Study group: The Bloch–Kato conjecture on special values of L-functions

Organisers: Andrew Graham and Damián Gvirtz

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Introduction

In 1979, Deligne noticed how results for special values of L-functions in the number field and elliptic curve cases (the latter being the famous and still open BSD conjecture) could be generalised. He predicted the special value $L(M, n)$ of the L-function attached to a motif M, up to a rational factor, at certain "critical" $n \in \mathbb{Z}$ where the L-function has neither pole nor zero. Beilinson removed the criticality assumption with a conjecture on the leading term $L(M, n)^*$, again only up to a rational number.

The topic of this study group is the conjecture by Bloch and Kato which determines this rational number and thus offers a precise value for $L(M, n)^*$. In particular, we want to see what the conjecture says in special cases, e.g. the refined Birch and Swinnerton-Dyer conjecture.

Organisation

The study group meets every Wednesday from 12:00-13:30. A Zoom link will be distributed via the mailing list. An up to date version of this document can be found on the study group website:

Schedule

1 Motives (20/01) – Wojtek Wawrów

Introduce the category of Chow motives, its various cohomological realisation functors (de Rham, Betti, p-adic) and compatible systems of realisations. [Sch94a, Jan95]

2 Motivic cohomology (27/01) – Alex Torzewski

Treating the unknown category of mixed motives or the known derived category of Voevodsky as a black box, motivate motivic cohomology. Construct it via Bloch's higher Chow groups and discuss the spectral sequence converging to algebraic K-theory (without going too much into K-theory itself). Define regulator maps.

Possible references: [MVW06, Wei13, Sou81]

3 Bloch–Kato Selmer groups (03/02) – Muhammad Manji

This talk is about Bloch–Kato Selmer groups, the Euler-Poincaré characteristic and exponential maps.

Possible references: [Bel09, §2]

4 Rank part of Bloch–Kato (10/02) – Dan Gulotta

Discuss the rank part of the Bloch–Kato conjecture and give examples. The Selmer groups are known from before but you will need to explain the L-function.

Possible references: [Bel09, §3 and 4]

5 Beilinson's conjecture (17/02) – Darya Schedrina

Define the Deligne period map and state Beilinson's conjecture for weights ≤ -2.

Possible references: [Nek94, §2, 6 and 7 but not $w = -1$], [Sch88]

6 Height pairings and Beilinson's conjecture for weight -1 (24/02) – Rob Rockwood

Discuss height pairings (starting from the Néron-Tate height), their appearance in Beilinson's conjecture for weight -1 and the relationship to the refined BSD conjecture up to a rational factor.

Possible references: [Nek94, Sch94b, Bei87]

7 The fundamental line (03/03) – Ashwin Iyengar

Define the fundamental line and its canonical measure. Everyone should see the exact sequence in [Kin03, Conj. 1.1.2].

Possible references: [Fon92, §4]

8 Statement of Bloch–Kato (10/03) – Jef Laga

Using the previous talk, state the Bloch–Kato conjecture and what it says in different weight cases. Specialise to the cases of 1-motives and smooth varieties

Possible references: [Fon92, §8 and 9], [Kin03, §1]

9 Tamagawa numbers and Tate-Shafarevich groups (17/03) – Netan Dogra

Introduce Tamagawa numbers and Tate-Shafarevich groups. Relate the predicted leading terms of refined BSD and the Bloch–Kato conjecture.

Possible references: [Fon92, §11], [BK90]

10 Artin motives and further applications (24/03) – Pak-Hin Lee

Try to break down the conjecture as concretely as possible in known special cases (Artin motives, CM elliptic curves or adjoint motives to modular forms).

Possible references: [Fon92, §10], [Kin03, §2 and 3]

References

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[BK90] Spencer Bloch and Kazuya Kato. *L*-functions and Tamagawa numbers of motives. In *The Grothendieck Festschrift, Vol. I*, volume 86 of *Progr. Math.*, pages 333–400. Birkhäuser Boston, Boston, MA, 1990.

[Fon92] Jean-Marc Fontaine. Valeurs spéciales des fonctions *L* des motifs. Number 206, pages Exp. No. 751, 4, 205–249. 1992. Séminaire Bourbaki, Vol. 1991/92.

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[Kin03] Guido Kings. The Bloch-Kato conjecture on special values of *L*-functions. A survey of known results. volume 15, pages 179–198. 2003. Les XXIIèmes Journées Arithmetiques (Lille, 2001).

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[Sou81] Christophe Soulé. On higher p-adic regulators. In *Algebraic K-theory, Evanston 1980 (Proc. Conf., Northwestern Univ., Evanston, Ill., 1980)*, volume 854 of *Lecture Notes in Math.*, pages 372–401. Springer, Berlin-New York, 1981.

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