ *is the pullback of*

$$-(n+k+1)H \oplus H^{\otimes d_1} \oplus H^{\otimes d_2} \oplus \ldots \oplus H^{\otimes d_k}$$

Complete intersections - normal bundle

Proposition

The stable normal bundle of $X_n(\underline{d})$ is the pullback of

$$-(n+k+1)H \oplus H^{\otimes d_1} \oplus H^{\otimes d_2} \oplus \ldots \oplus H^{\otimes d_k}$$

Proposition

$$c(\nu_{X_n(\underline{d})}) = (1+x)^{-(n+k+1)} \prod_{i=1}^{k} (1+d_i x)$$

$$p(\nu_{X_n(\underline{d})}) = (1-x^2)^{-(n+k+1)} \prod_{i=1}^{k} (1-d_i^2 x^2)$$

The Sullivan-conjecture - motivation

Proposition

The converse of Thom's theorem is false. That is, diffeomorphic complete intersections may have different multidegree.

Example (Libgober-Wood '82)

$X_3(16, 10, 7, 7, 2, 2, 2) \approx X_3(14, 14, 5, 4, 4, 4)$

Example (Crowley-N. '19)

$X_4(3^{(150)}, 7^{(89)}, 9^{(65)}, 15^{(1)}, 25^{(130)}) \approx X_4(5^{(261)}, 21^{(89)}, 27^{(64)})$

Proposition (Libgober-Wood '82)

In dimension $n > 2$, if two complete intersections are deformation equivalent, then they have the same multidegree.

Conjecture

Let $n \geq 3$. Two n-dimensional complete intersections are diffeomorphic if and only if they have the same

- *total degree*
- *Pontryagin-classes*
- *Euler-characteristic*

- p_i is a multiple of x^{2i}, so Pontryagin-classes can be compared.
- These conditions are necessary.

The Sullivan-conjecture - results

Theorem

The Sullivan-conjecture holds for $n = 3$.

This follows from the classification of simply-connected 6-manifolds with torsion-free homology done by Wall ('66) and Jupp ('73).

Theorem (Fang-Klaus '96, Fang-Wang '10)

The Sullivan-conjecture holds up to homeomorphism for $4 \leq n \leq 7$.

If $n = 4$, then the conjecture holds smoothly up to connected sum with a homotopy 8-sphere.

Theorem (Baraglia '20)

If $n = 4k + 1$, $X_n(\underline{d})$ and $X_n(\underline{d}')$ are spin, and they have the same invariants, then $\alpha(X_n(\underline{d})) = \alpha(X_n(\underline{d}'))$.

The Sullivan-conjecture - results

Theorem (Kreck-Traving '96)

In dimension n, the Sullivan-conjecture holds if the total degree is divisible by $p^{\lceil (2n+2p-1)/(2p-2) \rceil}$ *for every prime p with* $p(p-1) \leq n+1$.

For $n = 4$ the total degree must be divisible by 2^6.

Theorem (Crowley-N. '18)

The Sullivan-conjecture holds for $n = 4$.

Proof - homotopy spheres

Definition

Θ_n denotes the group of homotopy n-spheres:

- Its elements are smooth n-manifolds homeomorphic to S^n, up to orientation-preserving diffeomorphism.
- The group operation is connected sum.

Theorem (Kervaire-Milnor '63)

$\Theta_{4k} \cong \operatorname{Coker}\left(J_{4k} : \pi_{4k}(SO) \to \pi_{4k}^S\right) \quad$ *if* $k > 1$.

Definition

Σ^8 denotes the non-trivial element of $\Theta_8 \cong \mathbb{Z}_2$.

Proof - cases

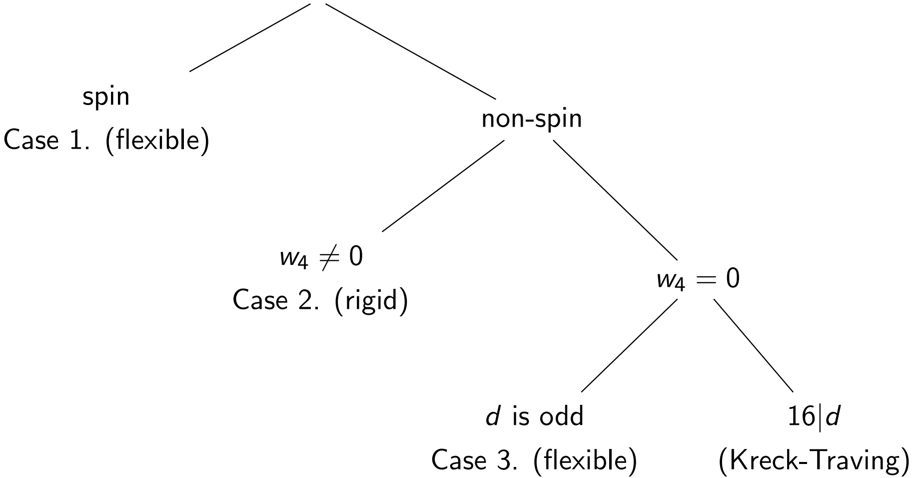
By Fang-Klaus, for $n = 4$ the conjecture fails only if there are complete intersections $X = X_4(\underline{d})$, $Y = X_4(\underline{d}')$ such that

- Y is diffeomorphic to $X \# \Sigma^8$, and
- X is not diffeomorphic to $X \# \Sigma^8$

So for every $X_4(\underline{d})$ we need to prove that either

- $X_4(\underline{d})$ is diffeomorphic to $X_4(\underline{d}) \# \Sigma^8$ (rigid case), or
- $X_4(\underline{d})$ is not diffeomorphic to $X_4(\underline{d}) \# \Sigma^8$, and $X_4(\underline{d}) \# \Sigma^8$ is not a complete intersection (flexible case)

Proof - cases


spin
Case 1. (flexible)
non-spin
$w_4 \neq 0$
Case 2. (rigid)
$w_4 = 0$
d is odd
Case 3. (flexible)
$16|d$
(Kreck-Traving)

Proof - flexible cases

Example (Kasilingam)

$\mathbb{C}P^4$ is flexible (ie. $\mathbb{C}P^4 \not\approx \mathbb{C}P^4 \# \Sigma^8$).

If $X_4(\underline{d})$ is flexible, then we need to show that $X_4(\underline{d}) \# \Sigma^8$ is not diffeomorphic to any complete intersection.

A way to do this is finding a special property that complete intersections have, but $X_4(\underline{d}) \# \Sigma^8$ doesn't.

Candidate property: every complete intersection is a smooth complex projective variety, therefore Kähler.

Question

Does $X_4(\underline{d}) \# \Sigma^8$ have a Kähler metric?

Proof - Case 1: spin (flexible)

Proposition

The S^1-bundle over a complete intersection $X_n(\underline{d})$ with first Chern class $c_1 = x$ has a framing such that it is framed nullcobordant.

Proof.

$X_n(\underline{d})$ is the transverse intersection of $X_{n+1}(\underline{d})$ and $\mathbb{C}P^{n+k}$ in $\mathbb{C}P^{n+k+1}$:

$$\begin{array}{ccc} \mathbb{C}P^{n+k} & \longrightarrow & \mathbb{C}P^{n+k+1} \\ \uparrow & & \uparrow \\ X_n(\underline{d}) & \longrightarrow & X_{n+1}(\underline{d}) \end{array}$$

- $X_{n+1}(\underline{d}) \setminus X_n(\underline{d})$ is stably parallelizable.
- The normal bundle of $X_n(\underline{d}) \subset X_{n+1}(\underline{d})$ has $c_1 = x$.

□

Proof - Case 1: spin (flexible)

Proposition

If $X_4(\underline{d})$ is spin, then the S^1-bundle over $X_4(\underline{d})\#\Sigma^8$ with $c_1 = x$ is not framed nullcobordant with any framing.

Outline of proof.

- The framed cobordism class of the S^1-bundle changes by $\Sigma^8 \times S^1$, with some framing.
- The framed cobordism class of $\Sigma^8 \times S^1$ depends on the framing of the S^1.
- The framings of S^1 are in bijection with $\pi_1(SO) \cong \pi_2(BSO) \cong \mathbb{Z}_2$, detected by w_2.
- If $w_2(X_4(\underline{d})) = 0$, then $\Sigma^8 \times S^1$ is not framed nullcobordant.
- The framed cobordism class of the S^1-bundle remains non-zero if we change the framing.

□

Proof - non-spin cases

The main tool in these cases is Kreck's modified surgery theory.
Setting:

- M_0 and M_1 are $2q$-manifolds, $q \geq 3$
- W is a cobordism between M_0 and M_1
- B is a fixed space with a bundle ξ over it
- $F : W \to B$ is a *normal map*: it is covered by a bundle map $\nu_W \to \xi$
- $\chi(M_0) = \chi(M_1)$
- f_0 and f_1 are q-connected, where $f_i = F\big|_{M_i}$

In this setting an obstruction $\theta(W, F)$ is defined.
$\theta(W, F)$ is *elementary* $\Leftrightarrow$ W can be replaced by an h-cobordism
If M_0 is a complete intersection*, then $\theta(W, F)$ is elementary.

Proof - Case 2: non-spin, $w_4 \neq 0$ (rigid)

Proposition

If $X_4(\underline{d})$ is non-spin and $w_4(X_4(\underline{d})) \neq 0$, then $X_4(\underline{d}) \approx X_4(\underline{d}) \# \Sigma^8$.

We apply modified surgery in the following setting:

- $B = \mathbb{C}P^\infty \times BString$ is the normal 3-type of $X_4(\underline{d})$
- $\xi = -5H \oplus H^{\otimes d_1} \oplus \ldots \oplus H^{\otimes d_k}$ is the bundle over $\mathbb{C}P^\infty$
- $f_0 : X_4(\underline{d}) \to B$ is the embedding $X_4(\underline{d}) \to \mathbb{C}P^\infty$
- $f_1 : X_4(\underline{d}) \# \Sigma^8 \to B$ is f_0 precomposed with a homeomorphism
- Needed: f_0 and f_1 are normally bordant

Proposition

The map $i : \mathbb{Z}_2 \cong \Theta_8 \to \Omega_8^{String}(\mathbb{C}P^\infty, \xi)$ is trivial

Proof - Case 2: non-spin, $w_4 \neq 0$ (rigid)

Definition

- Let T be the non-trivial D^6-bundle over S^2.
- $\Theta_8(S^2, 1) = \{g \in \mathrm{Diff}(\partial T) \mid H_*(g) = \mathrm{id}\} / \text{isotopy}$

Theorem (Crowley-N.)

$\Theta_8(S^2, 1) \cong \mathbb{Z}_4$.

If $X_4(\underline{d})$ is non-spin, then T can be embedded in $X_4(\underline{d})$, so a diffeomorphism of ∂T determines a cut-and-reglue operation.

The effect on the bordism class is given by a homomorphism $\Theta_8(S^2, 1) \to \Omega_8^{String}(\mathbb{C}P^\infty, \xi)$.

Moreover, the map $i : \Theta_8 \to \Omega_8^{String}(\mathbb{C}P^\infty, \xi)$ is the composition $\mathbb{Z}_2 \to \mathbb{Z}_4 \cong \Theta_8(S^2, 1) \to \Omega_8^{String}(\mathbb{C}P^\infty, \xi)$.

Proof - Case 2: non-spin, $w_4 \neq 0$ (rigid)

Proposition (Fang-Klaus)

If $w_2(X_4(\underline{d})) \neq 0$ and $w_4(X_4(\underline{d})) \neq 0$, then

$$\operatorname{Tor}\Omega_8^{String}(\mathbb{C}P^\infty, \xi) = \operatorname{Im}\big(i : \Theta_8 \to \Omega_8^{String}(\mathbb{C}P^\infty, \xi)\big)$$

So we have the following commutative diagram:

$$\begin{array}{ccc} \mathbb{Z}_2 \cong \Theta_8 & \xrightarrow{\quad i \quad} & \operatorname{Im}(i) \subset \Omega_8^{String}(\mathbb{C}P^\infty, \xi) \\ & \searrow \qquad \nearrow & \\ & \mathbb{Z}_4 \cong \Theta_8(S^2, 1) & \end{array}$$

This implies that the map i is trivial.

Proof - Case 3: odd d (flexible)

In this case the map $\Theta_8 \to \Omega_8^{String}(\mathbb{C}P^\infty, \xi)$ is non-trivial.

Suppose that $X_4(\underline{d})$ and $X_4(\underline{d}')$ have the same invariants.
We apply modified surgery in the following setting:

- $B = \mathbb{C}P^\infty \times BString$ is the normal 3-type of $X_4(\underline{d})$
- $\xi = -5H \oplus H^{\otimes d_1} \oplus \ldots \oplus H^{\otimes d_k}$ is the bundle over $\mathbb{C}P^\infty$
- $f_0 : X_4(\underline{d}) \to B$ is the embedding $X_4(\underline{d}) \to \mathbb{C}P^\infty$
- $f_1 : X_4(\underline{d}') \to B$ is the embedding $X_4(\underline{d}') \to \mathbb{C}P^\infty$

Proposition

f_0 and f_1 are normally bordant.

Proof - Case 3: odd d (flexible)

The embedding $X_4(\underline{d}) \to \mathbb{C}P^\infty$ factors through a degree-d map $X_4(\underline{d}) \to \mathbb{C}P^4$, covered by a bundle map

$$\begin{array}{ccc} \nu_{X_4(\underline{d})} & \longrightarrow & -5H \oplus H^{d_1} \oplus \ldots \oplus H^{d_k} \\ \downarrow & & \downarrow \\ X_4(\underline{d}) & \longrightarrow & \mathbb{C}P^4 \end{array}$$

Proof - Case 3: odd d (flexible)

Definition

A degree-d normal map is a diagram

$$\begin{CD} \nu_N @>{\bar{f}}>> \xi \\ @VVV @VVV \\ N @>{f}>> M \end{CD}$$

where ξ is a stable vector bundle, $\bar{f}$ is a bundle-morphism, and $\deg(f) = d$.

Definition

$\mathcal{N}_d(M)$ denotes the set of cobordism classes of degree-d normal maps into M.
$\mathcal{N}_d(M, \xi)$ denotes the set of cobordism classes of degree-d normal maps into M, with fixed bundle ξ over M.

Proof - Case 3: odd d (flexible)

Definition

A fibrewise degree-d map is a diagram

$$\begin{array}{ccc} S(\eta) & \xrightarrow{\bar{f}} & S(\eta') \\ & \searrow \quad \swarrow & \\ & M & \end{array}$$

where $S(\eta)$ and $S(\eta')$ are the sphere bundles of η and η', $\bar{f}$ is a fibre-preserving map, and its restriction to any fibre has degree d.

Definition

$\mathcal{F}_d(M)$ denotes the set of fibrewise degree-d maps over M up to fibrewise homotopy equivalence and stabilization.
$\mathcal{F}_d(M, \xi)$ denotes the set of fibrewise degree-d maps over M such that $\nu_M \oplus \eta' \oplus (-\eta) \cong \xi$, up to fibrewise homotopy equivalence and stabilization.

Proof - Case 3: odd d (flexible)

Theorem (Brumfiel-Madsen '76)

There is a space $(QS^0/O)_d$ such that $\mathcal{F}_d(M) \cong [M, (QS^0/O)_d]$ for every M.

Theorem (Hambleton-Madsen '86)

There is a bijection between $\mathcal{F}_d(M)$ and $\mathcal{N}_d(M)$, that restricts to a bijection between $\mathcal{F}_d(M, \xi)$ and $\mathcal{N}_d(M, \xi)$.

The degree-d normal map

$$\left[X_4(\underline{d}) \to \mathbb{C}P^4\right] \in \mathcal{N}_d(\mathbb{C}P^4, \xi)$$

corresponds to the following fibrewise degree-d map over $\mathbb{C}P^4$:

$$\left[S(H \oplus H \ldots \oplus H) \to S(H^{d_1} \oplus H^{d_2} \oplus \ldots \oplus H^{d_k})\right] \in \mathcal{F}_d(\mathbb{C}P^4, \xi)$$

Proof - Case 3: odd d (flexible)

We have the following diagram:

$$
\begin{array}{ccccc}
\mathcal{F}_d(\mathbb{C}P^4,\xi) & \longleftrightarrow & \mathcal{N}_d(\mathbb{C}P^4,\xi) & \longrightarrow & \Omega_8^{String}(\mathbb{C}P^\infty,\xi) \\
\big\downarrow & & \big\downarrow & & \\
\mathcal{F}_d(\mathbb{C}P^4) & \xleftrightarrow{\text{H-M}} & \mathcal{N}_d(\mathbb{C}P^4) & & \\
\text{B-M}\big\updownarrow & & & & \\
[\mathbb{C}P^4,(QS^0/O)_d] & & & &
\end{array}
$$

Θ_8 acts on $\mathcal{N}_d(\mathbb{C}P^4)$, and hence on $[\mathbb{C}P^4,(QS^0/O)_d]$.
It is enough to show that elements of $[\mathbb{C}P^4,(QS^0/O)_d]$ coming from complete intersections cannot differ by the action Σ^8.
Since $\Theta_8 \cong \mathbb{Z}_2$, it is enough to prove this 2-locally.

Proof - Case 3: odd d (flexible)

Proposition

After localization at 2 we have the following homotopy equivalences:
$((QS^0/O)_d)_{(2)} \simeq (G/O)_{(2)} \simeq (BO)_{(2)} \times (\mathrm{Coker}\, J)_{(2)}$

The first equivalence was proved by Brumfiel and Madsen, and holds for d odd.
The second equivalence comes from Sullivan's solution of the Adams conjecture.

Proposition

If two complete intersections have the same invariants, then they represent the same element in $[\mathbb{C}P^4, BO]_{(2)} \times [\mathbb{C}P^4, \mathrm{Coker}\, J]_{(2)}$.