Algebraic Stacks

a horizontal introduction

Damián Gvirtz

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Contents

1. Introduction

In 1969, Pierre Deligne and David Mumford published *The irreducibility of the space of curves of given genus* ([DM69]), containing two proofs of irreducibility for the above moduli spaces. It is the second proof that creates an importance of their paper beyond the scope of its main theorem. Following prior ideas by Giraud and Grothendieck in this proof, Deligne and Mumford introduce the language of algebraic stacks.

Since then, the field of algebraic stacks has gained a reputation for being notoriously difficult. See for example [HM98], where Harris and Morrison write on page 139 et seq. about the appearance of stacks in mathematical talks and their decision to omit the topic:

> Who hasn't heard these words, or their equivalent, spoken in a talk? And who hasn't fantasized about grabbing the speaker by the lapels and shaking him until he says what – exactly – he means by them? But perhaps you're now thinking that all that is in the past, and that at long last you're going to learn what a stack is and what they do. Fat chance. [...]
>
> Well that's how it goes in theory, anyway. In practice, there is one respect in which the language of stacks isn't wholly perfect; it's difficult to understand even the definition of a stack (we're speaking strictly of ourselves here). Actually, once you've absorbed the basic definitions, the rest is not so bad, but there's no question that the initial learning curve is steep, not to say vertical.

Without doubt, one issue that might have created this image amongst mathematicians is the original paper itself: The fourth section in which stacks are defined is a mere collection of "some results on algebraic stacks" with the promise to give the proofs "elsewhere". The reader can guess what happened next...

So for years, there were no good sources giving a detailed account on this subject. At the time of this writing, the situation has improved a bit. There is one standard reference [LMB00] but this is not suitable for an introduction as it proceeds at a quick pace. Also, Laumon and Moret-Bailly have a general perspective and do not cover all the results needed by Deligne and Mumford. The same holds for the Stacks Project [Sta15], a vast online encyclopedia building its way from commutative algebra to algebraic stacks that is still under construction and hopefully will add many more interesting details in the next years. Apart from this, there is an unfinished, collaborative draft [Beh06] which aims at beginners, as well as sketchy online notes from a course by Olsson[1].

In this essay, I try to give a non-vertical but horizontal introduction to stacks. My slogan will be the usual one for me:

[1] `http://stacky.net/wiki/index.php?title=Course_notes`

> Imagine yourself back at the point in time when you started to learn about stacks. With that level of knowledge, what would have been the introduction to stacks that you wished to read but that unfortunately did not exist?

Of course, this approach is subjective as my prior knowledge may not be congruent with that of others wanting to learn the theory of stacks. For example, the theory of Grothendieck topologies and descent results have been known to me before, but recognising that this might rather be a coincidence I decided to include a short overview on those in the appendix. In the end, I hope that with the help of clarifying footnotes, the rest is digestible with the usual Hartshorne or Vakil graduate education in algebraic geometry and some basic knowledge of categories.

In my own experience, almost nothing in mathematics (except for false theorems) is as damaging as unmotivated definitions. Therefore, I am going to spend the next section on motivating the later formalism. My hope is that by explaining the basic problems and philosophy at the beginning, most of what follows will look like *the natural way to do it*.

The third section deals with the abstract groundwork to express moduli problems in a categorical way and illuminates how this formulation gives "presheaves in categories" while the fourth section deals with adding the right "sheaf conditions". In the fifth section, the bridge to algebraic geometry is constructed by imposing algebraicity conditions on the abstract stacks from the previous section. Having finally arrived at the definition of the objects we are working with, the sixth section develops algebro-geometric properties for stacks.

As the grand final, we will apply a couple of propositions from the preceding section to prove the irreducibility result of [DM69]. This will follow quickly as soon as we have established some properties of *stable curves*. Since these are not essential to the topic of algebraic stacks, I have included these properties with explanations and references in an additional appendix section.

Several important aspects of algebraic stacks are omitted. The first one is the useful connection to algebraic groupoids which is an emphasis of [Beh06]. Also missing are the theory of modules on stacks, Artin stacks (in modern terminology "algebraic stacks"[2]) and Artin's criteria for algebraicity. Because we are interested in applications of quasi-separated Deligne-Mumford stacks, we will not need to develop a theory of algebraic spaces which are objects more general than schemes but less general than stacks. (That we can make this restriction is explained after the definition of Deligne-Mumford stacks.) The reader is nevertheless encouraged to read on algebraic spaces since this facilitates the understanding of algebraic stacks.

Throughout the text, the term "scheme" will mean "scheme over a fixed base scheme Λ" and fibre products are by default taken over the terminal object Λ unless otherwise specified.

[2] After Deligne-Mumford, M. Artin came up with a more general definition of algebraic stacks. This definition is mentioned in Section 5 but not used in this text.

2. Motivation

What is a stack? The shortest (but oversimplifying) answer to this question that one might encounter is to say that it is a sheaf which takes values in groupoids, i.e. categories where every morphism is an isomorphism. The aim of this chapter is to motivate how such objects can naturally arise and to build up an intuition for what makes them interesting to work with.

Precise definitions will be given in the following chapters and the daring reader (or those who have an aversion against lack of formalism) may skip this motivation. For those, however, who are new to the world of stacks I hope to provide an anchor to which they can hold amidst the abstract nonsense that is going to follow.

2.1. A moduli problem for ordered triangles

When trying to learn the language of stacks, an additional obstacle is posed by the algebro-geometric baggage which comes with the theory. We seize a suggestion by M. Artin to remove this weight and to firstly concentrate on the essential observations that make the use of stacks necessary. Investigating moduli spaces for plane triangles is a good way to do so. (For a longer treatment of triangles and moduli, read [Beh14].)

Moduli problems ask for spaces that parametrise a collection of geometric objects up to isomorphism. This means that every point of the wanted moduli space should correspond to one isomorphism class and the geometry of the space should "reflect", when two objects are "close" to each other. Suppose we now want to find such a moduli space for plane triangles up to isometry.

A first natural approach is to identify every triangle (actually every isomorphism class) by its corresponding side lengths (a, b, c). Of course, only those triples satisfying the triangle equations give rise to a triangle. We therefore arrive at an open subset of $\mathbb{R}^3$

$$\tilde{T} = \{(a, b, c) \in \mathbb{R}^3 | a, b, c > 0, a + b > c, b + c > a, c + a > b\}$$

$\tilde{T}$ has the shape of an infinite pyramid. A two-dimensional cross section along $a+b+c = 2$, i.e. points corresponding to perimeter 2 triangles, is shown in Figure 2.1. (If we had considered triangles up to similarity, this would be the complete picture.)

Even with the vague notion used above, $\tilde{T}$ is not going to be a moduli space for plane triangles, since for every triangle which is non-equilateral permutations of the vertices give rise to several points in $\tilde{T}$ representing the same isometry class. The number of points in $\tilde{T}$ standing for a triangle equals the index of its automorphism group in S_3.

The collection of objects that is in fact parametrised by our open subset are the ordered plane triangles (for which we remember the side labels). In fact, the geometry of $\tilde{T}$ goes beyond our simple bijection between points and isomorphism classes, and this additional geometric information really is what moduli problems ask for. It tells us which continuous families of ordered triangles exist:

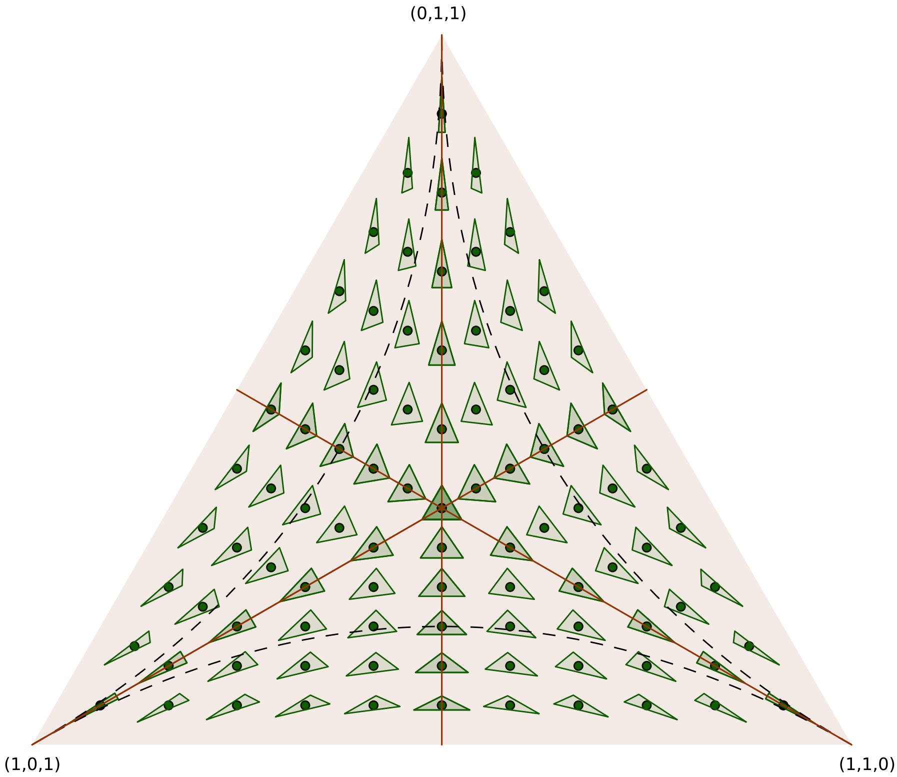

Figure 2.1: $\tilde{T}$ cut along perimeter 2 together with some chosen points and their corresponding triangles where a is the base, b the right side and c the left side. The solid lines indicate isosceles triangles, the dashed lines rectangular triangles.

Definition 2.1. A *family of triangles* is a (proper[3]) continuous fibre bundle map $\pi : \mathcal{S} \to S$ of topological spaces[4], together with a continuously varying metric[5] on the fibres such that each fibre is isometric to a plane triangle.

A *family of ordered triangles* is a family of triangles together with three sections $A, B, C : S \to \mathcal{S}$ (i.e. $\pi \circ A = 1_S$ and the same for B, C) which cut out exactly the triangle vertices in each fibre.

Sometimes the base may be omitted in the notation when it is implicitly clear.

[3]Properness is not really needed here since for a fibre bundle it is equivalent to the quasi-compactness of the fibres.

[4]This means there is a space F called the *fibre* and S can be covered by *trivialising opens* U s.t. there exist isomorphisms $\phi_U : \pi^{-1}(U) \cong U \times F$ with $\pi_1 \circ \phi_U = \pi|_U$ where π_1 is the first projection map.

[5]a continuous map $d : \mathcal{S} \times \mathcal{S} \to \mathbb{R}_{\geq 0}$ whose restriction to the fibre $\mathcal{S}_p := \pi^{-1}(p) \cong F$ at each point p is a metric

We can easily get families of ordered triangles by drawing paths in $\tilde{T}$ as in Figure 2.2. Furthermore, $\tilde{T}$ itself is the parametrising base of a family $\tilde{\mathcal{T}} \to \tilde{T}$ of ordered triangles which was sketched in Figure 2.1 where the triangles vary continuously along the fibres. ($\tilde{\mathcal{T}}$ could be realised as a subspace of $\mathbb{R}^3 \times \mathbb{R}^2$.)

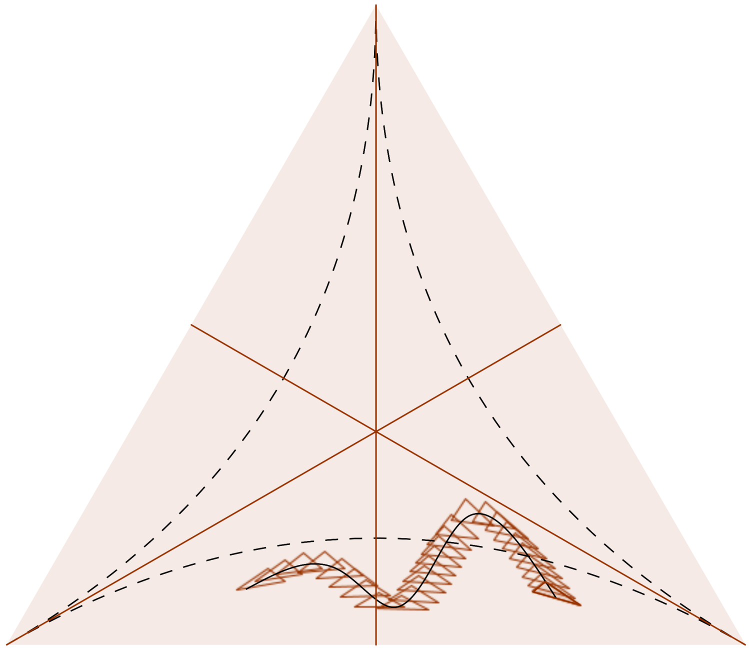

Figure 2.2: A path in $\tilde{T}$ giving rise to a family of ordered triangles.

The existence of the family $\tilde{\mathcal{T}}$ enables us to get families over any base S with a map $S \to \tilde{T}$ by pulling back along $S \to \tilde{T}$. In the case of a continuous path $[0,1] \to \tilde{T}$, this gives us families like the example above. Because the concept of pullbacks is so essential to stacks, we remind the reader of their categorical definition.

Definition 2.2. A commutative diagram in any category

$$\begin{tikzcd} X \times_Z Y \arrow[r] \arrow[d] & X \arrow[d, "g"] \\ Y \arrow[r, "f"] & Z \end{tikzcd}$$

is called *cartesian* if it satisfies the following universal property:

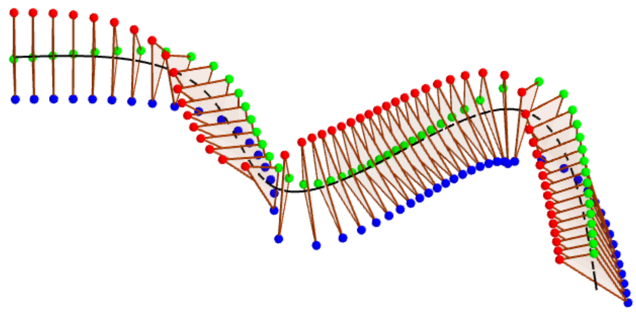

Figure 2.3: The family from Fig. Figure 2.2

For any T with commuting

$$\begin{tikzcd} T \arrow[r] \arrow[d] & X \arrow[d, "g"] \\ Y \arrow[r, "f"] & Z \end{tikzcd}$$

there is a unique map $T \to X \times_Z Y$ such that the following diagram commutes:

$$\begin{tikzcd} T \arrow[dr] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & X \times_Z Y \arrow[r] \arrow[d] & X \arrow[d, "g"] \\ & Y \arrow[r, "f"] & Z \end{tikzcd}$$

In this case $X \times_Z Y$ is called *fibre product of X and Y over Z*, *pullback* (written f^*X if g is implicitly clear) or *base change* of g along f. It is unique up to canonical isomorphism, if it exists.

For topological spaces the fibre product is realised by $\{(x, y) \in X \times Y | g(x) = f(y)\}$ (the fibre product of the spaces seen as sets) equipped with the obvious projections and the subspace topology of the product topology. It takes fibre bundles to fibre bundles preserving the fibres, hence the pullback of a family of triangles is a family again. Pulling back a family along the inclusion of an open set is the same as restricting the family to that set.

On the other hand, given a family of ordered triangles $\mathcal{S} \to S$, this always induces a so called *moduli map* of the base spaces $f : S \to \tilde{T}$ mapping $p \in S$ to the point corresponding to the isomorphism class of the fibre $\mathcal{S}_p$. One can see that the moduli map is continuous because the lengths of the triangle sides, which are the coordinates of $f(s) \in \tilde{T}$, vary continuously in the family $\mathcal{S} \to S$.

However, the crucial point here is that this moduli map determines the family $\mathcal{S}$. It can be recovered as being isomorphic[6] to the pullback $f^*\tilde{\mathcal{T}}$: Given any trivialising open U, there is only one canonical isomorphism of families $\mathcal{S}|_U \to f^*\tilde{\mathcal{T}}|_U$ over U because the sections A, B, C leave only one way to isometrically identify the fibres. As so often with this sort of problem, the uniqueness on the local level tells us that the local isomorphisms agree on overlaps and allows us to glue to a global isomorphism of families.

Later, we are going to encounter gluing of isomorphisms of families as the defining property of a prestack. The property needed to turn a prestack into a stack can also be observed in this example, it is the gluing (or *descent*) of families: Let $(U_i)_{i\in I}$ be a cover of S with open sets and $\mathcal{U}_i \to U_i$ be families with isomorphisms $\phi_{ij} : \mathcal{U}_i|_{U_{ij}} \to \mathcal{U}_j|_{U_{ij}}$ on the overlaps $U_{ij} := U_i \cap U_j$ satisfying the *cocycle condition* $\phi_{jk}\phi_{ij} = \phi_{ik}$. Then we can glue these families uniquely (up to isomorphism) to a family $\mathcal{S} \to S$ and isomorphisms of families $\phi_i : \mathcal{S}|_{U_i} \to \mathcal{U}_i$ with *transition functions* $\phi'_{ij} := \phi_j\phi_i^{-1}|_{U_{ij}} = \phi_{ij}$.

Concluding this subsection, we remark that we have found a topological space $\tilde{T}$ classifying continuous families of ordered triangles: It is the base of a *universal family* $\tilde{\mathcal{T}} \to \tilde{T}$ s.t. every family of ordered triangles $\mathcal{S} \to S$ arises as pullback from a canonical moduli map $S \to \tilde{T}$. Such a space is called *fine moduli space.*

2.2. A moduli problem for triangles

The main ingredient for $\tilde{T}$ to be a fine moduli space was the absence of non-trivial automorphisms. The fibres of families, which were ordered triangles, could be identified in one and only one way due to the remembered order. Now, if we return to our original interest, unordered triangles, we may worry that the existence of non-trivial automorphisms for isosceles triangles creates issues.

Indeed, these additional automorphisms render fine moduli spaces impossible. To see this, consider the two *isotrivial* families over S^1 as in Figure 2.4. Their fibres over all points are isometric to an equilateral triangle. The first family is the trivial one, the second rotates the fibre by 120° in one revolution (which is an automorphism of equilateral triangles). Their moduli maps to a moduli space will be constant. But only the first family is trivial and the pullback along the constant map. Thus, it is the second family which contradicts the existence of a fine moduli space.[7]

[6] as fibre bundles with the isomorphism inducing an isometry on fibres and being compatible in the obvious way with the sections labelling the vertices

[7] Note that the second family cannot be given the structure of a family of ordered triangles since it is not possible to choose a consistent labelling of vertices/edges.

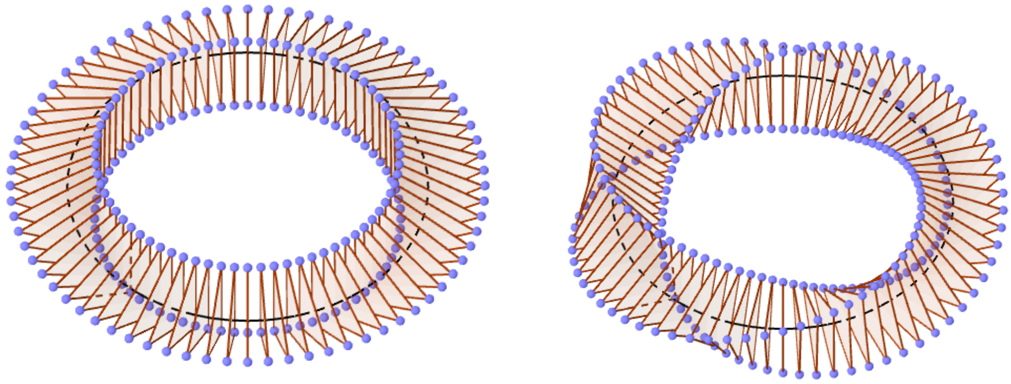

Figure 2.4: Two isotrivial families of triangles.

When mathematicians say that stacks solve the "problem of too many automorphisms", they are referring to the above problem. A point like the one corresponding to an equilateral triangles may have non-trivial automorphisms and stacks are designed to remember those.

A first naive approach to the moduli problem for plane unordered triangles may start with the fine moduli space $\tilde{T}$ and take the quotient with respect of the S_3-action permuting the labels. What this does to Figure 2.1 is identifying the six smaller triangles of a barycentric subdivision of $\tilde{T}$. (Each of them corresponds to an ordering of the sides, e.g. $a \leq b \leq c$.) Therefore, the result T (cross-sectioned along an isoperimeter plane) looks like one of these subtriangles and we do also get a quotient family $\mathcal{T}$ over T (see Figure 2.5). Of course, T cannot be a fine moduli space for triangles and $\mathcal{T}$ not a universal family, as shown above.

However, T is a so-called *coarse moduli space*: The points of T stand in bijection with isomorphism classes of objects (triangles in our case) and for any family of triangles $\mathcal{S} \to S$ we still get a continuous moduli map $S \to T$ with the same construction as before. Moreover, T is initial among such spaces. For any other space T' coming with a moduli map for any family, there is a continuous map $T \to T'$ such that the moduli map of any family into T' is given by composing the moduli map into T with $T \to T'$. This universal property implies the uniqueness of coarse moduli spaces. In our case, the map $T \to T'$ is given by the moduli map of the family $\mathcal{T} \to T$.

As a final note, we could have developed the previous moduli problem with smooth families over manifolds. In this case, $\tilde{T}$ would again be the fine moduli space and carry the structure of a smooth manifold. The quotient T on the other hand suddenly

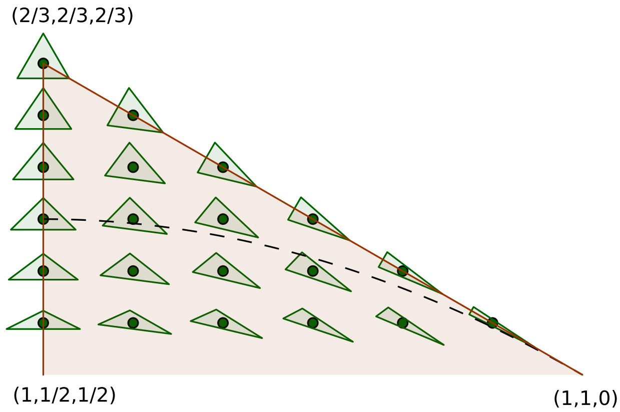

Figure 2.5: $T := \tilde{T}/S_3$ cut along $a+b+c=2$. The base is not in T, but the other two sides are.

exhibits a boundary and singular behaviour (which we could correct of course by a different parametrisation). Later on, the reader will understand that the quotient stack $[\tilde{T}/S_3]$ is automatically smooth. This hints at a great advantage of stacks. In cases, where the coarse moduli space is singular (as it happens with moduli of curves), the stack will be smooth and thus the better object to work with.

2.3. Grothendieck's philosophy of the functor of points

In this last motivational subsection, we reformulate our observations in a categorical framework. Grothendieck's philosophy was that instead of trying to understand an object C in a category $\mathfrak{C}$, one should try to understand the contravariant *functor of points* $h_C := C(\_) := \mathrm{Hom}_{\mathfrak{C}}(\_, C) : \mathfrak{C} \to \mathfrak{Sets}$. Denote the category of contravariant functors to $\mathfrak{Sets}$, i.e. set-valued presheaves, with natural transformations as morphisms by $PSh(\mathfrak{C})$. No information is lost by our replacement as the Yoneda lemma tells us:

Lemma 2.3 (Yoneda lemma)**.** *Let $\mathfrak{C}$ be a category and C an object therein. Then for any $F \in PSh(\mathfrak{C})$ there is a natural*[8] *isomorphism*

$$\mathrm{Hom}(h_C, F) \cong F(C)$$

Through the special case of $F = h_B$ for another object $B \in \mathfrak{C}$, we get the statement that $h : \mathfrak{C} \to PSh(\mathfrak{C}), C \mapsto h_C$ is a fully faithful embedding of $\mathfrak{C}$ into $PSh(\mathfrak{C})$. The interesting question to ask is which presheaves are isomorphic to h_C for a

[8] natural in F and C, if we see both sides as functor from $PSh(\mathfrak{C}) \times \mathfrak{C}$

suitable object C, or in other terms: Which contravariant functors $F \in PSh(\mathfrak{C})$ are *representable*? Due to Yoneda, a representing C always is unique up to isomorphism.

Solving the question of representability is strongly connected to our previous search for fine moduli spaces. Having defined a notion of families over a base in $\mathfrak{C}$ and a notion of isomorphism for families which are stable under pullback, we can define a presheaf F sending a base C to the set of isomorphism classes of families parametrised by C. A morphism $f : C \to D$ gets mapped to a morphism $F(D) \to F(C)$ pulling back any family over D along f. It is elementary to check that the existence of a fine moduli space M exactly means representability of F, where M is the representing object. The universal family in $F(M)$ is then given by the image of $h_M \cong F$ under the Yoneda isomorphism. The notion of a coarse moduli space M is equivalent to a natural transformation $\psi : F \to h_M$ s.t.

(i). $\psi_\Omega : F(\Omega) \to \mathrm{Hom}(\Omega, M)$ is a bijection for some notion of points Ω (i.e. just a point for our topological spaces, or $\operatorname{Spec}\mathbb{C}$ or all algebraically closed fields for schemes).

(ii). ψ is initial: For any other $\psi' : F \to M'$ there is $\phi : h_M \to h_{M'}$ with $\psi' = \psi \circ \phi$.

Let us for the rest of the subsection restrict to the category $\mathfrak{Sch}/\Lambda$ of schemes over a fixed base Λ. We are going to use the notion of Grothendieck topologies which is recapped in Appendix A. It is an important result that morphisms of schemes can be glued Zariski-locally and thus any representable presheaf F must be a sheaf over the big Zariski site Zar/Λ. Together with a second condition, for which we need a short definition, we can state a criterion for representability of F.

Definition 2.4. Let $\mathfrak{C}$ be (just for this definition) any category admitting fibre products.

(i). A morphism $F \to G$ of presheaves on $\mathfrak{C}$ is *representable*, if for any $X \in \mathfrak{C}$ and $h_X \to G$, the fibre product $h_X \times_G F$ (realised by componentwise fibre product) is representable by a scheme Y. (This gives a first projection $h_Y \to h_X$, or equivalently $Y \to X$).

(ii). Let P be a property of schemes stable under composition with isomorphisms from the left and right. We say that a representable morphism $F \to G$ like above *has property* P, if the projection $Y \to X$ has P.

(iii). A family of representable $F_i \to F$ that are open immersions (due to scheme morphisms $g_i : Y \to X$ for any Λ-scheme X) is called an *open cover*, if the images of the open immersions g_i cover X (for any Λ-scheme X).

Proposition 2.5. *A presheaf $F \in PSh(\mathfrak{Sch}/\Lambda)$ is representable iff the following are true:*

(i). F is a sheaf over Zar/Λ.

(ii). There exists an open cover of F by representable F_i.

Proof. see [GÏ0, Theorem 8.9]. □

The philosophy of this statement is that by looking at presheaves that admit an open cover by schemes, we do not get anything new. We just stick with the old concept of schemes.

The first historical step to algebraic stacks was to relax the condition on the cover, i.e. replace the Zariski site by the étale site. For any scheme C the functor h_C is also a sheaf over this finer site. And indeed, by doing so, we arrive at a more general concept than schemes.

Definition 2.6. An *algebraic space* is a sheaf F on $\acute{E}t/\Lambda$ s.t.

(i). The diagonal $F \to F \times_\Lambda F$ is representable.

(ii). There exists an étale covering of F by representable functors.

The first condition ensures that any morphism from a scheme h_C to F is representable (a fact we are going to see similarly for stacks). We will mention algebraic spaces from time to time but not treat them in detail.

For a standard reference in algebraic spaces, see [Knu71]. As an important remark, algebraic spaces are often required to be quasi-separated, i.e. have quasi-compact diagonal. An unwanted consequence of such a definition is that schemes are algebraic spaces via the Yoneda embedding if and only if they are quasi-separated. Algebraic spaces allow taking quotients by free group actions (in a certain sense) but do not solve our problem of "stacky points" resulting from non-trivial automorphisms. For this, the idea is to give the families over C the structure of a category instead of a set of isomorphism classes. Define a morphism between families $\mathcal{C} \to C$ and $\mathcal{D} \to D$ as a cartesian diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d] & \mathcal{D} \arrow[d] \\ C \arrow[r, "f"] & D \end{tikzcd}$$

In other words, $\mathcal{C}$ is the family $\mathcal{D}$ pulled back along f. Now let $F(C)$ be the subcategory of families over C with morphisms coming for which $f = 1_C$. Obviously, $1_C^*\mathcal{C} \cong \mathcal{C}$, so all morphisms in $F(C)$ are isomorphisms, which is the definition of a groupoid category. Taking the set of isomorphism classes of $F(C)$ brings us back to the set-theoretic structure.

With this observation, we have come to the point, where we see the reason for wanting something like a sheaf valued in groupoids, also known as stack. Because the natural concept for categories are not strict isomorphisms but isomorphisms up to natural isomorphism (also known as equivalences of categories), the correct, subtle definition for groupoid-valued sheaves we need will take us the next two sections.

A Deligne-Mumford stack will be a stack that does also admit an étale cover by schemes, analogously to the definition of algebraic spaces. Finally, in certain situation one needs to relax the étale cover condition to smoothness and gets an Artin stack (or algebraic stack in modern terminology) which can cover the case of quotients by group actions with non-finite stabilisers.

3. Fibred categories

The ingenious idea in the definition of stacks is to make the moduli problem its own solution and then add geometrical information to this description. In this section we will deal with the first step and see how moduli problems are nothing else but so-called fibred categories or, more specifically, categories fibred in groupoids. Stacks will be fibred categories with extra conditions. Roughly speaking from a moduli point of view, a category fibred in groupoids contains the families of the moduli problem as objects with pullbacks as morphism and has a functor to a base category sending a family to its parametrising base. After making the basic definitions, important examples are introduced.

3.1. Definition and first properties

Fix a base category $\mathfrak{S}$.

Definition 3.1. (i). A *category* $\mathfrak{X}$ *over* $\mathfrak{S}$ is a functor $p : \mathfrak{X} \to \mathfrak{S}$. A *morphism* of categories $p_{\mathfrak{X}} : \mathfrak{X} \to \mathfrak{S}$ and $p_{\mathfrak{Y}} : \mathfrak{Y} \to \mathfrak{S}$ over $\mathfrak{S}$ is a functor $F : \mathfrak{X} \to \mathfrak{Y}$ commuting with the projections $p_{\mathfrak{X}} = p_{\mathfrak{Y}} F$.

(ii). An element x in $\mathfrak{X}$ with $p(x) = S$ is said to *lie over* S. A diagram

$$\begin{tikzcd} x' \arrow[r, "\phi"] \arrow[d, mapsto, "p"] & x \arrow[d, mapsto, "p"] \\ S' \arrow[r, "f"] & S \end{tikzcd}$$

commutes if $p(\phi) = f$. In this case, ϕ *lies over* f.

(iii). The *fibre over* $S \in \mathfrak{S}$ is defined as the subcategory $\mathfrak{X}_S$ of objects over S and morphisms mapping to 1_S.

In the general theory, we try to consistently use small letters $x, y, z, \ldots$ for objects resp. Greek letters for morphisms in $\mathfrak{X}$, capital letters $S, T, U, \ldots$ for objects resp. small letters $f, g, h, \ldots$ for morphisms in $\mathfrak{S}$ and capital letters $F, G, H, \ldots$ for morphisms between categories over $\mathfrak{S}$.

The notion of a fibred category allows to talk about pulling back objects in a category over $\mathfrak{S}$ along a map in $\mathfrak{S}$. The problem that needs to be tackled here is that for two suitable maps f, g, the pullbacks $(gf)^*$ and f^*g^* are only naturally isomorphic in general.

Definition 3.2. Fix a category $\mathfrak{X}$ over $\mathfrak{S}$.

(i). A commutative diagram

$$\begin{tikzcd} x' \arrow[r, "\phi"] \arrow[d, mapsto] & x \arrow[d, mapsto] \\ S' \arrow[r, "f"] & S \end{tikzcd}$$

is *(strongly) cartesian* if any commutative

$$\begin{tikzcd}
z \arrow[d, mapsto] \arrow[rr, bend left] & x' \arrow[r, "\phi"] \arrow[d, mapsto] & x \arrow[d, mapsto] \\
Z \arrow[r] \arrow[rr, bend right] & S' \arrow[r, "f"] & S
\end{tikzcd}$$

can be extended uniquely and commutatively by a morphism $z \to x'$. Then, x' is called *pullback of x along f* and written as f^*x. It is determined up to canonical isomorphism. The morphism ϕ is called *cartesian.*

(ii). $\mathfrak{X}$ is a *fibred category* if given any x, S', S like above, there is a pullback f^*x. A *morphism* of fibred categories over $\mathfrak{S}$ is a morphism as categories over $\mathfrak{S}$ sending cartesian morphisms to cartesian morphisms.

By making choices, the pullback along $f : S' \to S$ in a fibred category can be easily made a functor $f^* : \mathfrak{X}_S \to \mathfrak{X}_T$: Choose (with the axiom of choice) a pullback $f^*x \to x$ for every $x \in \mathfrak{X}_S$. Then for every $\phi : x' \to x$ in $\mathfrak{X}_S$ the commutative diagram

$$\begin{tikzcd}
 & x' \arrow[dr, bend left] & \\
f^*x' \arrow[ur, bend left] \arrow[d, mapsto] & f^*x \arrow[r] \arrow[d, mapsto] & x \arrow[d, mapsto] \\
S' \arrow[r, equal] \arrow[rr, bend right] & S' \arrow[r, "f"] & S
\end{tikzcd}$$

gives a unique morphism $f^*x' \to f^*x$ in $\mathfrak{X}_{S'}$ (i.e. over $1_{S'}$). The uniqueness automatically proves the functoriality.

Definition 3.3. A choice of pullbacks is called a *cleavage.*

The identity functor is a possible choice of pullback along 1_S, so every other choice is naturally isomorphic to this one. The composition of pullbacks satisfies the cartesian property again and is itself a pullback. Therefore, we get the canonical isomorphism $(gf)^* \cong f^*g^*$. Moreover, by abstract nonsense, one can check that these isomorphisms satisfy the "cocycle condition", i.e.

$$\begin{tikzcd}
f^*(g^*h^*) \arrow[r, "\cong"] \arrow[d, equal] & f^*(hg)^* \arrow[r, "\cong"] & ((hg)f)^* \arrow[d, equal] \\
(f^*g^*)h^* \arrow[r, "\cong"] & (gf)^*h^* \arrow[r, "\cong"] & (h(gf))^*
\end{tikzcd}$$

commutes. Also

$$\begin{tikzcd} f^* \arrow[r, equal] \arrow[d, equal] & (1_S \circ f)^* \arrow[d, "\cong"] \\ f^* \circ 1_{\mathfrak{X}_S} \arrow[r, "\cong"] & f^* \circ 1_S^* \end{tikzcd}$$

and vice versa for the composition with identity from the other side. These properties make * a so-called *pseudo functor* and one may think of a fibred category as a collection of pullback functors with this condition. However, the first definition we gave is much easier to check.

The fibre over S of a fibred category serves as the value of our stack at S (cf. the motivational chapter) and the pullbacks as the restriction maps along covers. Since algebraic stacks always take values in groupoids (categories where all morphisms are isomorphisms), we define:

Definition 3.4. A *category fibred in groupoids* (*CFG*) is a fibred category such that all fibres are groupoids. A *morphism of CFGs* is a morphism of fibred categories.

Alternatively, many sources define CFGs directly by requiring the cartesian lifting property for all morphisms in the way the next lemma states.

Lemma 3.5. *Let $\mathfrak{X}$ be a category over $\mathfrak{S}$. $\mathfrak{X}$ is a CFG iff*

(i). Any morphism in $\mathfrak{S}$ can be lifted to one in the category $\mathfrak{X}$ lying over it.

(ii). For any commutative diagram

$$\begin{tikzcd} Z \arrow[dr, "g"] \arrow[dd, "h"'] & \\ & S \\ S' \arrow[ur, "f"] & \end{tikzcd}$$

and morphisms ϕ, ψ lying over f, g, there is a unique morphism γ lying over h s.t.

$$\begin{tikzcd} z \arrow[dr, "\psi"] \arrow[dd, dashed, "\gamma"'] & \\ & x \\ x' \arrow[ur, "\phi"] & \end{tikzcd}$$

commutes.

Proof. Rephrasing condition (ii) into one diagram, we see that it claims every lifted morphism is cartesian, while condition (i) says that lifts exist. Thus, these two conditions are equivalent to saying that $\mathfrak{X}$ is a fibred category in which every morphism is a pullback. But pullbacks along the identity are isomorphisms, thus the two conditions imply that $\mathfrak{X}$ is fibred in groupoids.

Conversely, suppose we are given a morphism $\psi : x' \to x$ over $f : S' \to S$. The universal property of pullbacks gives a morphism $x' \to f^*x$ lying over $1_{S'}$. If $\mathfrak{X}$ is fibred in groupoids, this has to be an isomorphism, hence $x' \to x$ itself is a pullback. □

This proof, which was not very difficult also makes clear that taking the subcategory of a fibred category $\mathfrak{X}$ comprising the same objects but only cartesian morphisms yields a CFG.

Corollary 3.6. *Let $\mathfrak{X}, \mathfrak{Y}$ be CFGs over $\mathfrak{S}$. Any morphism $\mathfrak{X} \to \mathfrak{Y}$ as categories over $\mathfrak{S}$ is a morphism of CFGs.*

Proof. All morphisms ϕ in CFGs are cartesian morphisms. □

3.2. Examples

Example 3.7. Let X be an object in $\mathfrak{S}$. We can define a CFG $\underline{X}$ over $\mathfrak{S}$ as "objects over X": The objects are morphisms $f : S' \to X$ of schemes (often denoted as S' with implicit structure morphism f) and morphisms $f \to g$ are scheme morphisms h with $f = g \circ h$. The projection p to $\mathfrak{S}$ maps f to S' and h to h. The pullback of $S \to X$ along $S' \to S$ is given by the composition $S' \to S \to X$. If $\mathfrak{S}$ has a terminal object Λ, then $\underline{\Lambda} \cong \mathfrak{S}$.

Example 3.8 (Grothendieck construction)**.** Via taking $F := h_X$, the previous example is a special case of the following CFG $\mathfrak{S}_F$ obtained from a presheaf

$$F : \mathfrak{S}^{opp} \to \mathfrak{Sets}$$

or more generally a (contravariant) functor $F : \mathfrak{S}^{opp} \to \mathfrak{Groupoids}$[9]. (Sets are the special case of groupoid categories where the only morphisms are the identity morphisms.)

In even greater generality, let $F : \mathfrak{S}^{opp} \to \mathfrak{Categories}$[10] be a functor (i.e. a presheaf of categories). Associate to F the following fibred category $\mathfrak{S}_F$ over $\mathfrak{S}$:

Objects are pairs (U, x) of objects U in $\mathfrak{S}$ and $x \in F(U)$. Morphisms from (U, x) to (V, y) are pairs (f, ϕ) of morphisms $f : U \to V$ and $\phi : x \to f^*y$, where we write $f^* := F(f)$. The composition of $(g, \psi : y \to g^*z) \circ (f, \phi : x \to f^*y)$ is defined as $(g \circ f, f^*(\psi) \circ \phi)$. The projection to $\mathfrak{S}$ forgets the second component of the pairs. This very general definition coincides with the one given in the first example, i.e. there is an obvious isomorphism between the CFGs constructed previously and in the general way.

[9] groupoid categories as objects with functors as morphisms
[10] categories as objects with functors as morphisms

Lemma 3.9. *$\mathfrak{S}_F$ is a fibred category. If F goes to the full subcategory of groupoids, $\mathfrak{S}_F$ is a CFG.*

Proof. This is simple checking, as long as one remembers that due to functoriality $(g \circ f)^* = f^* \circ g^*$. The pullback of (U, x) along f is given by $(f, 1_{f^*x})$ and its required cartesian lifting of

$$\begin{tikzcd}
(Z,z) \arrow[rr, bend left, "{(h,\phi)}"] \arrow[d, mapsto] & (V, f^*x) \arrow[r, "{(f,1_{f^*x})}"'] \arrow[d, mapsto] & (U,x) \arrow[d, mapsto] \\
Z \arrow[r, "g"] \arrow[rr, bend right, "h"'] & V \arrow[r, "f"] & U
\end{tikzcd}$$

can be achieved by (g, ϕ).

The fibre of $\mathfrak{S}_F$ at U is just $F(U)$. If this always is a groupoid, then $\mathfrak{S}_F$ is fibred in groupoids. □

On the other hand, for a fibred category $\mathfrak{X}$ we see that iff we can make a choice of pullbacks with $(g \circ f)^* = f^* \circ g^*$, then $\mathfrak{X}$ is isomorphic as a fibred category to some $\mathfrak{S}_F$ (namely $F : S' \mapsto \mathfrak{X}_{S'}, f \to f^*$), a situation called *splitting*.

If $\mathfrak{X}$ is fibred in discrete categories/sets, then there is always exactly one choice of pullback and $(g \circ f)^* = f^* \circ g^*$ holds automatically. Thus, we get an equivalence of the full subcategory of categories fibred in sets and the category of presheaves in sets. (A morphism between categories fibred in sets $\mathfrak{X} \to \mathfrak{Y}$ consists of maps between the fibres $\mathfrak{X}_S \to \mathfrak{Y}_S$ that are compatible with the pullbacks. These are the same data as for a natural transformation between the corresponding presheaves.) In the special case of functors h_S we see by Yoneda that $\mathrm{Hom}(S, S') = \mathrm{Hom}(\underline{S}, \underline{S}')$ i.e. the Grothendieck construction gives a fully faithful embedding of $\mathfrak{S}$ into categories fibred in sets.

Example 3.10. We take a *stable class of arrows A* in $\mathfrak{S}$, closed under base change and composition with isomorphisms. A morphism between two arrows f, g are two morphisms $\mathrm{dom}(f) \to \mathrm{dom}(g), \mathrm{codom}(f) \to \mathrm{codom}(g)$ making a commutative square with f, g. The resulting category $\mathfrak{S}_A$ has a projection functor to $\mathfrak{S}$ forgetting any information on the domain. This is a fibred category, in which the cartesian morphisms are given by cartesian squares (which we assume to exist). Of course, we may restrict to these arrows and get a CFG.

Examples are Aff/Λ and QAff/Λ, affine and quasi-affine arrows in $\mathfrak{Sch}/\Lambda$.

Example 3.11. Types of sheaves of modules that are stable under pullback form fibred categories. Examples are quasi-coherent sheaves $QCoh/\Lambda$, quasi-coherent sheaves of commutative algebras $QCohComm/\Lambda$, locally free sheaves of fixed rank...

Example 3.12 (Quotients)**.** Let $\mathfrak{S}$ have fibre products and a terminal object Λ. Given a Grothendieck topology on $\mathfrak{S}$ we can define a quotient $[X/G]$ for G a group object in $\mathfrak{S}$ acting on X[11]:

The objects of $[X/G]$ are morphisms $S \leftarrow E \rightarrow X$ such that $E \rightarrow S$ is a *principal G-bundle* or *G-torsor* and $E \rightarrow X$ is *equivariant.*

An equivariant morphism $f : X' \rightarrow X$ of objects acted on by G is one satisfying the commutativity[12] of

$$\begin{tikzcd} X' \times_\Lambda G \arrow[r, "f \times 1_G"] \arrow[d] & X \times_\Lambda G \arrow[d] \\ X' \arrow[r, "f"] & X \end{tikzcd}$$

A principal G-bundle is an object E over S with transitive, free action of G. Such actions can be pulled back along $f : S' \rightarrow S$ by defining the action of G on $X \times_S S'$ as before on X and trivially on S'. For a principal G-bundle we require that there is a cover $f_i : U_i \rightarrow S$ such that $f_i^* X$ is isomorphic to the trivial G-torsor over U_i. This means there is a G-equivariant isomorphism over U_i to the torsor $U_i \times G$ with $U_i \times G \rightarrow U_i$ being the first projection and G just acting on the second factor.

Remark 3.13*.* The following is a useful characterisation of triviality: An isomorphism of a principal G-bundle $E \rightarrow S$ to the trivial G-torsor $S \times G$ gives rise to a section $S \rightarrow E$ corresponding to the identity section mapping $s \mapsto (s, e)$ where e is the neutral element[13]. On the other hand, given a section $\sigma : S \rightarrow E$, we get an isomorphism $S \times G \rightarrow E$ given by $(s, g) \mapsto \sigma(s)g$.

A morphism $(E' \rightarrow S', E' \rightarrow X) \rightarrow (E \rightarrow S, E \rightarrow X)$ is given by a G-equivariant morphism $E' \rightarrow E$ and $S' \rightarrow S$ forming a cartesian diagram

$$\begin{tikzcd} E' \arrow[r] \arrow[d] & E \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

and compatible with the morphisms to X:

$$\begin{tikzcd} E' \arrow[r] \arrow[d] & E \arrow[dl] \\ X & \end{tikzcd}$$

[11] Group object means that $G(T)$ can be given the structure of a group for all T making h_G a functor to $\mathfrak{Groups}$. An action (on the right) is a morphism $X \times_\Lambda G \rightarrow X$ inducing actions $X(T) \times G(T) \rightarrow X(T)$. The reader may just think of algebraic groups for instance.

[12] or equivalently inducing G-equivariant maps $f_T : X'(T) \rightarrow X(T)$ $(f_T(x'g) = f_T(x')g$ for all $x' \in X'(T), g \in G(T))$.

[13] As we will do again later, we (ab)use the point-wise notation which under the Yoneda embedding shall be interpreted in this case as $h_T \rightarrow h_T \times h_G, f \mapsto (f, e)$ where e is the map to the neutral element.

The projection functor to $\mathfrak{S}$ sends $(E \to S, E \to X)$ to S and a morphism like above to the base morphism $S' \to S$.

Indeed, this yields a CFG. Pullbacks are given by the fibre product in $\mathfrak{S}$: If $f : S' \to S$ is a morphism in $\mathfrak{S}$ and $(E \to S, E \to X)$ an object in $[X/G]$ lying over S, then we have already seen that G acts on $f^*E = E \times_{S,f} S'$. Furthermore, a trivialising cover $U_i \to S$ pulls back to a cover $f^*U_i \to S'$, hence f^*E is a principal G-bundle. The morphism $E \times_{S,f} S' \to E$ is equivariant since G acts trivially on S' and $f^*E \to X$ can be defined as composition of two equivariant maps $f^*E \to E \to X$. Because we required cartesian diagrams in our definition of morphisms in $[X/G]$, any morphism in $[X/G]$ is cartesian, thus we have constructed not just a fibred category but a CFG.

Example 3.14. Note the special case of $BG := [X/G]$ where $X = \Lambda$ in $\mathfrak{S}$ (e.g. the point $\operatorname{Spec} k$ in $\mathfrak{Sch}/k$). Then, BG classifies principal G-bundles since there is only one choice for $E \to X$. On the other hand, the categorical quotient[14] X/G exists and is just X again. Therefore, even if the categorical quotient exists, $\underline{X/G}$ is very different from $[X/G]$ in general.

Example 3.15. Let $\mathfrak{S} = \mathfrak{Sch}/S$. Fix an integer $g \geq 0$. There is a CFG $\mathcal{M}_g$ of smooth curves of genus g. Objects are proper, flat families $C \to T$ in $\mathfrak{S}$ whose geometric fibres are connected curves of arithmetic genus g. Morphisms $(C' \to T') \to (C \to T)$ are pairs $(C' \to C, T' \to T)$ such that the four maps form a cartesian diagram. Composition can be defined componentwise (as "composition" of cartesian diagrams is cartesian). The pullback property for all morphisms follows directly from the universal property of fibre products. Projection to $\mathfrak{S}$ is given by just remembering the base objects/morphisms.

Example 3.16. Slightly decorating, we can define a CFG $\mathcal{M}_{g,n}$ of n-pointed curves ($n \geq 0$). It is defined almost like the previous example but objects contain the additional data of n disjoint sections $\sigma_1, \ldots, \sigma_n : T \to C$ and morphisms must commute with sections of same index.

Example 3.17. Let $2g-2+n > 0$. A very important object in the paper by Deligne-Mumford is the compactification $\overline{\mathcal{M}}_{g,n}$. The objects are proper, flat families $C \to T$ with n disjoint sections. The family's geometric fibres C_s are stable n-pointed curves of genus g, i.e.

(i). C_s is a connected curve of arithmetic genus g.

(ii). The only singularities in C_s are ordinary double points.

(iii). The sections pick non-singular points.

(iv). Each rational component of C_s has at least three *marked points* (i.e. a point which is chosen by sections or singular).

(see Appendix B for a motivation and facts on stable curves we need).

[14] a morphism $\pi : X \to X/G$ that is G-invariant ($\pi \circ \sigma = \pi \circ p_2$ for σ the group action and $p_2 : X \times G \to X$ the projection) and through which every other such morphism factors.

3.3. 2-categories

As seen in the last subsection, fibred categories (and therefore stacks) are a special type of categories and morphisms between them are functors. Thus, the answer to the question when two stacks are "structurally equal" is not the notion of isomorphism, a morphism having an inverse, but of categorical equivalence, a morphism having an inverse up to natural isomorphism. It is now time to deal with this problem, that we have been avoiding so far, properly within the framework of 2-categories.

The idea of 2-categories is that besides morphisms between objects which are called 1-morphisms in this setting, the 2-category also knows morphisms between 1-morphisms which we call 2-morphisms. As a prototypical example the reader may think of $\mathfrak{Categories}$, categories with functors as 1-morphisms and natural transformations as 2-morphisms. (In fact, all our 2-categories arise from this prototype.) To get a definition of 2-categories, one can take $\mathfrak{Categories}$ and write down all the properties satisfied in there:

Definition 3.18. A *2-category* $\mathfrak{C}$ consists of:

(i). a class of objects $Ob(\mathfrak{C})$.

(ii). for every pair X, Y of objects a category $\mathrm{Hom}(X,Y)$ whose objects $F : X \to Y$ we call *1-morphisms* and whose morphisms we write as $\alpha : F \Rightarrow F'$ and call *2-morphisms*. The composition of 2-morphisms is called *vertical composition*.

(iii). a functor
$$\mathrm{Hom}(Y,Z) \times \mathrm{Hom}(X,Y) \to \mathrm{Hom}(X,Z)$$
The image of 1-morphism pairs (F,G) on the left is denoted by $F \circ G$ and called composition of 1-morphisms. The image of 2-morphism pairs (β, α) on the left is called *horizontal composition*.

such that

(i). $Ob(\mathfrak{C})$ with 1-morphism and composition of 1-morphisms forms a category.

(ii). Horizontal composition is associative.

(iii). 1_{1_X} is a unit of horizontal composition for all X.

A 2-category is a *(2,1)-category*, if all 2-morphisms are isomorphisms.

A conceptually better way is to define the notion of *monoidal* and afterwards *enriched* categories and define inductively 0-categories as sets (of some kind - see the next remark) and $(n+1)$-categories as categories enriched over $n - \mathfrak{Categories}$.

Remark 3.19. This is the only remark we are going to make about foundational issues. Any reader who worries about sets and classes, is advised to check that one of the proposed solutions works. Any reader who does not worry may stop reading here.

Modern set theory as defined by ZFC has been carefully designed to avoid pathological examples like "the set of all sets"[15]. However, Grothendieck's approach to replace mathematical objects by much larger objects brings this system to its limits. The objects of naturally arising categories like $\underline{X}$ simply do not form sets! So how is it possible to make statements quantifying over CFGs, when they are not mathematical objects in the sense of ZFC?

The approach proposed in [AGV$^+$72] is to "blame" set theory for being too restrictive and not encompassing the correct notion of "collection" (see also [Mac98, I.6]). They define a *universe* as a set closed under the usual operations that mathematicians perform with sets (working with members, pairs, power sets and unions). The existence of a universe containing ω would provide a model for ZFC and therefore cannot be proved in ZFC. Grothendieck et al. add the independent axiom to ZFC that every objects is contained in a universe and always work with respect to sets in a universe.

In our setting, we would postulate the existence of three universes $U_0 \in U_1 \in U_2$. U_0 would be the universe containing ordinary sets, the sets in U_1 would be reserved for the classes and categories like $\mathfrak{Sets}, \mathfrak{Groups}, \ldots$, and U_2 for 2-classes and 2-categories. Constructions with sets will always stay in U_0, constructions with categories in U_1.

Those who want to stay within ZFC may instead look into [Sta15, Section 000H]. The approach there is to bound the cardinality of involved sets and do a good amount of bookkeeping.

Definition 3.20. A *sub-2-category* of a 2-category $\mathfrak{C}$ is a subclass of objects ob $\mathfrak{C}$ and a subcategory of $\mathrm{Hom}(X, Y)$ for all X, Y which together form a 2-category. If the Hom-categories are the same as in $\mathfrak{C}$, we speak of a *full sub-2-category*.

We can always construct a sub-2-category which is a $(2, 1)$-category by removing all 2-morphisms that are not 2-isomorphisms. Examples of 2-categories we know are:

Example 3.21. $\mathfrak{Categories}$, the prototypical example.

Example 3.22. $\mathfrak{Groupoids}$ can be given the structure of a full sub-2-category of $\mathfrak{Categories}$.

Example 3.23. One can make $\mathfrak{Categories}/\mathfrak{S}$, the categories over $\mathfrak{S}$, a 2-category. 2-morphisms are defined as those natural transformations $\alpha : F \Rightarrow G$ such that $\alpha_x : Fx \to Gx$ lies over the identity, i.e. in a fibre, for all x. With the same definition for 2-morphisms, we get that fibred categories $\mathfrak{Fib}/\mathfrak{S}$ and CFGs $\mathfrak{CFG}/\mathfrak{S}$ are 2-categories. In fact, $\mathfrak{CFG}/\mathfrak{S}$ is a (2,1)-category since all α_x lie over the identity, thus are isomorphisms in a fibre. $\mathfrak{Categories}/\mathfrak{S}$ and $\mathfrak{Fib}/\mathfrak{S}$ are also (2,1)-category by

[15] which cannot exist since its power set would have higher cardinality

restricting to natural isomorphisms. Finally, we may also take the full sub-2-category of *categories fibred in sets* whose fibres are all discrete categories (sets). Since the only possible morphisms between objects in a fibre are identities, all 2-morphisms in this 2-category have to be identities.

2-categories have a notion of equivalence weaker than isomorphism: A morphism $F : X \to Y$ in a 2-category is an *equivalence* if there are $G : Y \to X$ and isomorphisms $F \circ G \Rightarrow 1_Y$, $G \circ F \Rightarrow 1_X$. In this case, X and Y are called *equivalent* (write $X \simeq Y$). We show that equivalence of fibred categories is nothing else than (fibrewise) equivalence of categories:

Proposition 3.24. *Let $F : \mathfrak{X} \to \mathfrak{Y}$ be a morphism of categories over $\mathfrak{S}$. The following are equivalent:*

(i). F is an equivalence of categories over $\mathfrak{S}$.

(ii). F is an equivalence of categories.

If the categories are fibred, the two conditions are also equivalent to:

(iii). $F_S : \mathfrak{X}_S \to \mathfrak{Y}_S$ is an equivalence of categories for all base objects S.

Proof. $(i) \Rightarrow (ii), (ii) \Rightarrow (iii)$ are trivial. Below, we will show $(iv) \Rightarrow (i)$ where (iv) means: F is fully faithful and an equivalence on fibres. Clearly, $(ii) \Rightarrow (iv)$. Thus it remains to additionally show $(iii) \Rightarrow (iv)$.

$(iii) \Rightarrow (iv)$: Full faithfulness of F_S for all S implies full faithfulness of F.

Given $\phi' : Fx' \to Fx$ in $\mathfrak{Y}$ over $f : S' \to S$, we need a unique $\phi : x' \to x$ with $F\phi = \phi'$. We have cartesian morphisms $\eta : f^*x \to x$ and $\eta' : Ff^*x \to Fx$. This gives a unique factorisation:

$$\begin{tikzcd}
Fx' \arrow[r, "\psi'"] \arrow[d, mapsto] \arrow[rr, bend left, "\phi'"] & Ff^*x \arrow[r, "\eta'"] \arrow[d, mapsto] & Fx \arrow[d, mapsto] \\
S' \arrow[r, equal] \arrow[rr, bend right] & S' \arrow[r, "f"] & S
\end{tikzcd}$$

The morphism ψ' lies in the fibre over S', so there is a unique $\psi : x' \to f^*x$ with $F\psi = \psi'$. Then $F(\eta\psi) = \eta'\psi' = \phi'$. On the other hand any ϕ with $F\phi = \phi'$ gives rise to a factorisation as above and therefore it is determined.

$(iv) \Rightarrow (i)$: Let us construct the inverse G for F required in (i) now. Assuming axiom of choice, for every y in $\mathfrak{Y}$ lying over S, choose Gy in $\mathfrak{X}_S$ with isomorphism $\alpha_y : y \to F(Gy)$ over 1_S (F is essentially surjective). For every $\phi : y' \to y$, there is a unique ϕ' making

$$\begin{tikzcd}
y' \arrow[r, "\phi"] \arrow[d, "\alpha_{y'}"', "\cong"] & y \arrow[d, "\alpha_y"', "\cong"] \\
F(Gy') \arrow[r, "\phi'"] & F(Gy)
\end{tikzcd}$$

commute. ϕ' has a unique preimage $G\phi$ by full faithfulness of F. This definition of G is functorial due to uniqueness of $G\phi$.

$1_{\mathfrak{Y}} \Rightarrow FG$ is given by α. This is a natural transformation due to the above diagram and lies over 1_S, hence it is a 2-morphism of fibred categories. For $1_{\mathfrak{X}} \Rightarrow GF$ we need isomorphisms $\beta_x : x \to G(Fx)$ for all x in $\mathfrak{X}$ over S. By full faithfulness of F, the isomorphism $\alpha_x : Fx \to FGFx$ over 1_S has a unique inverse image under F lying over 1_S and this is the required β_x. It is a natural transformation because after applying F the diagram

$$\begin{tikzcd} x' \arrow[r, "\phi"] \arrow[d, "\beta_{x'}"', "\cong"] & x \arrow[d, "\beta_x"', "\cong"] \\ GF(x') \arrow[r, "GF\phi"] & GF(x) \end{tikzcd}$$

commutes and thus, by full faithfulness had to commute before applying F. □

The Proposition shows that in our definition of 2-morphisms for categories over categories, the restriction to specific natural transformations did not change the notion of equivalence. Up to now, we could have worked with more 2-morphisms. However, these 2-morphisms make it possible to formulate a 2-Yoneda lemma clarifying the relation between fibres and $\underline{S}$-valued points:

Proposition 3.25 (2-Yoneda lemma)**.** *Let $\mathfrak{X} \to \mathfrak{S}$ be a fibred category and S an object in $\mathfrak{S}$. There is a functor $\mathbf{y}_S$*

$$\mathrm{Hom}(\underline{S}, \mathfrak{X}) \to \mathfrak{X}_S$$

which is an equivalence of categories.

Proof. $\mathbf{y}_S$ sends a morphism $F : \underline{S} \to \mathfrak{X}$ to $F(1_S)$ and a 2-morphism $\alpha : F \Rightarrow G$ to $\alpha_{1_S} : F(1_S) \to G(1_S)$. This is functorial: $1_F \mapsto 1_{F1_S} = 1_{F(1_S)}$ and $\beta \circ \alpha \mapsto (\beta \circ \alpha)_{1_S} = \beta_{1_S} \circ \alpha_{1_S}$.

Make a choice of pullbacks such that $1_S^* x = x$. For an object x in $\mathfrak{X}_S$ we can define $F : \underline{S} \to \mathfrak{X}$ sending an object $f : S' \to S$ to the pullback of x along f. On morphisms

$$\begin{tikzcd} T \arrow[r, "\phi"] \arrow[d, "f"] & T' \arrow[dl, "g"] \\ S & \end{tikzcd}$$

define $F(\phi)$ as composition $f^*x = (g \circ \phi)^* x \cong \phi^* g^* x \to g^* x$. Then $F(1_f) = 1_{f^*x}$ and for $\psi : g \to h$

$$\begin{tikzcd} & \phi^* g^* x \arrow[rr] \arrow[dr, "\cong"] & & g^* x \arrow[d, "\cong"] \\ f^* x \arrow[ur, "\cong"] \arrow[dr, "\cong"'] & & \phi^* \psi^* h^* x \arrow[r] & \psi^* h^* x \arrow[d] \\ & (\psi\phi)^* h^* x \arrow[ur, "\cong"'] \arrow[rr] & & h^* x \end{tikzcd}$$

commutes due to the pseudo-functoriality of *. Thus, F is in $\mathrm{Hom}(\underline{S}, \mathfrak{X})$.

$$\mathbf{y}_S : F \mapsto F(1_S) = 1_S^* x = x$$

which proves essential surjectivity.

For full faithfulness, consider a 2-morphism $\alpha : F \to G$ in $\mathrm{Hom}(\underline{S}, \mathfrak{X})$. This means that one is given $\alpha_f : F(f) \to G(f)$ in $\mathfrak{X}_T$ for any $f : T \to S$ such that

$$\begin{tikzcd}
F(f) \arrow[r, "F(h)"] \arrow[d, "\alpha_f"] & F(g) \arrow[d, "\alpha_g"] \\
G(f) \arrow[r, "G(h)"] & G(g)
\end{tikzcd}$$

commutes for all $h : f \to g$. Suppose $\alpha_{1_S} = \beta_{1_S}$. This yields for any $f : T \to S$ (by considering $f \to 1_S$) a commutative diagram

$$\begin{tikzcd}
 & F(1_S) \arrow[dr, bend left, "\alpha_{1_S}=\beta_{1_S}"] & \\
F(f) \arrow[ur, bend left, "F(f\to 1_S)"] \arrow[r, dashed, "\alpha_f \vee \beta_f"] \arrow[d, mapsto] & G(f) \arrow[r, "G(f\to 1_S)"] \arrow[d, mapsto] & G(1_S) \arrow[d, mapsto] \\
T \arrow[r, equal] \arrow[rr, bend right, "f"'] & T \arrow[r, "f"] & S
\end{tikzcd}$$

Because $f \to 1_S$ is cartesian ($\underline{S}$ is a CFG) and G is a morphism of fibred categories, $G(f \to 1_S)$ is cartesian too, therefore $\alpha_f = \beta_f$. Conversely, for given $\alpha_{1_S} = \beta_{1_S}$, the last diagram uniquely defines α_f for any $f : T \to S$. The resulting α is indeed a natural transformation:

For another $g : T' \to S$ and $h : f \to g$, we know from above that in

$$\begin{tikzcd}
F(f) \arrow[r, "F(h)"] \arrow[d, "\alpha_f"] & F(g) \arrow[r, "F(f\to 1_S)"] \arrow[d, "\alpha_g"] & F(1_S) \arrow[d, "\alpha_{1_S}"] \\
G(f) \arrow[r, "G(h)"] & G(g) \arrow[r, "G(f\to 1_S)"] & G(1_S)
\end{tikzcd}$$

the right and outer square commute and we want commutativity of the left square. This is implied by the diagram

$$\begin{tikzcd}
F(f) \arrow[r, dashed, "\alpha_g \circ F(h)", "\vee G(h)\circ\alpha_f"'] \arrow[d, mapsto] & G(g) \arrow[r, "G(g\to 1_S)"] \arrow[d, mapsto] & G(1_S) \arrow[d, mapsto] \\
T \arrow[r, "h"] \arrow[rr, bend right, "f"'] & T' \arrow[r, "g"] & S
\end{tikzcd}$$

$\square$

Due to the 2-Yoneda lemma, we will freely switch between objects in a fibre and morphisms from a scheme into the fibred category.

The following corollary sheds light on the relation between preshaves in categories and fibred categories.

Corollary 3.26. *Any fibred category $\mathfrak{X}$ is equivalent to a split one, i.e. $\mathfrak{S}_F$ for some presheaf in categories.*

Proof. We construct $F : \mathfrak{S} \to \mathfrak{Categories}$ as follows: For objects S let

$$F(S) := \mathrm{Hom}(\underline{S}, \mathfrak{X})$$

and for morphisms $f : S \to S'$ take the morphism $F(f) : \mathrm{Hom}(\underline{S'}, \mathfrak{X}) \to \mathrm{Hom}(\underline{S}, \mathfrak{X})$ induced by precomposition with $\bar{f} : \underline{S} \to \underline{S}'$.

Now construct the obvious functor $G : \mathfrak{S}_F \to \mathfrak{X}$ taking objects (U, x) to $x(1_U)$ and morphisms $(f : U \to V, \phi : x \to y \circ \bar{f})$ to $x(1_U) \to y(\bar{f}(1_U)) = y(f) \to y(1_V)$. The 2-Yoneda lemma yields that

$$G_U : F(U) = \mathrm{Hom}(\underline{U}, \mathfrak{X}) \to \mathfrak{X}_U$$

is an equivalence. Now the claim follows from Proposition 3.24. □

Warning: Although every fibred category is equivalent to a split one, the category of fibred categories is not equivalent to that of presheaves in categories. The reason is that the morphisms between presheaves always have to respect the splittings while those of fibred categories do not.

We end our discussion of 2-categories by revising our notion of commutative diagrams.

Definition 3.27. A diagram in a 2-category is called *2-commutative*, if its 1-morphisms commute up to given 2-isomorphisms and these 2-isomorphisms commute in the induced diagram taking 1-morphisms (and their compositions) as vertices.

In fact, most diagrams in the world of stacks do only 2-commute.

3.4. 2-fibre products

Because the natural way to think about relations in 2-categories is up to 2-isomorphisms, the 2-fibre product has to take this into consideration, too. The notion of 2-fibre products in 2-categories that are non-strict[16] can be a bit tricky as there are different non-equivalent definitions involving 2-isomorphisms or mere 2-morphisms. Therefore we restrict to (2,1)-categories.

[16] To our 2-categorical horror, there are also various strict 2-fibre products determining up to isomorphism. However, conceptually we are interested in a notion stable under equivalence.

Definition 3.28. A 2-commutative diagram in a (2,1)-category

$$\begin{tikzcd} X \times_Z Y \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & Z \end{tikzcd}$$

is called *2-cartesian*, if it satisfies the following universal property:

For any T with 2-commuting

$$\begin{tikzcd} T \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & Z \end{tikzcd}$$

there is a map $\gamma : T \to X \times_Z Y$ such that the following diagram 2-commutes

$$\begin{tikzcd} T \arrow[dr] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & X \times_Z Y \arrow[r] \arrow[d] & X \arrow[d] \\ & Y \arrow[r] & Z \end{tikzcd}$$

and if γ' is a second such morphism, then there is a unique 2-isomorphism $\gamma \Rightarrow \gamma'$ making the overall diagram 2-commute. In this case $X \times_Z Y$ is called *2-pullback* or *2-fibre product* and unique up to equivalence.

Like fibre products in categories, 2-fibre products satisfy commutativity, associativity and invariance under taking a product with an object over itself, but only up to equivalence.

For a morphism $X \to S$, there is a canonical diagonal morphism $\Delta_X : X \to X \times_S X$ induced by the identity. This morphism will play a very prominent role in the theory of representable morphisms. It is also useful in recovering arbitrary 2-fibre products:

Lemma 3.29. *Let $X \to Y$, $X' \to Y'$, $Z \to Y$, $Z \to Y'$ be morphisms over an object S.*

(i). $(X \times_S X') \times_{Y \times_S Y'} Z$ is equivalent to $(X \times_Y Z) \times_Z (X' \times_{Y'} Z)$.

(ii). $(X \times_S X') \times_{Y \times_S Y} Y$ is equivalent to $X \times_Y X'$. (Remember: Any fibre product can be realised by the diagonal morphism.)

Proof. The second statement follows from the first with $Y = Y' = Z$. For the first statement consider the 2-commutative diagram

$$
\begin{tikzcd}
(X \times_Y Z) \times_Z (X' \times_{Y'} Z) \arrow[d, dashed, shift left] \arrow[dr, dashed] \arrow[ddr] \arrow[drr, bend left] & & & & \\
(X \times_S X') \times_{Y \times_S Y'} Z \arrow[u, dashed, shift left] \arrow[r] \arrow[dr, dashed] \arrow[drr, bend left] \arrow[rr, dashed, bend left] & X \times_S X' \arrow[ddd, bend right] \arrow[ddrr, dashed] \arrow[rrr, bend left] & X' \times_{Y'} Z \arrow[rr] \arrow[d] & & X' \arrow[dd] \\
& X \times_Y Z \arrow[r] \arrow[dd] & Z \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& & & Y \times_S Y' \arrow[r] \arrow[d] & Y' \arrow[d] \\
& X \arrow[rr] & & Y \arrow[r] & S
\end{tikzcd}
$$

in which the dashed arrows come from universal properties. □

Lemma 3.30. *Let $F : \mathfrak{X} \to \mathfrak{Z}$ and $G : \mathfrak{Y} \to \mathfrak{Z}$ be morphisms of categories over $\mathfrak{S}$. A 2-fibre product $\mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y}$ in the (2,1)-category of categories over $\mathfrak{S}$ is given by the following construction:*

(i). Objects are quadruples (S, x, y, ϕ) with x resp. y objects in $\mathfrak{X}_S$ resp. $\mathfrak{Y}_S$ and $\phi : F(x) \to G(y)$ an isomorphism in $\mathfrak{Z}_S$.

(ii). Morphisms $(a, b) : (S, x, y, \phi) \to (S', x', y', \phi')$ are morphisms $a : x \to x'$, $b : y \to y'$ mapping to the same $S \to S'$ in $\mathfrak{S}$ and forming a commutative square

$$
\begin{tikzcd}
F(x) \arrow[r, "\phi"] \arrow[d, "F(a)"] & G(y) \arrow[d, "G(b)"] \\
F(x') \arrow[r, "\phi'"] & G(y')
\end{tikzcd}
$$

Composition is given componentwise.

Projections p_1, p_2 are given by forgetful functors.

Proof. The isomorphism $\alpha : Fp_1 \Rightarrow Gp_2$ is given by $\alpha_{(S,x,y,\phi)} := \phi : F(x) \to G(y)$.

Let $p' : \mathfrak{T} \to \mathfrak{S}$, $p_1' : \mathfrak{T} \to \mathfrak{X}$, $p_2' : \mathfrak{T} \to \mathfrak{Y}$, $\alpha' : Fp_1' \to Gp_2'$ be a test object. Define $\gamma : \mathfrak{T} \to \mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y}$ as $\gamma(t) := (p'(t), p_1'(t), p_2'(t), \alpha_t')$ on objects and $(p_1'(\psi), p_2'(\psi))$ on morphisms. If $\gamma'(t) := (p'(t), p_1''(t), p_2''(t), \alpha_t'')$ is a second such morphism, then the isomorphisms

$$a : p_1''(t) = p_1(\gamma'(t)) \to p_1'(t)$$
$$b : p_2''(t) = p_2(\gamma'(t)) \to p_2'(t)$$

induce a morphism $(a, b) : \gamma'(t) \Rightarrow \gamma(t)$. It is easy to check that this is functorial and the only choice possible. □

Lemma 3.31. *With the above construction $(X \times_{\mathfrak{Z}} \mathfrak{Y})_S = \mathfrak{X}_S \times_{\mathfrak{Z}_S} \mathfrak{Y}_S$ as categories (over the one point category S).*

Proof. Trivial. ☐

Lemma 3.32. *The (2,1)-category of fibred categories is closed under 2-fibre products.*

Proof. Just check that $(X \times_{\mathfrak{Z}} \mathfrak{Y})$ in categories over $\mathfrak{S}$ has pullbacks: For an object (S, x, y, ϕ) and $f : S' \to S$ choose pullbacks $a : f^*x \to x$ and $b : f^*y \to y$. Then $F(a), G(b)$ are cartesian morphisms to $F(x) \cong G(y)$ and therefore we have a unique isomorphism $f^*\phi : F(f^*x) \to G(f^*y)$ such that

$$\begin{tikzcd}
F(f^*x) \arrow[r, "f^*\phi"] \arrow[d, "F(a)"] & G(f^*y) \arrow[d, "G(b)"] \\
F(x) \arrow[r, "\phi"] & G(y)
\end{tikzcd}$$

commutes. Hence, we have found a morphism

$$(a, b) : (S', f^*x, f^*y, f^*\phi) \to (S, x, y, \phi)$$

This is cartesian: Any $(a', b') : (T, x', y', \phi') \to (S, x, y, \phi)$ gives unique $c : x' \to f^*x$, $d : y' \to f^*y$ such that

$$\begin{tikzcd}
F(x') \arrow[d, "F(c)"] \arrow[dd, bend right=60, "F(a')"'] & G(y') \arrow[d, "G(d)"'] \arrow[dd, bend left=60, "G(b')"] \\
F(f^*x) \arrow[r, "f^*\phi"] \arrow[d, "F(a)"] & G(f^*y) \arrow[d, "G(b)"'] \\
F(x) \arrow[r, "\phi"] & G(y)
\end{tikzcd}$$

commutes. Inserting ϕ' does not destroy the commutativity due to the uniqueness of $F(x') \to G(f^*y)$. ☐

Corollary 3.33. *The (2,1)-category of categories fibred in groupoids or sets are closed under 2-fibre products.*

Proof. Use the last two lemmata. ☐

Lemma 3.34. *For F, G, H presheaves in discrete categories/sets:*

$$\mathfrak{S}_F \times_{\mathfrak{S}_H} \mathfrak{S}_G \cong \mathfrak{S}_{F \times_H G}$$

In particular, for objects in $\mathfrak{S}$: $\underline{X} \times_{\underline{Z}} \underline{Y}$ is isomorphic to $\underline{X \times_Z Y}$

Proof. The only 2-morphisms of categories fibred in sets are identities. ☐

There are more interesting examples of 2-fibre products.

Example 3.35. Take a quotient $[X/G]$ and a scheme S. Any morphism $\underline{S} \to [X/G]$ determines an object E in $[X/G]_S$. Let $\underline{X} \to [X/G]$ be the morphism sending $f : T \to X$ to the trivial torsor $T \times G$ with equivariant map to X given by $(t, g) \mapsto f(t)g$. We claim that

$$\begin{array}{ccc} \underline{E} & \longrightarrow & \underline{X} \\ \big\downarrow & & \big\downarrow \\ \underline{S} & \longrightarrow & [X/G] \end{array}$$

is 2-cartesian.

Objects in the 2-fibre product are given by scheme morphisms $T \to S$, $T \to X$ and an isomorphism $E_T \cong T \times G$ in $[X/G]_T$. An object in $\underline{E}$ is a scheme morphism $T \to E$. This induces a section $T \to E_T$ and thus an isomorphism $E_T \cong T \times G$. We conclude that the above diagram is 2-commutative.

The induced functor $F : \underline{E} \to \underline{S} \times_{[X/G]} \underline{X}$ is given on objects by sending $T \to E$ to the object given by $T \to E \to S$, $T \to E \to X$ and the isomorphism $E_T \cong T \times G$ described above. On the other hand, objects in the fibre product give a section $T \to E_T$ which by composition with $E_T \to E$ gives an object in $\underline{E}$. This proves essential surjectivity of F.

Morphisms $(T' \to E) \to (T \to E)$ over E get send by F to the same morphism over S and X. This satisfies the compatibility condition of morphisms in the 2-fibre product:

$$\begin{array}{ccc} T' \times G & \xrightarrow{\cong} & E_{T'} \\ \big\downarrow & & \big\downarrow \\ T \times G & \xrightarrow{\cong} & E_T \end{array}$$

On the other hand, $T' \to T$ over S and X sin the 2-fibre product satisfying this compatibility give rise to a 2-commutative diagram:

$$\begin{array}{ccccccc} T' & \longrightarrow & T' \times G & \xrightarrow{\cong} & E_{T'} & \longrightarrow & E \\ \big\downarrow & & \big\downarrow & & \big\downarrow & & \big\| \\ T & \longrightarrow & T \times G & \xrightarrow{\cong} & E_T & \longrightarrow & E \end{array}$$

Following the horizontal arrows, one finds $T' \to T$ over E with image under F equal to the original morphism. This proves full faithfulness.

4. Stack conditions

From now on, fix a Grothendieck topology on the base category $\mathfrak{S}$. In this section, we will state two conditions that define when a fibred category is a stack. As motivated in the first chapter, these will be gluing of morphisms and descent on objects.

4.1. Preheaves of morphisms and prestacks

Given a fibred category $\mathfrak{X} \to \mathfrak{S}$ and objects S in $\mathfrak{S}$, x and y in $\mathfrak{X}_S$, make a choice of pullbacks. We define a presheaf in sets on $\underline{S}$:

$$\underline{\mathrm{Hom}}_{\mathfrak{X}}(x,y) : (f : T \to S) \mapsto \mathrm{Hom}_{\mathfrak{X}_T}(f^*x, f^*y)$$

For $g : f \to f'$ in $\underline{S}$, define the restriction map

$$\mathrm{Hom}_{\mathfrak{X}_T}(f'^*x, f'^*y) \to \mathrm{Hom}_{\mathfrak{X}_T}(f^*x, f^*y) \cong \mathrm{Hom}_{\mathfrak{X}_T}(g^*f'^*x, g^*f'^*y)$$

as $\phi \mapsto g^*\phi$, the unique morphism $f^*x \to f^*y$ that forms a commutative diagram with the canonical $f^*x \to f'^*x$, $f^*x \to f'^*x$ and ϕ. These choices are independent of the cleavage, i.e. for a different choice $(f^\bullet x, f^\bullet y)$ instead of (f^*x, f^*y) the canonical isomorphisms $\mathrm{Hom}_{\mathfrak{X}_T}(f^\bullet x, f^\bullet y') \cong \mathrm{Hom}_{\mathfrak{X}_T}(f^*x, f^*y)$ satisfy the commutativity relation of a natural isomorphism. (If they did not, this would contradict the uniqueness of $g^*\phi$.)

It is also clear that for two equivalent fibred categories $\mathfrak{X}$ and $\mathfrak{Y}$,

$$\underline{\mathrm{Hom}}_{\mathfrak{X}}(x,y) \cong \underline{\mathrm{Hom}}_{\mathfrak{Y}}(x,y)$$

and that a morphism $\mathfrak{Y} \to \mathfrak{Z}$ induces $\underline{\mathrm{Hom}}_{\mathfrak{Y}}(x,y) \to \underline{\mathrm{Hom}}_{\mathfrak{Z}}(x,y)$.

Lemma 4.1. $\underline{\mathrm{Hom}}_{\mathfrak{X}}(x,y)$ *is in fact a presheaf.*

Proof. By the previous remarks we can replace $\mathfrak{X}$ by an equivalent split category (Corollary 3.26) and choose pullbacks that split. But this means $g^*f^* = (f \circ g)^*$, so the claim that the restriction maps are compatible is obvious. $\square$

Definition 4.2. A fibred category $\mathfrak{X}$ is a *prestack*, if

(A) for every choice of S, x, y the presheaf $\underline{\mathrm{Hom}}_{\mathfrak{X}}(x,y)$ is a sheaf.

The 2-category of prestacks is defined as full sub-2-category of fibred categories.

Note that isomorphisms glue to isomorphisms since if two morphisms are locally inverse to each other, then there composition yields the identity morphism locally and thus globally. Hence, for any prestack there is a subsheaf

$$\underline{\mathrm{Isom}}_{\mathfrak{X}}(x,y) \subset \underline{\mathrm{Hom}}_{\mathfrak{X}}(x,y)$$

Lemma 4.3. *The (2,1)-category of prestacks is closed under 2-fibre products.*

Proof. Let $F : \mathfrak{X} \to \mathfrak{Z}, G : \mathfrak{Y} \to \mathfrak{Z}$ be morphisms of prestacks. From the explicit construction of the 2-fibre product and cartesian morphisms therein, one sees that setting $z = F(x) \cong G(y), z' = F(x') \cong G(y')$

$$\underline{\mathrm{Hom}}_{\mathfrak{X}\times_{\mathfrak{Z}}\mathfrak{Y}}((S,x,y,\phi),(S',x',y',\phi')) = \underline{\mathrm{Hom}}_{\mathfrak{X}}(x,x') \times_{\underline{\mathrm{Hom}}_{\mathfrak{Z}}(z,z')} \underline{\mathrm{Hom}}_{\mathfrak{Y}}(y,y')$$

The fact that the category of sheaves is closed under fibre products in the category of presheaves finishes the proof. $\square$

Lemma 4.4. *If F is a presheaf in sets, then $\mathfrak{S}_F$ is a prestack iff F is separated*[17].

Proof. For $f : T \to S$, setting $x' = f^*x, y' = f^*y$, the prestack condition translates to exactness of

$$\mathrm{Hom}_{\mathfrak{X}_T}(x', y') \to \prod_i \mathrm{Hom}_{\mathfrak{X}_{T_i}}(x'|_{T_i}, y'|_{T_i}) \rightrightarrows \prod_{i,j} \mathrm{Hom}_{\mathfrak{X}_{T_{ij}}}(x'|_{T_{ij}}, y'|_{T_{ij}})$$

where as a usual convention $T_{ij} := T_i \times_T T_j$ and pullback is suggestively written as restriction ($x|_{T_i} := f_i^* x$). Since the Hom-categories are discrete, the equaliser of the right arrow is the whole $\prod_i \mathrm{Hom}_{\mathfrak{X}_{T_i}}(x'|_{T_i}, y'|_{T_i})$ and this means that we get an equality $x' = y'$ iff we have equalities $x'|_{T_i} = y'|_{T_i}$ for all i. We get the first sheaf condition for all x, y by taking $f = 1_S$, thus $x' = x, y' = y$. □

Example 4.5. In general, the prestack condition is related to descent on morphism which is possible in subcanonical sites. Let A be a a stable set of arrows as in Example 3.10 on a subcanonical site. Then the fibred category $\mathfrak{S}_A$ is a prestack:
Let $(T_i \to T)$ be a covering. Take two arrows $S \to T$, $S' \to T$ in A

$$S_i := S \times_T T_i$$

$$S_{ij} := S \times_T T_{ij} = S_i \times_S S_j$$

and analogously for S'. Given morphisms $f_i : S_i \to S'_i$ over T_i such that f_i and f_j agree on $S_{ij} \cong S_{ji}$, we need to find a unique $f : S \to S'$ over T agreeing with f_i on S_i for all i.

The compositions $g_i : S_i \to S'_i \to S'$ agree on S_{ij}. Now $h_{S'}$ is a sheaf by assumption, so there is a unique morphism $f : S \to S'$ restricting to the g_i. It is indeed a morphism over T since $(S_i \to S'_i \to S' \to T) = (S_i \to S'_i \to T_i \to T)$ agree on S_{ij} and h_T is a sheaf.

Example 4.6. Working with the previous example, we see that $[X/G]$ and $\mathcal{M}_{g,n}$ are prestacks for any subcanonical topology. For morphisms constructed by local data, we have additional conditions like equivariance and compatibility with sections but these are equivalent to commutativity of some diagram which can be checked locally.

4.2. Stacks

The missing condition for stacks is descent on objects. For this, we define a category of descent data. Let $\mathfrak{X} \to \mathfrak{S}$ be a fibred category. For fibre products we denote by p_n the projection to the n-th component and by p_{nm} the projection to the product of the n-th and m-th component etc. Also write fibre products $T_i \times_T T_j$ as T_{ij} etc.

Definition 4.7. Let $(T_i \to T)$ be a family of morphisms in $\mathfrak{S}$.

[17] F is *separated* if it satisfies the first sheaf condition: Given two sections $x, y \in F(T)$ such that their pullback to a cover $(f_i : T_i \to T)$ agrees, the sections are equal.

(i). A *descent datum on $\mathfrak{X}$ relative to* $(T_i \to T)$ is a family of objects x_i in $\mathfrak{X}_{T_i}$ for all i and isomorphisms $\phi_{ij} : p_1^* x_i \to p_2^* x_j$ in all $\mathfrak{X}_{T_{ij}}$ satisfying the cocycle condition

$$\begin{tikzcd} p_{12}^* p_1^* x_i \arrow[r, "p_{12}^*\phi_{ij}"] \arrow[d, "\cong"] & p_{12}^* p_2^* x_j \arrow[r, "\cong"] & p_{23}^* p_2^* x_j \arrow[d, "p_{23}^*\phi_{jk}"] \\ p_{13}^* p_1^* x_i \arrow[r, "p_{13}^*\phi_{ik}"] & p_{13}^* p_3^* x_k \arrow[r, "\cong"] & p_{23}^* p_3^* x_k \end{tikzcd}$$

in $X_{T_{ijk}}$ where the $\cong$ denote canonical isomorphisms.

(ii). A *morphism of descent data* $(x_i, \phi_{ij}) \to (x_i', \phi_{ij}')$ relative to $(T_i \to T)$ is a family of morphisms $\phi_i : x_i \to x_i'$ such that

$$\begin{tikzcd} p_1^* x_i \arrow[r, "\phi_{ij}"] \arrow[d, "p_1^*\phi_i"] & p_2^* x_j \arrow[d, "p_2^*\phi_j"] \\ p_1^* x_i' \arrow[r, "\phi_{ij}'"] & p_2^* x_j' \end{tikzcd}$$

commutes. The composition of descent data morphisms works componentwise.

We define $DD((T_i \to T))$ as the resulting category.

This definition is nothing else but the gluing data of families that we know from the motivational section with categorical fibre products replacing intersections and pullbacks replacing restrictions. Descent data can be pulled back along covers:

Definition 4.8. Let $(T_i \to T)_{i \in I}, (T_j' \to T')_{j \in I'}$ be two families of morphisms in $\mathfrak{S}$. Let $\alpha : I' \to I$, $f : T' \to T$ and $f_j : T_j' \to T_{\alpha(j)}$ constitute a morphism $(T_j' \to T') \to (T_i \to T)$ as defined in Definition A.2.

Then we define a *pullback functor of descent data*

$$f^* : DD((T_i \to T)) \to DD((T_j' \to T'))$$

given by

$$(x_i, \phi_{ij}) \mapsto (f_j^* x_{\alpha(j)}, (f_j \times f_{j'})^* \phi_{\alpha(j)\alpha(j')})$$

on objects and by $(\phi_i) \mapsto (f_j^* \phi_{\alpha(j)})$ on morphisms.

Lemma 4.9. *This is indeed well-defined and functorial.*

Proof. Functoriality is clear. Well-definedness is easy as soon as one notes that

$$\begin{tikzcd} T_{ijk}' \arrow[r, "p_{12}'"] \arrow[d, "f_i \times f_j \times f_k"] & T_{ij}' \arrow[r, "p_1'"] \arrow[d, "f_i \times f_j"] & T_i' \arrow[d, "f_i"] \\ T_{ijk} \arrow[r, "p_{12}"] & T_{ij} \arrow[r, "p_1"] & T_i \end{tikzcd}$$

commutes and analogously for the other projections. Then, one is able to shift the f's in the cocycle condition to the front as $f_i \times f_j \times f_k$ or to the front as $f_i \times f_j$ in the required commuting relation of descent data morphisms. $\square$

Now we can define what it means for descent data to glue to a global object.

Definition 4.10. (i). For x lying over T we have the *trivial descent datum* $(x, (1_x))$ relative to (1_T).

(ii). For x lying over T and $(T_i \to T)$, we get a *canonical descent datum* by pulling back the trivial descent datum.

(iii). A descent datum is called *effective*, if it is isomorphic to the canonical descent datum of an object x.

Having developed the above formalism we define a stack.

Definition 4.11. A *stack* is a prestack such that

(B) for every cover $(T_i \to T)$ in the chosen Grothendieck topology, all descent data relative to $(T_i \to T)$ are effective.

The 2-category of stacks is defined as full sub-2-category of the 2-category of prestacks. A stack that is fibred in groupoids/sets is called a *stack in groupoids/sets*.

Stack condition (B) says that local objects with compatible overlaps glue together to a global object. Then stack condition (A) implies that given two global objects x, y having isomorphic canonical descent data, the local isomorphisms glue together to a global one, $x \cong y$. We can reformulate the two stack conditions in categorical terms:

Lemma 4.12. *Let $\mathfrak{X}$ be a fibred category. For any family $\mathcal{T} = (f_i : T_i \to T)$, there is a canonical functor $j_{\mathcal{T}} : \mathfrak{X}_T \to DD((T_i \to T))$ sending objects to their canonical descent data and morphisms to the pullbacks of the morphisms between the trivial descent data.*

(i). $\mathfrak{X}$ is a prestack iff $j_{\mathcal{T}}$ is fully faithful for all covers $\mathcal{T}$.

(ii). $\mathfrak{X}$ is a stack iff $j_{\mathcal{T}}$ is an equivalence for all covers $\mathcal{T}$.

Proof. (i). If $\phi : x \to y$ is a morphism in $\mathfrak{X}_T$, it gets mapped to $(f_i^*\phi : f_i^*x \to f_i^*y)$ in $DD((T_i \to T))$. Now the sheaf condition on $\underline{\mathrm{Hom}}_{\mathfrak{X}}(x, y)$ states exactly that all families $(\phi_i : f_i^*x \to f_i^*y)$ agreeing when pulled back $(f_j^*\phi_i = f_i^*\phi_j)$ come from a global $\phi : x \to y$ ($j_{\mathcal{T}}$ is surjective on Hom-sets) and that this ϕ is unique (j_T is injective on Hom-sets).

(ii). Stack condition (B) means that $j_{\mathcal{T}}$ is essentially surjective for all covers $\mathcal{T}$.

□

Warning Replacing covers $(T_i \to T)$ by the disjoint union $\coprod T_i \to T$ as it is often done in Grothendieck topologies involves a minor problem: A priori, it is not clear that descent relative to $(T_i \to \coprod T_i)$ is effective. For this, it suffices to check that the base change functors $\mathfrak{X}_T \to \mathfrak{X}_{T_i}$ induce an equivalence of categories $\mathfrak{X}_T \to \prod \mathfrak{X}_{T_i}$.

Lemma 4.13. *If F is a presheaf in sets, then $\mathfrak{S}_F$ is a stack iff F is a sheaf.*

Proof. We already know that stack condition (A) is equivalent to separatedness, the first sheaf condition. Stack condition (B) on the other hand is the second sheaf condition that objects being equal on the overlaps glue together. □

Lemma 4.14. *The (2,1)-category of stacks is closed under 2-fibre products.*

Proof. For the 2-fibre product $\mathfrak{X} \times_{F,\mathfrak{Z},G} \mathfrak{Y}$ of stacks as prestacks, we have to verify stack condition (B). A descent datum relative to $(T_i \to T)$ in the 2-pullback category looks like $((T_i, x_i, y_i, \phi_i), (\psi_{ij}, \chi_{ij}))$. From this we get descent data (x_i, ψ_{ij}) and (y_i, χ_{ij}) which glue to give objects x and y. Via ϕ_i we have isomorphisms

$$F(x)|_{T_i} \cong F(x|_{T_i}) \cong F(x_i) \to G(y_i) \cong G(y|_{T_i}) \cong G(y)|_{T_i}$$

in $\mathfrak{Z}_{T_i}$ and one can check that these glue together on overlaps to give an isomorphism $\phi : F(x) \to G(y)$. The canonical datum of (T, x, y, ϕ) relative to $(T_i \to T)$ is then indeed canonically isomorphic to the given one. □

Corollary 4.15. *The (2,1)-categories of stacks in groupoids/sets are closed under 2-fibre products.*

4.3. Examples

Example 4.16. The CFG $\underline{S}$ is a stack in groupoids for any object S in $\mathfrak{S}$ iff the chosen Grothendieck topology is subcanonical.

Example 4.17. Algebraic spaces are étale sheaves in sets and can therefore be seen as stacks with respect to the étale topology.

Non-trivial examples usually need some result from faithfully flat descent theory.

Example 4.18. For G a quasi-affine group scheme, $[X/G]$ is a stack for the fpqc topology. We have already checked it is a prestack. Stack condition (B) follows by descent of quasi-affine morphisms which lets us construct a torsor $E \to S$ for given torsors $E_i \to S_i$ and a covering $(S_i \to S)$. All conditions on the action, local triviality and the existence of a global equivariant morphism to X can be checked locally by descent on morphisms.

If in our definition of G-torsors, we had allowed algebraic spaces instead of just schemes, we would not have to restrict to the quasi-affine case ([Sta15, Tag 036Z]).

Example 4.19. We could directly show that $M_{g,n}$ is a stack but later we will see that it is equivalent to a quotient stack and is thus covered by the previous example.

5. Deligne-Mumford stacks

This chapter introduces the last condition we need in order to make a stack an algebraic object, namely that of a smooth cover. All algebraic stacks are stacks in groupoids, so from now on, let "stack" always mean "stack in groupoids". Another convention often made is to call stacks "isomorphic" when in fact they are just equivalent, but we are not going to adapt this.

Also fix the base site as $\acute{E}t/\Lambda$. There are some minor differences between definitions found in the literature and we are going to mention these. The first one is that [Sta15] use the coarser fppf topology as a more natural topology for descent. (All our stacks will be stacks in both senses.)

5.1. Representable morphisms

As seen in the motivational section, where we talked about morphisms of functors having geometric properties like being an open immersion, we need the notion of a representable morphism of stacks.

Definition 5.1. A morphism $\mathfrak{X} \to \mathfrak{Y}$ of stacks is *representable* if for every scheme S and $\underline{S} \to \mathfrak{Y}$ the 2-fibre product $\mathfrak{X} \times_{\mathfrak{Y}} \underline{S}$ is *representable*, i.e. there is a scheme T such that the fibre product is equivalent to $\underline{T}$.

Alternative definition This definition is sufficient for our scope aiming at Deligne-Mumford stacks. For Artin stacks however, one should just require representability by algebraic spaces, i.e. there is an algebraic space T such that the fibre product is equivalent to $\mathfrak{S}_T$.

Definition 5.2. Let P be a property of morphisms of schemes stable under base change and local on the target (in our chosen site, i.e. surjective, smooth, étale,... in the étale topology). We say a representable morphism $\mathfrak{X} \to \mathfrak{Y}$ *has property* P if for every $\underline{S} \to \mathfrak{Y}$, the morphism of schemes $\mathfrak{X} \times_{\mathfrak{Y}} \underline{S} \to \underline{S}$ has property P.

As a useful remark, note that by the 2-Yoneda lemma $F : \mathfrak{X} \to \mathfrak{Y}$ being a monomorphism in the sense of representable morphism, coincides with F being fully faithful. (F is fully faithful iff $F_T : \mathrm{Hom}(\underline{T}, \mathfrak{X}) \to \mathrm{Hom}(\underline{T}, \mathfrak{Y})$ is fully faithful iff $\underline{T} \times_{\mathfrak{X}} \mathfrak{Y} \to \underline{T}$ is a monomorphism.)

Lemma 5.3.

(i). The composition of representable morphisms is representable.

(ii). If a morphism and its target are representable, then so is its domain.

(iii). Arbitrary base change of representable morphisms is representable.

(iv). Products of representable morphisms are representable.

(v). If $G \circ F$ and G are representable, then so is F.

Proof. The proof is trivial just plays around with equivalences of 2-fibre products. We give the main steps.

(i). Use $\mathfrak{X} \times_{\mathfrak{Z}} \underline{S} \simeq \mathfrak{X} \times_{\mathfrak{Y}} (\mathfrak{Y} \times_{\mathfrak{Z}} \underline{S})$.

(ii). Special case of (*i*).

(iii). Use same equivalence but the other way around.

(iv). Combine (*i*) and (*iii*).

(v). Get a 2-cartesian diagram

$$\begin{tikzcd}
\mathfrak{X} \times_{\mathfrak{Y}} \underline{S} \arrow[r] \arrow[d] & \underline{S} \arrow[d] & \\
\mathfrak{X} \times_{\mathfrak{Z}} \underline{S} \arrow[r] \arrow[d] & \mathfrak{Y} \times_{\mathfrak{Z}} \underline{S} \arrow[r] \arrow[d] & \underline{S} \arrow[d] \\
\mathfrak{X} \arrow[r] & \mathfrak{Y} \arrow[r] & \mathfrak{Z}
\end{tikzcd}$$

in which the objects in the second row are representable and therefore $\mathfrak{X} \times_{\mathfrak{Y}} \underline{S}$ is.

□

Corollary 5.4. *Let P be a property as above. The composition, base change and product of morphisms having P, has P again.*

Proof. Use the same identities as in Lemma 5.3. □

Lemma 5.5. *If P' is a second property stable under base change and local on the target and $P \Rightarrow P'$ for morphisms of schemes, then also $P \Rightarrow P'$ for representable morphisms of stacks.*

Proof. Trivial. □

Lemma 5.6. *Let P be a property as above and $\mathfrak{X} \to \mathfrak{Y}$ a representable morphism of stacks. Then it suffices to check P for base change along one representable $\underline{T} \to \mathfrak{Y}$ which is étale and surjective. If P is smooth local on the target, étale can be relaxed to smooth.*

Proof. Let $\underline{T}' \to \mathfrak{Y}$ be a scheme over $\mathfrak{Y}$. Look at the diagram

$$\begin{tikzcd}
\underline{T}' \times_{\mathfrak{Y}} \mathfrak{X} \arrow[d] & \underline{T}' \times_{\mathfrak{Y}} \underline{T} \times_{\mathfrak{Y}} \mathfrak{X} \arrow[l] \arrow[r] \arrow[d] & \underline{T} \times_{\mathfrak{Y}} \mathfrak{X} \arrow[d] \\
\underline{T}' & \underline{T}' \times_{\mathfrak{Y}} \underline{T} \arrow[l] \arrow[r] & \underline{T}
\end{tikzcd}$$

where the left and the right square are 2-cartesian. The right vertical arrow has P by assumption, then the middle vertical arrow has P by base change. Finally, by locality on the target, the left vertical arrow has P. □

We are interested the most in representability of morphisms of the form $\underline{S} \to \mathfrak{X}$ because covers by schemes/algebraic spaces will look like this. The next proposition illuminates the earlier, cryptic remark about the importance of the diagonal morphism.

Proposition 5.7. *For a stack $\mathfrak{X}$ over $\mathfrak{S}$, the following are equivalent:*

(i). The diagonal morphism $\Delta_{\mathfrak{X}/\mathfrak{S}}$ is representable.

(ii). For all schemes S, T and $\underline{S} \to \mathfrak{X}$, $\underline{T} \to \mathfrak{X}$, the fibre product $\underline{S} \times_{\mathfrak{X}} \underline{T}$ is representable.

(iii). For all schemes S, all $\underline{S} \to \mathfrak{X}$ are representable.

(iv). For all schemes S and $F : \underline{S} \to \mathfrak{X}, G : \underline{S} \to \mathfrak{X}$, $\underline{S} \times_{F,\mathfrak{X},G} \underline{S}$ is representable.

(v). For all schemes S and objects x, y in the fibre $\mathfrak{X}_S$, the sheaf $\mathrm{Isom}_{\mathfrak{X}}(x, y)$ is representable by a scheme over S.

Proof. $(ii) \Rightarrow (iii) \Rightarrow (iv)$ are trivial.

$(i) \Rightarrow (ii)$ because $\underline{S} \times_{\mathfrak{X}} \underline{T} \simeq \underline{S \times T} \times_{\mathfrak{X}\times\mathfrak{X}} \mathfrak{X}$ by Lemma 3.29.

For $(iv) \Rightarrow (v) \Rightarrow (i)$ it suffices to prove the claim

$$\mathfrak{S}_{\mathrm{Isom}_{\mathfrak{X}}(x,y)} \simeq \mathfrak{X} \times_{\mathfrak{X}\times\mathfrak{X},(x,y)} \underline{S}$$

$(v) \Rightarrow (i)$ follows directly, $(iv) \Rightarrow (v)$ after identifying

$$\mathfrak{X} \times_{\mathfrak{X}\times\mathfrak{X},(x,y)} \underline{S} \simeq ((\underline{S \times S}) \times_{(x\times y),\mathfrak{X}\times\mathfrak{X}} \mathfrak{X}) \times_{\underline{S\times S}} S \simeq (\underline{S} \times_{x,\mathfrak{X},y} \underline{S}) \times_{\underline{S\times S}} \underline{S}$$

Proof of claim Objects in the product on the right hand side are of the form[18]

$$(z, f : T \to S, (\alpha : z \to f^*x, \beta : z \to f^*y))$$

Note that $(z, f, (\alpha, \beta)) \cong (f^*x, f, (1_{f^*x}, \beta\alpha^{-1}))$ via $(\alpha, 1_f)$, a well-defined isomorphism due to the commutativity of

$$\begin{array}{ccc} (z, z) & \xrightarrow{(\alpha,\beta)} & (f^*x, f^*y) \\ \downarrow{\scriptstyle (\alpha,\alpha)} & & \| \\ (f^*x, f^*x) & \xrightarrow{(1_{f^*x},\beta\alpha^{-1})} & (f^*x, f^*y) \end{array}$$

Let $\mathfrak{Y}$ denote the full subcategory containing objects of the form $(f^*x, f, (1_{f^*x}, \phi))$. By the above isomorphism $\mathfrak{X} \times_{\mathfrak{X}\times\mathfrak{X},(x,y)} \underline{S} \simeq \mathfrak{Y}$. Lying over any $h : g \to f$ there is

[18] For simplicity of notation, we omit the base object T in the quadruples.

at most one morphism from $(g^*x, g, (1_{g^*x}, \psi))$ to $(f^*x, f, (1_{f^*x}, \phi))$, namely the one given by $g^*x \to f^*x$ and $h : g \to f$. It has to satisfy the compatibility

$$\begin{array}{ccc} (g^*x, g^*x) & \xrightarrow{(1_{g^*x},\psi)} & (g^*x, g^*y) \\ \downarrow & & \downarrow \\ (f^*x, f^*x) & \xrightarrow{(1_{f^*x},\phi)} & (f^*x, f^*y) \end{array}$$

which means $\psi = h^*\phi$. Therefore $\mathfrak{Y}$ is fibred in sets[19] and the pullback of

$$(f^*x, f, (1_{f^*x}, \phi))$$

along h is given by $(g^*x, g, (1_{g^*x}, h^*\phi))$. Thus we see that

$$(\phi : f^*x \to f^*y) \mapsto (f^*x, f, (1_{f^*x}, \phi))$$

defines a natural isomorphism of sheaves $\mathrm{Isom}_{\mathfrak{X}}(x, y) \cong \mathrm{Hom}(\_, \mathfrak{Y})$. $\square$

The claim used in the proof is at least as important as the proposition itself since it establishes the connection between properties of the diagonal and properties of the isomorphism sheaves.

5.2. Algebraicity

Having seen why the diagonal morphism is important, we state the definition of a Deligne-Mumford stack. The reader may compare it with the definition of algebraic spaces Definition 2.6.

Definition 5.8. A stack $\mathfrak{X}$ is a *Deligne-Mumford stack* (DM stack) if

(i). The diagonal $\mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ is representable.

(ii). There exist a scheme S and an étale, surjective morphism $\underline{S} \to \mathfrak{X}$ (representable by (i)) called an *atlas* or *presentation*.

The (2,1)-category of Deligne-Mumford stacks is defined as full sub-2-category of the (2,1)-category of stacks (in groupoids).

Warning and alternative definition The above definition is not quite right. It would be better to only require that the diagonal is representable by algebraic spaces. Then, we can define an *Artin stack* or in modern terminology[20] *algebraic stack* by weakening the morphism in (ii) to be smooth and surjective. For quasi-separated Deligne-Mumford stacks (see below) which are our main interest, representability

[19]The original 2-fibre product is therefore a *stack in setoids* (groupoids where every automorphism is the identity). By collapsing isomorphism classes, every category fibred in setoids can be replaced with an equivalent one fibred in sets, like we did with $\mathfrak{Y}$ in this case.

[20]In their original paper [DM69], Deligne and Mumford call their stacks "algebraic".

of the diagonal by algebraic spaces implies representability by schemes (see Corollary 5.14), so the above definition is sufficient. Very often, the condition of quasi-separatedness is included into the definition of DM stacks.

All statements we make here could also be proven with the broader definition of representability by algebraic spaces as soon as the required properties of algebraic spaces would be established.

When trying to define quasi-separatedness, we certainly want the diagonal morphism to be quasi-compact. However, unlike in the case of schemes and algebraic spaces, the diagonal itself is not automatically separated. Thus, we add this condition.

Definition 5.9. A stack is *quasi-separated* if the diagonal morphism is quasi-compact and separated.

Alternative definition Note that [Sta15] has the weaker condition of a quasi-compact and quasi-separated diagonal morphism.

Example 5.10. For algebraic spaces F (in particular schemes), $\mathfrak{S}_F$ is a DM stack. An atlas is given by the étale atlas of the algebraic space (which is just $\underline{X} \to \underline{X}$ for a scheme X). If F is quasi-separated as a scheme/algebraic space, F is quasi-separated as a stack. The *void stack* $\emptyset$ is represented by the void scheme.

Lemma 5.11. *The diagonal of a Deligne-Mumford stack is unramified*[21].

Proof. Taking an atlas $\underline{S} \to \mathfrak{X}$ and a test scheme $\underline{T} \to \mathfrak{X}\times\mathfrak{X}$, we get a 2-commutative diagram

$$\begin{tikzcd}
\mathfrak{X} \times_{\mathfrak{X}\times\mathfrak{X}} \underline{T} \times_{\mathfrak{X}} \underline{S} \arrow[rrr] \arrow[dr] \arrow[ddd] & & & \underline{T} \times_{\mathfrak{X}\times\mathfrak{X}} \underline{S\times S} \arrow[dl] \arrow[ddd] \\
& \underline{S} \arrow[r] \arrow[d] & \underline{S\times S} \arrow[d] & \\
& \mathfrak{X} \arrow[r] & \mathfrak{X}\times\mathfrak{X} & \\
\mathfrak{X} \times_{\mathfrak{X}\times\mathfrak{X}} \underline{T} \arrow[rrr] \arrow[ur] & & & \underline{T} \arrow[ul]
\end{tikzcd}$$

All outer squares except for the top one are 2-cartesian, hence the top one is 2-cartesian too. But therefore, the top arrow is an immersion. Because the two outer down arrows are surjective étale ($\underline{S} \times \underline{S} \to \mathfrak{X} \times \mathfrak{X}$ is étale, see proof of Lemma 5.3), the bottom arrow is unramified too. □

Corollary 5.12. *A quasi-separated Deligne-Mumford stack has diagonal of finite type. Thus, the diagonal is quasi-finite and quasi-affine.*

[21] in the sense of Raynaud, so in particular locally of finite type

Proof. The quasi-compactness condition of quasi-separatedness together with locally of finite type from the preceding statement implies the stack is of finite type. Of finite type and unramified imply quasi-finite. And quasi-finite and separated imply quasi-affine. □

Remark 5.13. In more generality, Artin stacks have diagonals of locally of finite type (cf. [LMB00, 4.2]). Unramifiedness and the implications in the corollary are also valid for representability by algebraic spaces, thus it follows that the diagonal of a quasi-separated DM stack is quasi-affine, even with the weaker notion of representability by algebraic spaces. But this implies the diagonal is representable by schemes, because all algebraic spaces quasi-affine over schemes are themselves schemes.

Corollary 5.14. *For x an object over quasi-compact S in a quasi-separated Deligne-Mumford stack $\mathfrak{X}$, x has only finitely many automorphisms over S.*

Proof. We can assume connectedness of S. Because $\mathrm{Isom}_{\mathfrak{X}}(x,x)$ over S is quasi-finite and separated, any section $S \to \mathrm{Isom}_{\mathfrak{X}}(x,x)$ is determined by choosing the image of a fixed and an inclusion of the residue fields ([Sta15, Tag 024V]), for which we have only finitely many options. □

Corollary 5.14 shows that Deligne-Mumford stacks can only capture stacky points with finite stabilisers. Moreover, in the presence of nontrivial sections, the diagonal cannot be an embedding, a crucial difference between stacks and schemes/algebraic spaces!

Lemma 5.15. *The (2,1)-category of DM stacks is closed under 2-fibre products.*

Proof. Let $\mathfrak{X}, \mathfrak{Y}$ be DM stacks over $\mathfrak{Z}$, a stack in groupoids with representable diagonal. (More is not needed for $\mathfrak{Z}$.) The analogous proof also works for algebraic stacks. Representability of the diagonal is implied by the cartesian diagram in the proof of Lemma 4.3. If $\underline{S} \to \mathfrak{X}$, $\underline{T} \to \mathfrak{Y}$ are atlases, then $\underline{S} \times_{\mathfrak{Z}} \underline{T}$ is representable by a scheme and

$$\underline{S} \times_{\mathfrak{Z}} \underline{T} \to \mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y}$$

is étale and surjective as product of étale and surjective maps. □

5.3. Important criteria

In order to check that a stack is a Deligne-Mumford stack, we need to check two further conditions, one on the diagonal and one about the existence of étale covers. The next proposition makes it easier to check the diagonal conditions.

Proposition 5.16. *Let $\mathfrak{X}$ be a stack, S a scheme and x, y objects in $\mathfrak{X}_S$. Assume that there exists an étale cover $f : S' \to S$ such that $\mathrm{Isom}_{\mathfrak{X}}(f^*x, f^*y)$ is represented by a scheme T', quasi-affine over S'. Then $\mathrm{Isom}_{\mathfrak{X}}(x,y)$ is represented by a scheme T, quasi-affine over S. As a direct consequence, if we can find such a cover for all S, the diagonal morphism of $\mathfrak{X}$ is quasi-affine (hence quasi-compact and separated).*

Proof. Set $F = \mathrm{Isom}_{\mathfrak{X}}(x, y)$. The assumption gives isomorphisms $\gamma : h_{T'} \cong F \times_{h_S} \times h_{S'}$ given on sections by $g \mapsto (\gamma_1(g), \gamma_2(g))$. Hence we get isomorphisms

$$h_{S'} \times_{h_S} h_{T'} \cong h_{S'} \times_{h_S} F \times_{h_S} h_{S'} \cong h_{T'} \times_{h_S} h_{S'}$$

given by $(f, g) \mapsto (f, \gamma_1(g), \gamma_2(g)) \mapsto (\gamma^{-1}(\gamma_1(g), f), \gamma_2(g))$. These satisfy the cocycle condition

$$\begin{tikzcd}
h_{S'} \times_{h_S} h_{T'} \times_{h_S} h_{S'} \arrow[rr] \arrow[dr] & & h_{T'} \times_{h_S} h_{S'} \times_{h_S} h_{S'} \\
& h_{S'} \times_{h_S} h_{S'} \times_{h_S} h_{T'} \arrow[ur] &
\end{tikzcd}$$

as can be checked: The top arrow maps $(f_1, g, f_2) \mapsto (\gamma^{-1}(\gamma_1(g), f_1), \gamma_2(g), f_2)$, while the composition of the other two maps

$$\begin{aligned}
(f_1, g, f_2) &\mapsto (f_1, \gamma_2(g), \gamma^{-1}(\gamma_1(g), f_2)) \\
&\mapsto (\gamma^{-1}(\gamma_1(\gamma^{-1}(\gamma_1(g), f_2)), f_1), \gamma_2(g), \gamma_2(\gamma^{-1}(\gamma_1(g), f_2))) \\
&= (\gamma^{-1}(\gamma_1(g), f_1), \gamma_2(g), f_2)
\end{aligned}$$

Thus, after reversing the Yoneda embedding, one can apply faithfully flat descent for quasi-affine morphisms and gets a scheme T quasi-affine over S such that $h_T \times_{h_S} h_{S'} \cong h_{T'}$.[22]

Applying the sheaf condition for $S' \to S$ to F and h_T, we see that they are both equalisers of the same sets (modulo compatible isomorphism), so $F(S) \cong h_T(S)$ over $h_S(S) = 1_S$ and for any $S'' \to S$ values of both sheaves are given by pulling back along $h_{S''} \to h_S$. It follows that $F \cong h_T$. $\square$

Example 5.17. This can be used to check (A) for the quotient stack $[X/G]$ of a quasi-affine group scheme G acting on a quasi-separated scheme X: It suffices to check that after trivialising, $\mathrm{Isom}(S \times G, S \times G)$ is represented by a scheme quasi-affine over S. We freely use element notation $s \in S, t \in T, g \in G$ with trust in the reader's ability to translate this into set-theoretic statements for Hom-functors.

Let $f : T \to S$ be a morphism. Then an isomorphism $i : T \times G \to T \times G$ which is G-equivariant has the form $(t, g) \mapsto (t, \alpha(t)g)$ for some $\alpha : T \to G$. The equivariant morphisms $S \times G \to X$ are of the form $(s, g) \mapsto x(s)g, (s, g) \mapsto y(s)g$ for some $x, y : S \to X$. Then, compatibility of i with the structure morphisms to X means

$$x(f(t))g = y(f(t))(\alpha(t)g)$$

for all $g \in G$, so particularly for $g = e$ the neutral elements (and then for all other g). Therefore $\mathrm{Isom}(S \times G, S \times G)$ is represented by $(S \times G) \times_{X \times X} X$ where $S \times G \to X \times X$

[22] The construction of descent data from sheaves is a more general statement: For an fppf covering $(S_i \to S)$, there is an equivalence of categories between descent data $(T_i/S_i, \phi_{ij})$ and sheaves F on S such that $h_{S_i} \times_{h_S} F$ are representable (see [Sta15, Tag 02W4] - they do not work with fpqc sites because of problems when bounding cardinalities).

via $(s,g) \mapsto (x(s), y(s)g)$ and $X \to X \times X$ the diagonal morphism:

$$\begin{aligned} & \mathrm{Hom}_S(T, (S \times G) \times_{X \times X} X) \\ = \; & \{a : T \to S, \alpha : T \to G, \beta : T \to X | a = f, x(a(t)) = \beta(t) = y(a(t))\alpha(t)\} \\ = \; & \{\alpha : T \to G | x(f(t)) = y(f(t))\alpha(t)\} \end{aligned}$$

This scheme is quasi-affine over S because $S \times G$ is quasi-affine over S and

$$(S \times G) \times_{X \times X} X \to (T \times G)$$

is a base change of $X \to X \times X$, locally quasi-finite and separated (as for every scheme) and quasi-compact (due to X quasi-separated), hence quasi-affine.

The next proposition deals with the condition of finding an étale cover. More precisely, it states that it is sufficient to find a smooth, surjective cover as long as the diagonal is unramified.

Proposition 5.18. *Let $\mathfrak{X}$ be a stack over a Noetherian scheme $\underline{\Lambda}$. Assume that*

(i). The diagonal $\Delta_{\mathfrak{X}}$ is representable and unramified.

(ii). There exists a smooth, surjective cover $\underline{U} \to \mathfrak{X}$.

(iii). U is of finite type over Λ.

Then there exists a scheme T of finite type over Λ and an étale, surjective cover $\underline{T} \to \mathfrak{X}$ making $\mathfrak{X}$ a Deligne-Mumford stack.

Like [Edi00, 2.1], we are only proving the case where the residue fields of Λ are perfect, e.g. $\Lambda = \mathbb{Z}$. Noetherian and finite type properties can be dropped from the statement ([Sta15, Tag 06N3]).

Proof. The proof idea can be described as follows:

Pick a closed point $\operatorname{Spec} k = u \in U$. It is enough to cut out a closed subscheme Z_u of an étale neighbourhood of U which pulled back along the atlas is a neighbourhood of u.[23] The global étale cover will then be a finite disjoint union of such Z_u by Noetherianess. By assumption the fibre $\underline{u} \times_{\mathfrak{X}} \underline{U}$ is smooth over k. The problem is the relative dimension which we have to *slice* down to 0. After showing that $\underline{u} \times_{\mathfrak{X}} \underline{U} \to \underline{U}$ is unramified using the unramified diagonal (and in our case extra assumptions), we can restrict locally to one (étale) branch $V' \to U'$ of this morphism and take as Z_u a subscheme intersecting the fibre transversely. This situation is pictured in Figure 5.1.

Now to the details: Choose a closed point $u = \operatorname{Spec} k \in U$ and take the fibre $\underline{V} = \underline{u} \times_{\mathfrak{X}} \underline{U}$ which is smooth over k. Set z the point in V induced by $(1_u, u \to U)$.

[23] We could try to find an étale slice Z_u that contains u but in fact, not every point in U is contained in such a slice (cf. [Beh06, Exercise 5.7] for an example).

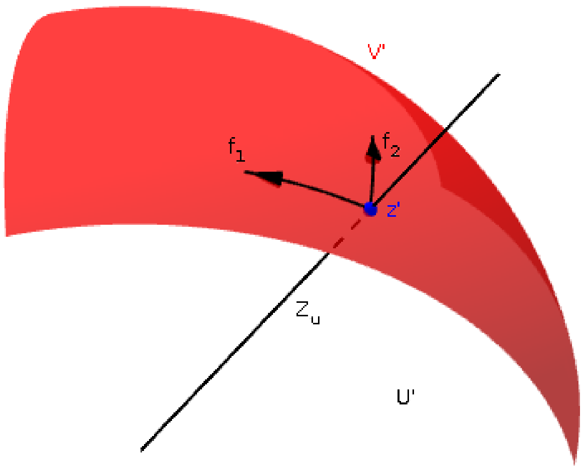

Figure 5.1: Illustration of the slicing in the proof of Proposition 5.18

Next, prove that $V \to U$ is unramified. For this note that $V \to u \times_\Lambda U$ arises as base change of the diagonal along $\underline{u} \times_{\underline{\Lambda}} \underline{U} \to \mathfrak{X} \times_{\underline{\Lambda}} \mathfrak{X}$, hence is unramified. $u \to U \to \Lambda$ is also unramified since U is of finite type over Λ, meaning $k(u)$ is a finite (separable) extension of a (perfect) residue field of Λ. Therefore $u \times_\Lambda U \to U$ is unramified. Composition $V \to u \times_\Lambda U \to U$ yields the claim.

The result in [GD67, 18.4.8] lets us restrict locally around z to a branch of $V \to U$, i.e. we have étale neighbourhoods V' and U' of V and U respectively with $z' \in U$ mapping to z, $u' \in U$ mapping to u and a closed immersion $V' \hookrightarrow U'$ giving a commutative diagram

$$\begin{tikzcd} V' \arrow[r, hook] \arrow[d] & U' \arrow[d] \\ V \arrow[r] & U \end{tikzcd}$$

The local ring $\mathcal{O}_{V',z'}$ is regular, therefore after shrinking V' the point z' is cut out by a regular sequence of sections of $\mathcal{O}_{V'}$ $f_1, \dots, f_n$. Lifting these to sections of $\mathcal{O}_{U'}$, gives a closed subscheme Z_u of U'.

We show that $\underline{Z_u} \to \mathfrak{X}$ is étale in a neighbourhood of z'. It suffices to check after base change along U (Lemma 5.6). For this, we see that Z_u is cut out of U' by a regular sequence, which means it is a regular immersion and so is the flat base change $\underline{Z_u} \times_\mathfrak{X} \underline{U} \to \underline{U'} \times_\mathfrak{X} \underline{U}$. Therefore by smoothness of $\underline{U'} \times_\mathfrak{X} \underline{U} \to \underline{U'}$ and [GD67, 17.12.1], $\underline{Z_u} \times_\mathfrak{X} \underline{U} \to \underline{U}$ is smooth in a neighbourhood of z'. To compute the relative dimension, note that the fibre $\underline{z'} \times_{\underline{U}} \underline{Z_u} \times_\mathfrak{X} \underline{U}$ is isomorphic to $\underline{V'} \times_{\underline{U'}} \underline{Z_u} \cong \underline{z'}$, a point, in an étale neighbourhood.

Having found étale $\underline{Z_u} \to \mathfrak{X}$ for each $u \in U$, each of finite type over Λ like U and U', the disjoint union of all of them is étale and surjective because the pullback along $\underline{U} \to \mathfrak{X}$ is surjective: Its image is open in U and contains every closed point. Because U is Noetherian (of finite type over a Noetherian scheme), it is enough to pick finitely many $\underline{Z_u} \times_{\mathfrak{X}} \underline{U}$ still covering U. $\square$

Example 5.19. Return to the case of a smooth quasi-affine group scheme acting on a quasi-separated scheme. This is a typical example of how finding a smooth, surjective atlas can be much easier than finding an étale one: The base change of the canonical morphism $\underline{X} \to [X/G]$ along $\underline{S} \to [X/G]$ has been computed as a principal homogeneous G-bundle $E \to S$ in Example 3.35.

If G is étale (over Λ), we are done since $\underline{X} \to [X/G]$ is étale surjective. If not, we need some extra condition to conclude the diagonal is unramified and apply our criterion. Beside X being of finite type and Λ having perfect residue fields, which we assumed in our proof of the criterion, the crucial condition is that the stabilisers of geometric points $x \to X$ under the action of G shall be *finite and geometrically reduced.*

Because of finiteness, the fibre of $\mathrm{Isom}_{\mathfrak{X}}(E, E) \to S$ at any point $\operatorname{Spec} k$ in S can have only finitely many points, hence is a disjoint union of field extensions of k. Now a point $\operatorname{Spec} K$ over $\operatorname{Spec} k$ is separable because it is geometrically reduced ([Sta15, Tag 030W]). Having seen that every fibre is a disjoint union of separable extensions, unramifiedness of all Isom-schemes, hence of the diagonal, follows by [Sta15, Tag 02G8].

6. Properties of stacks and stack morphisms

In this chapter, we generalise geometric properties known from schemes (or algebraic spaces) to algebraic stacks. On the one hand, the objective is to develop the properties that we need for the irreducibility of moduli stacks of curves in the next chapter.

On the other hand, we will go a bit further than necessary, in order to get used to working with stacks. Many statements made are also valid for Artin stacks, if one makes the usual replacements (algebraic spaces instead of schemes, smooth instead of étale).

6.1. Properties of stacks

A first broad definition can be made for properties P of schemes which are local in the étale topology (e.g. normal, locally Noetherian,...).

Definition 6.1. Let P be a property of schemes which is étale local. We say that a Deligne-Mumford stack has P, if there is an étale, surjective atlas satisfying P.

Lemma 6.2. *If one atlas has P, then any étale surjective atlas has P.*

Proof. Let $\underline{S} \to \mathfrak{X}$ be the atlas satisfying P. If $\underline{S}' \to \mathfrak{X}$ is a second atlas, then the projection maps of $\underline{S}' \times_{\mathfrak{X}} \underline{S}$ are étale surjective base changes. Therefore,

$$(\underline{S}' \times_{\mathfrak{X}} \underline{S} \to \underline{S} \to \mathfrak{X}) = (\underline{S}' \times_{\mathfrak{X}} \underline{S} \to \underline{S}' \to \mathfrak{X})$$

has P, so by locality $\underline{S'} \to \mathfrak{X}$ has P. □

Corollary 6.3. *If P, P' are properties like above, and $P \Rightarrow P'$ for schemes, then $P \Rightarrow P'$ also for DM stacks.*

Proof. Trivial. □

Other non-local properties need to be dealt with in a different way.

Definition 6.4. A DM stack $\mathfrak{X}$ is *quasi-compact*, if there exists a quasi-compact atlas $\underline{S} \to \mathfrak{X}$.

Definition 6.5. A DM stack $\mathfrak{X}$ is *Noetherian*, if it is quasi-compact, locally Noetherian and quasi-separated(!).

To define connectedness, we have the notion of a disjoint union:

Definition 6.6. Given a family of categories $(\mathfrak{X}_i)_{i\in I}$ over $\mathfrak{S}$, define their *disjoint union* $\coprod_{i\in I} \mathfrak{X}_i$ as the category having as objects (i, x), where $i \in I, x \in \mathfrak{X}_i$, and as morphisms $(i, x) \to (j, y)$ the morphisms $x \to y$ for $i = j$ but no morphisms for $i \neq j$. The projection to $\mathfrak{S}$ is defined in the obvious way.

It is also obvious that all our defined 2-categories are closed under disjoint unions and not very difficult to see that this definition coincides with our notion of connectedness for schemes.

Definition 6.7. A DM stack is *connected*, if it cannot be written as the disjoint union of two non-void (necessarily) DM stacks.

Lemma 6.8. *Let $\mathfrak{X}$ be a locally Noetherian, DM stack. Then $\mathfrak{X}$ can be written in a unique way as the disjoint union of connected DM stacks. These are called* connected components *of $\mathfrak{X}$.*

Proof. Writing a DM stack as disjoint union gives rise to writing its (locally Noetherian) atlas as disjoint union. □

The second important topological property in the world of schemes is irreducibility. For this, we define the notion of substacks.

Definition 6.9. A morphism of stacks $\mathfrak{X} \to \mathfrak{Y}$ is an *open immersion/closed immersion/immersion*, if it is representable and an open immersion/closed immersion/immersion in the sense of representable morphisms. An *open/closed/locally closed substack* of $\mathfrak{X}$ is a strictly[24] full subcategory whose inclusion is an open immersion/closed immersion/immersion.

Note that the substack definition is *not* stable under equivalence. However, we have the following lemma:

[24] If x is an object in the subcategory and $x \cong y$, then y is in the subcategory too.

Lemma 6.10. *If $i : \mathfrak{X} \to \mathfrak{Y}$ is an open immersion/closed immersion/immersion, then there is a unique open/closed/locally closed substack of $\mathfrak{Y}$ through which i factors.*

Proof. Trivial. $\square$

Lemma 6.11. *Given two open/closed/locally closed substacks $\mathfrak{U}, \mathfrak{U}' \subset \mathfrak{X}$, there intersection as categories (taking objects contained in both and all morphisms between them) yields an open/closed/locally closed substack again.*

Proof. The constructed subcategory is strictly full. Note that $\mathfrak{U} \cap \mathfrak{U}'$ is equivalent to $\mathfrak{U} \times_{\mathfrak{X}} \mathfrak{U}'$, hence a DM stack. It is indeed an open/closed/locally closed substack since given atlases $\underline{S} \to \mathfrak{X}$ and $\underline{T} = \underline{S} \times_{\mathfrak{X}} \mathfrak{U} \to \mathfrak{U}, \underline{T}' = \underline{S} \times_{\mathfrak{X}} \mathfrak{U}'$, an atlas of $\mathfrak{U} \times_{\mathfrak{X}} \mathfrak{U}' \to \mathfrak{U}'$ is given by $\underline{T} \times_{\mathfrak{X}} \underline{T}'$ which maps to $\underline{S}$ via the composition of two (open/closed) immersions. $\square$

Definition 6.12. A DM stack is called *irreducible*, if any two open substacks in it have non-void intersection.

Pulling back an open substack along an atlas gives an open subscheme. An important observation to be able to think about substacks in terms of atlases is that the other direction works too:

Lemma 6.13. *Let $\underline{S} \to \mathfrak{X}$ be the atlas of a DM stack and $U \subset S$ an open subscheme. Then there exists an open substack $\mathfrak{Y}$ of $\mathfrak{X}$ with atlas $\underline{U} \to \mathfrak{Y}$ giving $\underline{U} \hookrightarrow \underline{S} \to \mathfrak{X}$ after composition with $\mathfrak{Y} \to \mathfrak{X}$.*

Proof. One can define $\mathfrak{Y}$ as the full subcategory of $\mathfrak{X}$ whose fibre category over an arbitrary scheme T are the objects x over T such that $\underline{T} \times_{x,\mathfrak{X}} \underline{U} \to \underline{T}$ is surjective. This is stricty full. Because for $f : T' \to T$ the two squares

$$\begin{array}{ccccc} \underline{T}' \times_{f^*x,\mathfrak{X}} \underline{U} & \longrightarrow & \underline{T} \times_{x,\mathfrak{X}} \underline{U} & \longrightarrow & \underline{U} \\ \downarrow & & \downarrow & & \downarrow \\ \underline{T}' & \xrightarrow{\quad f \quad} & \underline{T} & \xrightarrow{\quad x \quad} & \mathfrak{X} \end{array}$$

are 2-cartesian, $\mathfrak{Y}$ has cartesian arrows, hence is fibred in groupoids. (Surjectivity is stable under base change.) From this diagram, it also follows that we have representable Isom-sheaves (the same as before) and that $\mathfrak{Y}$ is closed under descent on objects (choosing f to be an étale cover and using surjectivity is local on the target). Thus, we have shown every property of a DM stack except for the étale atlas which we will do at the end.

Objects in $\underline{T} \times_{x,\mathfrak{X}} \mathfrak{Y}$ are of the form $(f : T' \to T, y \in Ob(\mathfrak{X}_{T'}), f^*x \cong y)$ with surjective $\underline{T}' \times_{y,\mathfrak{X}} \underline{U} \to \underline{T}'$. This gives a diagram like the last one but with f^*x replaced by y. The middle vertical arrow is smooth by stability under base change, so in particular open and we can take its image T_0 in T, an open subscheme.

If f does not land in T_0, then $\underline{T'} \times_{y,\mathfrak{X}} \underline{U} \to \underline{T'}$ must not be surjective due to 2-commutativity of the left square. If on the other hand f lands in T_0, then we can replace the base T by T_0 and get that $\underline{T'} \times_{y,\mathfrak{X}} \underline{U} \to \underline{T'}$ is surjective as base change of a surjective morphism. Concluding from the just established logical equivalence, $\underline{T} \times_{x,\mathfrak{X}} \mathfrak{Y}$ is equivalent to $\underline{T_0}$, therefore $\mathfrak{Y} \to \mathfrak{X}$ will be a representable open immersion.

Setting $g = (\underline{U} \hookrightarrow \underline{S} \to \mathfrak{X})$, we know that $\underline{U} \times_{g,\mathfrak{X},g} \underline{U}$ is surjective, a section being given by the diagonal morphism. This means, by definition of $\mathfrak{Y}$, g is in $\mathfrak{Y}_U$. Therefore, there exists a 2-commutative diagram

$$\begin{tikzcd} \underline{U} \arrow[r] \arrow[d, "g'"] & \underline{S} \arrow[d] \\ \mathfrak{Y} \arrow[r] & \mathfrak{X} \end{tikzcd}$$

The morphism $\underline{U} \times_{g,\mathfrak{X}} \mathfrak{Y} \to \underline{U}$ is an open immersion with a section $(1, g')$, thus it is an isomorphism. But this means g' is the base change of g along $\mathfrak{Y} \to \mathfrak{X}$. This implies it is étale. It is surjective because for $\underline{T} \to \mathfrak{Y}$

$$\begin{tikzcd} \underline{U} \times_{\mathfrak{X}} \underline{T} \arrow[r] \arrow[d] & \underline{U} \times_{\mathfrak{X}} \mathfrak{Y} \arrow[r] \arrow[d] & \underline{U} \arrow[d] \\ \underline{T} \arrow[r] & \mathfrak{Y} \arrow[r] & \mathfrak{X} \end{tikzcd}$$

is 2-cartesian and by construction the left vertical arrow is surjective. □

With the preceding lemma, arbitrary unions of open substacks can be defined:

Lemma 6.14. *Let $(\mathfrak{X}_i)$ be a family of open substacks of $\mathfrak{X}$. Then there exists an open substack $\bigcup \mathfrak{X}_i \subset \mathfrak{X}$ with open immersions $\mathfrak{X}_i \to \bigcup \mathfrak{X}_i$ for all i such that the $\mathfrak{X}_i$ cover $\bigcup \mathfrak{X}_i$, i.e. $\coprod \mathfrak{X}_i \to \bigcup \mathfrak{X}_i$ is surjective.*

Proof. Take an atlas $\underline{S} \to \mathfrak{X}$ and open immersions $\underline{S_i} \subset S$ corresponding to $\mathfrak{X}_i \subset \mathfrak{X}$. Then the open subscheme $\bigcup S_i \subset S$ gives rise to the open substack $\bigcup \mathfrak{X}_i \subset \mathfrak{X}$. This has the desired properties as

$$\begin{tikzcd} \underline{\coprod S_i} \arrow[r] \arrow[d] & \coprod \mathfrak{X}_i \arrow[d] \\ \underline{\bigcup S_i} \arrow[r] \arrow[d] & \bigcup \mathfrak{X}_i \arrow[d] \\ \underline{S} \arrow[r] & \mathfrak{X} \end{tikzcd}$$

is 2-cartesian and $\coprod S_i \to \bigcup S_i$ is surjective. □

A direct description of $(\bigcup \mathfrak{X}_i)_T$ is that it contains all objects x over T with $\coprod(\underline{T} \times_{x,\mathfrak{X}} \mathfrak{X}_i) \to \underline{T}$ surjective.

For matters of completeness, we state without proof a lemma relating open and closed substacks as well as a direct corollary.

Lemma 6.15. *Let $\mathfrak{U} \subset \mathfrak{X}$ be an open substack. Then there exists a unique closed substack $\mathfrak{Z} \subset \mathfrak{X}$ such that for all atlases $\underline{S} \to \mathfrak{X}$, $\underline{S} \times_{\mathfrak{X}} \mathfrak{Z}$ is the closed reduced complement of $\underline{S} \times_{\mathfrak{X}} \mathfrak{U}$ in $\underline{S}$. If $\mathfrak{U} = \emptyset$, we write $\mathfrak{Z} = \mathfrak{X}_{red}$.*

Proof. see [LMB00, 4.10]. □

Corollary 6.16. *Let $\mathfrak{X}$ be a locally Noetherian DM stack. Up to permutation, there exists a unique family $\mathfrak{X}_i$ of closed reduced substacks, the* irreducible components *such that none is contained in another and they cover $\mathfrak{X}_{red}$.*

Proof. Trivial. □

6.1.1. Points of a stack

Without doubt, the notion of substacks "smells like topological spaces". Indeed, we will now introduce a topological space associated to a DM stack.

Definition 6.17. Let $\mathfrak{X}$ be a DM stack. Denote by $|\mathfrak{X}|$ the set $\coprod_k \mathfrak{X}_{\mathrm{Spec}(k)}$ where k ranges over all fields (over Λ) modulo the following equivalence relation: Two elements x, x' over k respectively k' are equivalent iff there exists an extension K of k and k' such that the pullbacks of x and x' are isomorphic. Elements in $|\mathfrak{X}|$ are called *points of* $\mathfrak{X}$.

A morphism $F : \mathfrak{X} \to \mathfrak{Y}$ of DM stacks gives rise to a map $|F| : |\mathfrak{X}| \to |\mathfrak{Y}|, x \mapsto F \circ x$ which is independent of the chosen representative since fields that pull back to an extension over $\mathfrak{X}$ do so over $\mathfrak{Y}$. Under this association, 2-commutative diagrams become commutative. As a special case, if F is an equivalence, then $|F|$ is bijective.

If $\underline{S} \to \mathfrak{X}$ is an atlas, every point $x : \underline{\mathrm{Spec}\, k} \to \mathfrak{X}$ comes from $\underline{S}$. To see this, just take a point in the scheme $\underline{S} \times_{\mathfrak{X}} \underline{\mathrm{Spec}\, k}$. Hence, we only need to care about residue fields of S. In particular, the stack notion of point coincides with that known from schemes. (From the above remark, one may also reassure oneself that there are no set-theoretic issues in defining $|\mathfrak{X}|$.)

Lemma 6.18. *(i). A DM stack is empty iff its point set is.*

(ii). A representable morphism of DM stacks F is surjective iff $|F|$ is[25].

(iii). The natural map $|\mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y}| \to |\mathfrak{X}| \times_{|\mathfrak{Z}|} |\mathfrak{Y}|$ is surjective.

(iv). For any DM stack $|\mathfrak{X}| = |\mathfrak{X}_{red}|$.

(v). If $\mathfrak{X}, \mathfrak{Y}$ are locally closed substacks of $\mathfrak{Z}$ and $\mathfrak{X}$ is reduced or $\mathfrak{Y}$ is open (complementary cases), then $\mathfrak{X} \subset \mathfrak{Y}$ iff $|\mathfrak{X}| \subset |\mathfrak{Y}|$.

Proof. The proofs are all straightforward. □

[25] This also holds for non-representable morphisms with the notion introduced in the next subsection.

The topology on $|\mathfrak{X}|$ is defined in the obvious way, namely we take as open sets all $|\mathfrak{U}|$ where $\mathfrak{U} \subset \mathfrak{X}$ is an open substack. This is called the *Zariski topology* of the stack.

Lemma 6.19. *If $F : \mathfrak{X} \to \mathfrak{Y}$ is a morphism of DM stacks, then $|F|$ is continuous.*

Proof. Choose two atlases giving a 2-commutative diagram

$$\begin{array}{ccc} \underline{S} & \longrightarrow & \underline{T} \\ \downarrow & & \downarrow \\ \mathfrak{X} & \longrightarrow & \mathfrak{Y} \end{array}$$

This gives a commutative diagram

$$\begin{array}{ccc} |S| & \longrightarrow & |T| \\ \downarrow & & \downarrow \\ |\mathfrak{X}| & \longrightarrow & |\mathfrak{Y}| \end{array}$$

and from our discussion of open substacks we know that the vertical arrows are continuous and open. Also, the top arrow between spaces underlying schemes is continuous. Thus, the preimage of an open $|\mathfrak{V}| \subset |\mathfrak{Y}|$ in $|S|$ is an open V. Because the left vertical arrow also is surjective, we know that $|F|^{-1}(|\mathfrak{V}|)$ is the image of V in $|\mathfrak{Y}|$, which is open. □

The continuity of $|F|$ allows us to use notions like connectedness, irreducibility and density in an intuitive, topological way. We may also define *closed, open, dominant and universally closed* morphisms of DM stacks in the obvious way. Quasi-compactness, which we defined earlier, behaves as expected:

Lemma 6.20. *A DM stack $\mathfrak{X}$ is quasi-compact iff $|\mathfrak{X}|$ is quasi-compact.*

Proof. If $\underline{S} \to \mathfrak{X}$ is a quasi-compact atlas, then $|\mathfrak{X}|$ is the image of the quasi-compact set $|S|$ under a continuous map, hence quasi-compact. On the other hand, having $|\mathfrak{X}|$ quasi-compact, implies that we can choose finitely many elements from a cover of S by quasi-compact opens which map surjectively to $|\mathfrak{X}|$ and therefore give a quasi-compact atlas. □

Corollary 6.21. *(i). $|\mathfrak{X}|$ admits a base of open quasi-compacts.*

(ii). For an open, surjective morphism F of DM stacks, e.g. an atlas, every open quasi-compact in the target is the image of an open quasi-compact.

Proof. (i). This is clear since it is valid for an atlas $|S|$ and $|S| \to |\mathfrak{X}|$ is open.

(ii). Follows from (i).

□

As always, there are more interesting facts about the Zariski topology (like soberness, Chevalley's theorem and constructible sets...). However, we stop at this point and end with a lemma which is of importance for Deligne-Mumford's irreducibility proof.

Lemma 6.22. *A locally Noetherian, normal, connected DM stack $\mathfrak{X}$ is irreducible. (Hence, for an arbitrary normal, locally Noetherian DM stack, the connected and irreducible components coincide.)*

Proof. Choose a (normal) atlas $\underline{S} \to \mathfrak{X}$. Cover S by locally Noetherian, affine, connected, normal opens U_i. Then it follows that the U_i are spectra of normal domains (see [Sta15, Tag 030C]), hence irreducible. The images $|\mathfrak{U}_i|$ of the $|U_i|$'s are irreducible by continuity and cover $|\mathfrak{X}|$. So $|\mathfrak{X}|$ is locally irreducible and connected, hence irreducible. □

6.2. Properties of stack morphisms

The reader may check that for suitable properties P all definitions made coincide with the one made for representable morphisms. A first broad definition can be applied to properties P *étale local on source-and-target*, i.e. for any commutative diagram of schemes

$$\begin{tikzcd} S_i \arrow[r] \arrow[d, "f_i"] & S \arrow[d, "f"] \\ T_i \arrow[r] & T \end{tikzcd}$$

with étale covers $S_i \to S, T_i \to T$, $P(f)$ iff $\forall i : P(f_i)$.[26] Examples are flat, smooth, unramified, étale, normal, locally of finite type/presentation...

Definition 6.23. A morphism of DM stacks $f : \mathfrak{X} \to \mathfrak{Y}$ is said to *have property P* assumed étale local on source-and-target, iff for some atlases $\underline{S} \to \mathfrak{X}, \underline{T} \to \mathfrak{Y}$ and morphisms $\underline{S} \to \underline{T}$ with 2-commutative diagram

$$\begin{tikzcd} \underline{S} \arrow[r] \arrow[d] & \mathfrak{X} \arrow[d] \\ \underline{T} \arrow[r] & \mathfrak{Y} \end{tikzcd}$$

$S \to T$ has P.

Lemma 6.24. *If some choice of atlases $\underline{S} \to \underline{T}$ like above satisfies P, so does any choice.*

[26] This is the Deligne-Mumford definition. The one found in [Sta15, Tag 04QW] is a bit stronger in that it assumes stability under postcomposition with open immersions.

Proof. Let $\underline{S}' \to \underline{T}'$ be a second choice. Then consider the 2-commutative diagram

$$\begin{array}{ccccc} \underline{S} & \longleftarrow & \underline{S} \times_{\mathfrak{X}} \underline{S}' & \longrightarrow & \underline{S}' \\ \downarrow & & \downarrow & & \downarrow \\ \underline{T} & \longleftarrow & \underline{T} \times_{\mathfrak{Y}} \underline{T}' & \longrightarrow & \underline{T}' \end{array}$$

where the horizontal arrows are étale covers and use locality. □

Lemma 6.25. *Let P, P' be properties étale local on source-and-target.*

(i). If P is stable under composition/base change as property of schemes, then it is as property of stacks.

(ii). If $P \Rightarrow P'$ as properties of schemes, then also as properties of stacks.

Proof. Trivial. □

Definition 6.26. A morphism $F : \mathfrak{X} \to \mathfrak{Y}$ of DM stacks is *quasi-compact* if for any quasi-compact scheme S and $\underline{S} \to \mathfrak{Y}$, $\underline{S} \times_{\mathfrak{Y}} \mathfrak{X}$ is a quasi-compact stack.

By looking at open subschemes, we see that this notion of quasi-compactness coincides with the one we have for schemes.

Lemma 6.27. *Quasi-compactness is stable under base change and composition.*

Proof. Trivial. □

It is also not difficult to see that a morphism F is quasi-compact iff the preimages of open quasi-compacts under $|F|$ are quasi-compact again.

Definition 6.28. A morphism of DM stacks is *of finite type/presentation* if it is locally of finite type/presentation and quasi-compact.

For separated morphisms, we want a notion that is stronger than our notion of quasi-separated morphisms (over $\underline{\Lambda}$) but circumventing the issue of not having a diagonal immersion.

Definition 6.29. A morphism $F : \mathfrak{X} \to \mathfrak{Y}$ is *separated* if its diagonal $\Delta_F : \mathfrak{X} \to \mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{X}$ is proper. As usual, if $\mathfrak{X} \to \underline{\Lambda}$ is separated, we say $\mathfrak{X}$ is separated.

A DM stack is separated iff the diagonal morphism is finite iff it is quasi-separated and universally closed. This follows from the equivalence of the following properties for scheme morphisms (proved in [Sta15, Tag 02LS]):

(i). finite

(ii). proper and finite fibres

(iii). universally closed, separated, locally of finite type and finite fibres

Definition 6.30. A morphism of DM stacks is *proper* if it is separated, of finite type and universally closed.

This is the definition of [LMB00], the original one in [DM69] is rather ad hoc.

There are valuative criteria for stacks which we only give in the case of morphisms $\mathfrak{X} \to \underline{\Lambda}$. For the formulation of the criteria, assume $\mathfrak{X}$ to be quasi-separated and of finite type and Λ Noetherian.

Proposition 6.31 (Valuative criterion for separation)**.** *$\mathfrak{X}$ is separated iff the following is true: For any complete discrete valuation ring R with fraction field K and algebraically closed residue field, a morphism $\operatorname{Spec}(R) \to \Lambda$, and two choices g_1, g_2 for g making the diagram*

$$\begin{tikzcd} & \mathfrak{X} \arrow[d] \\ \underline{\operatorname{Spec}(R)} \arrow[ur, "g"] \arrow[r] & \underline{\Lambda} \end{tikzcd}$$

2-commute, any 2-isomorphism between the restrictions of g_1, g_2 to $\operatorname{Spec}(K)$ can be extended to a 2-isomorphism between g_1 and g_2. Moreover, it suffices to consider g_1, g_2 whose restrictions factor through a given dense open substack of $\mathfrak{X}$.

Proof. For separation, we have to check that base changes of the diagonal morphism $\Delta_{\mathfrak{X}}$ along some $(g_1', g_2') : \underline{S} \to \mathfrak{X} \times \mathfrak{X}$ (the naming g_1', g_2' is chosen suggestively) are proper. Applying the valuative criterion for properness of scheme morphisms (e.g. in [Gro61, 7.3.8]) gives a unique lift in the diagram

$$\begin{tikzcd} \underline{\operatorname{Spec}(K)} \arrow[r] \arrow[d] & \operatorname{Isom}_{\mathfrak{X}}(g_1', g_2') \arrow[d] \\ \underline{\operatorname{Spec}(R)} \arrow[ur, dashed] \arrow[r, "f"] & \underline{S} \end{tikzcd}$$

But this lifting is exactly the extension needed in the statement of the proposition. (One direction gives a lifting of isomorphisms for f^*g_1', f^*g_2', the other direction follows by choosing $S = \operatorname{Spec}(R), f = 1_S$.)

In order to see the last remark, note that properness of the representable diagonal can be checked for base changes along atlases $\underline{T} \to \mathfrak{X}$. A dense open in $\mathfrak{X}$ gives a dense open in the atlas which in turn gives by base change a dense open in the above valuative criterion for schemes. In this case, it is known that the criterion can be checked on a dense open (cf. [Gro61, 7.3.10 (ii)]). $\square$

Proposition 6.32 (Valuative criterion for properness)**.** *Assume $\mathfrak{X}$ is separated. $\mathfrak{X}$ is proper iff the following is true: For any complete discrete valuation ring R with fraction field K and algebraically closed residue field and a 2-commutative diagram*

$$\begin{tikzcd} \underline{\operatorname{Spec}(K)} \arrow[r] \arrow[d] & \mathfrak{X} \arrow[d] \\ \underline{\operatorname{Spec}(R)} \arrow[r] & \underline{\Lambda} \end{tikzcd}$$

there exists a finite extension K'/K with integral closure R' of R in K' and an extension of the above diagram

$$\begin{tikzcd}
\underline{\mathrm{Spec}(K')} \arrow[r] \arrow[d] & \underline{\mathrm{Spec}(K)} \arrow[r] \arrow[d] & \mathfrak{X} \arrow[d] \\
\underline{\mathrm{Spec}(R')} \arrow[r] \arrow[urr] & \underline{\mathrm{Spec}(R)} \arrow[r] & \underline{\Lambda}
\end{tikzcd}$$

which is 2-commutative. (All morphisms between spectra shall be induced by the canonical inclusions.) Furthermore, it suffices to check this for $\underline{\mathrm{Spec}(K)} \to \mathfrak{X}$ factoring through a fixed, dense open substack.

Proof. The proof requires some work (especially a stacky version of Chow's lemma, see [LMB00, 7.12]) which would go beyond the scope of this introduction. It is however important to remember that unlike the scheme version, the lift in this valuative criterion is only guaranteed after switching to a larger field. See [Edi00] with an example due to Vistoli for why this is necessary. □

At this point, we are almost ready to proceed to Deligne and Mumford's irreducibility proof. There is only one little proposition missing which is the central lemma making their proof work.

Proposition 6.33. *Assume Λ is Noetherian. Let $F : \mathfrak{X} \to \underline{\Lambda}$ be a stack morphism of finite type, proper and flat. If F has geometrically normal fibres, then the number of connected components in the geometric fibres is locally constant.*

Proof. We show why this is true for a scheme morphism $f : X \to \Lambda$. Recall the Stein factorisation

$$X \xrightarrow{f'} Y \xrightarrow{g} \Lambda$$

where $Y = \mathrm{Spec}\, f_*(\mathcal{O}_X)$, f' is proper with $f'_*\mathcal{O}_X = \mathcal{O}_Y$ and g is finite. Now by Zariski's connectedness theorem, the fibres of f' are connected, so the connected components of f stand in bijection with those of g.

Applying [Gro63, 7.8.10] says that for a proper flat morphism with geometrically normal fibres of finite type the finite morphism g in the Stein factorisation is in fact étale. But then by [GD67, 18.8.2] the number of points in the geometric fibres of g is locally constant. □

7. Irreducibility of the moduli stack of genus g curves

The central theorem in [DM69] is that the coarse moduli spaces M_g and $\bar{M}_g$ of smooth resp. stable genus $g > 1$ curves are irreducible. This is a remarkable property since it allows us to intersect any two non-empty, open conditions on the moduli spaces and automatically get curves satisfying both (e.g. smoothness, having no automorphisms, reducedness).

For the purpose of this chapter, set $\Lambda = \operatorname{Spec}\mathbb{Z}$. In order to use the theory of DM stacks, we first have to verify that $\mathcal{M}_g$ and $\overline{\mathcal{M}}_g$ are DM stacks (which we have not done so far). Deligne and Mumford do this directly but with what we have shown before, it is enough to see that these CFGs are equivalent to quotient stacks. Again, the reader is referred to consult Appendix B for facts about stable curves.

Proposition 7.1. *There exists an equivalence of categories $\overline{\mathcal{M}}_g \simeq [\bar{H}_{g,\nu}/G]$, where $G := \mathrm{PGL}(N+1)$, taking $\mathcal{M}_g$ to $[H_{g,\nu}/G]$.*

Proof. The equivalence can be constructed explicitly:

Given a family of stable curves $\pi : C \to S$, we need to give a principal G-bundle $E \to S$. Our construction for E is to take the bundle *associated*[27] to the projective bundle $\mathbb{P}_\pi := \mathbb{P}(\pi_*(\omega_{C/S}^{\otimes\nu}))$. Next, we construct the equivariant map to $\bar{H}_{g,\nu}$: Pullback $C \to S$ along $\eta : E \to S$ and get a family of stable curves

$$\pi' : C \times_S E \to E$$

In general, for all $f : S' \to S$, we have a canonical isomorphism (due to [Har10, 8.10])

$$\omega_{C \times_S S'} \cong f^*(\omega_{C/S})$$

Therefore, there is a canonical isomorphism $\mathbb{P}_{\pi'} \cong \eta^* \mathbb{P}_\pi$ which gives $C \times_S E \to E$ the structure of a family of ν-canonically embedded, stable curves (cf. [Har10, 7.12]). But this means we get an equivariant map $E \to \bar{H}_{g,\nu}$.

This finishes the construction of the functor on objects. For morphisms

$$\begin{array}{ccc} C' & \xrightarrow{\phi'} & C \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ S' & \xrightarrow{\phi} & S \end{array}$$

we see $\pi'_*(\omega_{C'/S'}) \cong \phi^*\pi_*(\omega_{C/B})$, which gives rise to a morphism of the associated principal G-bundles. The reader may check that this indeed is functorial.

To see this is an equivalence, remember from our discussion of fibred categories that it is enough to show equivalence on fibres. A non-trivial automorphism of C over S must induce a non-trivial automorphism on $\mathbb{P}_\pi$ since it acts non-trivially on the ν-canonical embedding of C. On the other hand, a non-trivial element α of G leaving the embedding invariant cannot act trivially on it because the fixed points of α always form a (proper) linear subspace. We deduce that our functor is faithful and full.

[27] On a trivialising open U of the projective bundle P, the associated G-bundle is given by $U \times G$. Transitions are defined by the transition functions of P (which are elements in G).

Finally, for essential surjectivity, take an $S \leftarrow E \rightarrow \bar{H}_{g,\nu}$ in $[\bar{H}_{g,\nu}/G]$. By pulling back the universal G-equivariant family Y over $\bar{H}_{g,\nu}$, we get a cartesian diagram

$$\begin{array}{ccc} C' & \longrightarrow & Y \\ \downarrow & & \downarrow \\ E & \longrightarrow & \bar{H}_{g,\nu} \end{array}$$

where $C' \rightarrow E$ is a family of ν-canonically embedded curves and the bottom arrow is by assumption G-equivariant, so G acts equivariantly and freely on $C' \rightarrow E$. We would like to take a quotient family $C'/G \rightarrow E/G = S$. To do this, note that locally on trivialisations of E the quotient of C' exists and can be given by some (quasi-coherent) sheaves of ideals. These sheaves satisfy the cocycle condition, so that descent theory yields a global sheaf of ideals defining C'/G. Finally, $C'/G \times_S E \cong C'$. So the quotient family maps isomorphically to $S \leftarrow E \rightarrow \bar{H}_{g,\nu}$ and we are done. $\square$

Remark 7.2. This proof also works for $\overline{\mathcal{M}}_{g,n} = [H_{g,n,\nu}/G]$. The technique used at the very end is the common technique for descent of projective morphisms (see [Vis05, 4.3.3]) and reduces to descent of quasi-coherent sheaves because we have a canonical projective embedding via an ample line bundle. In a similar way, we could have proven descent on $\overline{\mathcal{M}}_g$ directly. For $g = 1$ however, there is no canonical embedding, in fact descent fails in this case.

Corollary 7.3. *$\overline{\mathcal{M}}_g$ and $\mathcal{M}_g$ are smooth, quasi-separated DM stacks of finite type over* $\operatorname{Spec}\mathbb{Z}$.

Proof. Stable curves over an algebraically closed field k have finite automorphism group. Also, because there are non non-trivial vector fields on a stable curve, i.e. no non-trivial $k[\epsilon]/(\epsilon^2)$-valued points, the isomorphism sheaves are reduced. Thus, all conditions to form a quotient stack as in Example 5.19 are satisfied.

Using the equivalence $\overline{\mathcal{M}}_g \simeq [\bar{H}_{g,\nu}/G]$ exhibits $\overline{\mathcal{M}}_g$ as a DM stack of finite type over $\operatorname{Spec}\mathbb{Z}$. It is smooth since $\bar{H}_{g,\nu}$ is. $\square$

$H_{g,\nu}$ being a dense open in $\bar{H}_{g,\nu}$ implies the same for $\mathcal{M}_g$ in $\overline{\mathcal{M}}_g$. (In fact, knowing that the complement of $H_{g,\nu}$ is a divisor with normal crossings, the same holds for the stacks.)

Proposition 7.4. *$\overline{\mathcal{M}}_g$ is separated.*

Proof. Use Proposition B.3 and the stacky valuative criterion for separation. $\square$

Proposition 7.5. *$\overline{\mathcal{M}}_g$ is proper.*

Proof. Use the Stable Reduction Theorem (Proposition B.4) and the stacky valuative criterion for properness. $\square$

We can now put everything together and prove the promised irreducibility result.

Proposition 7.6. *The geometric fibres of $\mathcal{M}_g$ and $\overline{\mathcal{M}}_g$ are irreducible.*

Proof. $\mathcal{M}_g$ is a dense open in $\overline{\mathcal{M}}_g$, it remains to show irreducibility for the latter. First, note that the fibres are smooth over their corresponding algebraically closed field, hence normal. From the previous chapter, Lemma 6.22, we conclude that it is enough to show the fibres are connected.

But by Proposition 6.33, all geometric fibres have the same number of connected components. The proposition follows now after proving irreducibility of $\overline{\mathcal{M}}_g \times \underline{\operatorname{Spec}\mathbb{C}}$ by Teichmüller theory arguments over $\mathbb{C}$, which were already known before [DM69][28]. $\square$

Remark 7.7. A step omitted in any sources the author has seen is to remark why irreducibility of the geometric fibres of $\overline{\mathcal{M}}_g$ implies the originally intended irreducibility of $\bar{M}_g$, the coarse moduli space of stable curves over an algebraically closed field k.

The argument is easy: There is a morphism $\overline{\mathcal{M}}_g \times \underline{\operatorname{Spec}\mathbb{C}} \to \underline{\bar{M}_g}$ sending a family to its moduli map into M_g. The induced map $|\overline{\mathcal{M}}_g \times \underline{\operatorname{Spec}\mathbb{C}}| \to |\bar{M}_g|$ is bijective and continuous, so its image, $|\bar{M}_g|$, is irreducible.

Of course, one may argue that the irreducibility of the stack fibres is the more interesting result in the end.

Remark 7.8. Finally, let's emphasise again how crucial the use of stacks was in this proof. Indeed, trying to formulate the same with the coarse moduli spaces goes blatantly wrong. This is not because the geometric framework does not exist but because elementary assumptions to apply theorems do not hold: Even $M_g(\mathbb{C})$ is not smooth in general (see [HM98], p.53). Only after passing to the better-behaving stacks and developing the algebro-geometric framework for them, the proof works.

[28]Deligne and Mumford work out a more general theory, the statement we need is the special case of Theorem (5.13) setting $G = 1$.

A. Grothendieck topologies

The notion of presheaves is not restricted to topological spaces, it exists for any category, in fact *presheaf* is just another term for contravariant functor. (As the reader will know, this coincides with presheaves on a topological space, if we take the category of open sets with inclusions as morphisms.)

Alexandre Grothendieck's main insight leading to so-called Grothendieck topologies was that in order to get a theory of sheaves, only one piece of data is missing. One has to specify what the covers of an object in the category are for which the sheaf axioms shall hold:

Definition A.1. Let $\mathfrak{S}$ be a category which admits all fibre products we need in what follows. A *Grothendieck topology* on $\mathfrak{S}$ is a class C of *coverings*, i.e. families of morphisms $(U_i \to U)_{i\in I}$ s.t.

(i). If $(U_i \to U)_{i\in I}$ is in C and $V \to U$ a morphism, then $(U_i \times_U V \to V)_{i\in I}$ is in C.

(ii). If $(U_i \to U)_{i\in I}$ and for all $i \in I$ $(V_{ij} \to U_i)_{j\in J}$ are in C, so is $(V_{ij} \to U_i \to U)_{i\in I, j\in J}$.

(iii). If ϕ is an isomorphism, then the single elements family (ϕ) is in C.

$\mathfrak{S}$ together with a chosen Grothendieck topology is called a *site* and sometimes also denoted by $\mathfrak{S}$. If C and C' are two topologies in S and $C' \supset C$, we say C' is finer than C.

These properties are directly inspired by coverings of a topological space by open immersions. The first base change axiom corresponds to intersecting an open cover with an open subset V to get a cover of V, while the second axiom corresponds to locally refining a cover.

Definition A.2. Let $\mathcal{U} = (U_i \to U)_{i\in I}$, $\mathcal{V} = (V_j \to V)_{j\in J}$ be families of morphisms (with fixed target) in $\mathfrak{S}$. A *morphism of families with fixed target* is a morphism $U \to V$, and a map $\alpha : I \to J$ with morphisms $U_i \to V_{\alpha(i)}$ for all i such that

$$\begin{array}{ccc} U_i & \longrightarrow & V_{\alpha(i)} \\ \downarrow & & \downarrow \\ U & \longrightarrow & V \end{array}$$

commutes. If $U \to V$ is an identity morphism, $\mathcal{U}$ is called a *refinement* of $\mathcal{V}$.

Definition A.3. A *morphism of sites* is a functor $f : \mathfrak{S} \to \mathfrak{S}'$ of the underlying categories together such that

(i). If $(U_i \to U)_{i\in I}$ is a covering in $\mathfrak{S}$, then $(f(U_i) \to f(U))_{i\in I}$ (f applied to the morphisms) is a covering in $\mathfrak{S}'$.

(ii). If $(U_i \to U)_{i\in I}$ is a covering and $V \to U$ a morphism in $\mathfrak{S}$, then

$$f(U_i \times_U V) \to f(U_i) \times_{f(U)} f(V)$$

is an isomorphism.

Definition A.4. Let $\mathfrak{S}$ be a site and $\mathfrak{C}$ a category with products.

(i). The category of $\mathfrak{C}$-valued *presheaves* on $\mathfrak{S}$ is the category of contravariant functors $F : \mathfrak{S} \to \mathfrak{C}$ (with natural transformations between them).

(ii). A $\mathfrak{C}$-valued *sheaf* on $\mathfrak{S}$ is a $\mathfrak{C}$-valued presheaves F such that for all coverings $(U_i \to U)_{i\in I}$

$$F(U) \longrightarrow \prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \times_U U_j)$$

is exact, i.e. an equaliser diagram. The category of such sheaves is the full subcategory of presheaves that are sheaves and is called the *topos* of the site.

We mention that for any Grothendieck topology, one can define a cohomology theories of sheaves by deriving the sections functor or by Čech cohomology.

A.1. Examples

In any category, h_S is a presheaf for all objects S, the presheaf *represented* by S. There is a *canonical topology* on S which is the finest topology such that all representable presheaves are sheaves. It is constructed by taking all so-called *families of universally effective epimorphisms* as coverings which yield a Grothendieck topology as one can easily check.

Definition A.5. (i). A *family of epimorphisms* is a family $(U_i \to U)_{i\in I}$ of morphisms such that $\mathrm{Hom}(U, S) \to \prod_i \mathrm{Hom}(U_i, S)$ is injective for all objects S.

(ii). A family of epimorphisms is *effective* if all representable presheaves satisfy the sheaf conditions with respect to it.

(iii). A family of epimorphisms $(U_i \to U)_{i\in I}$ is *universally effective* if for all $V \to U$, $(U_i \times_U V \to V)_{i\in I}$ is effective.

Any topology in which representable presheaves are sheaves, i.e. is coarser than the canonical topology, is called *subcanonical.* Unfortunately, apart from some examples like (continuous) G-sets for a (profinite) group G, the canonical topology is difficult to characterise. However, in the case of schemes the subcanonical fpqc topology will be a good replacement, fine enough for most purposes.

From now on, fix a category $\mathfrak{S}=\mathfrak{Sch}/\Lambda$. We can define several Grothendieck topologies. The general procedure is always the same. Fix a property P of morphisms in $\mathfrak{S}$. Then the corresponding *big site* on $\mathfrak{S}$ has as its covers all families $(U_i \to U)_{i\in I}$ of morphisms satisfying P such that their (always) open images cover

U. We can also define a *small site* by restricting the category to objects with a fixed morphism to Λ satisfying P, or an *affine site* by restricting to affine schemes. Here is a list of properties giving important Grothendieck topologies:

(i). open immersion, for the big Zariski site Zar/Λ.

(ii). étale[29], for the big étale site $\acute{E}t/\Lambda$. The étale topology imitates phenomenons from the classical topology over $\mathbb{C}$ like cohomology with coefficients, for which the Zariski topology is too coarse.

(iii). smooth, for the big smooth site $Smooth/\Lambda$.

(iv). flat and locally of finite presentation (fidèlement plate de présentation finie = faithfully flat and of finite presentation), for the big fppf site $Fppf/\Lambda$. The term "faithfully" comes from the fact that for morphisms of ring spectra flat and surjective is equivalent to faithfully flat.

Each topology in this list is finer than the preceding one. For any covering $(U_i \to U)_{i\in I}$ we can take the disjoint union of the U_i and the maps and get a covering consisting of one single map.

There is an even finer topology, the subcanonical fpqc topology (fidèlement plate quasi-compacte), which like the fppf topology arises in the theory of descent. The naive definition, going by the name, as "$\coprod_i U_i \to U$ is faithfully(=surjective) flat and quasi-compact" works for descent theory but is problematic because it does not even include all Zariski covers. It is better to require $\coprod_i U_i \to U$ to be flat and satisfy the subtle condition: Every quasi-compact open in U is the image of a quasi-compact open.

A.2. Descent results for fpqc morphisms

The basic question of descent theory is: Given local objects defined on a cover and data describing how to patch each two of these objects on the intersection of the domains for which they are defined, can we patch the local objects to a global objects?

Of course, the answer must be *no*, as long as the identifications on the intersections are not compatible with each other. The resulting condition is called cocycle condition and explained in the motivational section as well as defined in the general context of fibred categories in Section 4.2.

We now state the classical results of descent theory for fpqc morphisms that we need. The proofs can be found in [Vis05], Chapter 4, or [Sta15, Tag 0238].

Proposition A.6. *$QCoh/\Lambda$ is a stack for the fpqc topology.*

[29] smooth of relative dimension 0 or equivalently flat and unramified

The proof idea is to reduce to the affine case of a cover given by a faithfully flat ring map $A \to B$. Now translating what it means to have descent data in the opposite category of quasi-coherent modules, we get a category $Mod_{A\to B}$ and have to show that the functor associating to each A-module its descent datum in $Mod_{A\to B}$ is an equivalence. In this algebraic context, the inverse can be constructed explicitly.

For fibre products we denote by p_n the projection to the n-th component and by p_{nm} the projection to the product of the n-th and m-th component etc. Also write fibre products $U_i \times_U U_j$ as U_{ij} etc. For the reader who is not well-acquainted with stacks, we spell out what descent on objects means in this case without using the language of stacks.

Corollary A.7. *Let there be given a quasi-coherent $\mathcal{O}_{U_i}$-sheaf $\mathcal{F}_i$ for each U_i. Furthermore assume we have isomorphisms $\phi_{ij} : p_1^*\mathcal{F}_i \to p_2^*\mathcal{F}_j$ of quasi-coherent $\mathcal{O}_{U_{ij}}$-modules satisfying the cocycle condition*

$$\begin{tikzcd} p_{12}^*p_1^*\mathcal{F}_i \arrow[r, "p_{12}^*\phi_{ij}"] \arrow[d, "\cong"] & p_{12}^*p_2^*\mathcal{F}_j \arrow[r, "\cong"] & p_{23}^*p_2^*\mathcal{F}_j \arrow[d, "p_{23}^*\phi_{jk}"] \\ p_{13}^*p_1^*\mathcal{F}_i \arrow[r, "p_{13}^*\phi_{ik}"] & p_{13}^*p_3^*\mathcal{F}_k \arrow[r, "\cong"] & p_{23}^*p_3^*\mathcal{F}_k \end{tikzcd}$$

Then there exists a quasi-coherent $\mathcal{O}_U$-sheaf $\mathcal{F}$ and isomorphisms $\lambda_i : f_i^*\mathcal{F} \to \mathcal{F}_i$ such that

$$p_2^*\lambda_j = \phi_{ij} \circ p_1^*\lambda_i$$

Moreover, the pair $(\mathcal{F}, \lambda)$ is unique up to canonical isomorphism.

From the prototypical example of quasi-coherent sheaves of modules, most other examples of fpqc descent are deduced. For example, we can add additional structure defined by commutativity of sheaf diagrams. Descent on quasi-coherent sheaves respects this structure because if morphisms agree locally, they must agree globally by the prestack condition.

Proposition A.8. *$QCohComm/\Lambda$ is a stack for the fpqc topology.*

Using the relative Spec construction to get an equivalence between quasi-coherent sheaves of commutative algebras and affine morphisms over Λ, one finds:

Proposition A.9. Aff/Λ *is a stack for the fpqc topology.*

Finally, this can be extended to quasi-affine morphisms.

Proposition A.10. QAff/Λ *is a stack for the fpqc topology.*

Spelling out descent for the last two without using stack terminology gives:

Corollary A.11. *Let there be given a (quasi-)affine morphism $X_i \to U_i$ for each U_i. Furthermore assume we have isomorphisms $\phi_{ij} : X_i \times_U U_j \to U_i \times_U X_j$ over U_{ij} satisfying the cocycle condition*

$$\begin{tikzcd} X_i \times_U U_j \times_U U_k \arrow[rr, "p_{13}^*\phi_{ik}"] \arrow[dr, "p_{12}^*\phi_{ij}"'] & & U_i \times_U U_j \times_U X_k \\ & U_i \times_U X_j \times_U U_k \arrow[ur, "p_{23}^*\phi_{jk}"'] & \end{tikzcd}$$

Then there exists a (quasi-)affine morphism $X \to U$ and isomorphisms $\lambda_i : U_i \times_U X \to X_i$ such that for all i, j

$$1_{U_i} \times_U \lambda_j = \phi_{ij} \circ (\lambda_i \times_U 1_{U_j})$$

Moreover, the pair (X, λ) is unique up to canonical isomorphism.

B. Stable curves

Remember the definition of *stable n-pointed curves* in Example 3.17. The intuition behind this definition could be as follows: We want to construct a compactification of the moduli space of smooth curves of genus g. It is expected that as limit cases, we have to allow nodal degeneracies as in Figure B.1 for Riemann surfaces. It is reassuring to know that these degenerations do not change the genus: If $g_1, \ldots, g_n$ are the genera of the irreducible components meeting in δ nodes, then it is shown in [HM98, 3.2] that

$$g = \sum_{i=1}^{n} g_i + \delta - n + 1$$

However, one should not take all nodal curves because this would result in "too many" limit points, so the moduli spaces would have to be highly non-separated. Indeed, to compactify we do only need the stable curves while for non-stable examples like the third one in Figure B.1 (which corresponds to adding a $\mathbb{P}^1$ by blowing up at a point) the degeneracy can be "collapsed", as the reader may be able to imagine. The only way a $\mathbb{P}^1$ component can be essential to taking a limit and cannot be collapsed is when it intersects three other components. This explains the additional condition of the definition.

To understand the n-pointed version, note that if there are less than two chosen points on a Riemann sphere intersecting one other component, the sphere may be collapsed, the same for a sphere with no chosen point intersecting two components.

Another way to motivate the definition is to say that we still want to have a finite automorphism group for genus $g > 1$ and the condition of the definition to have three marked points on each $\mathbb{P}^1$ is exactly the one ensuring this. (For genus $g > 1$ components, the automorphism groups are finite, genus 1 components have to meet at least another component, eliminating degrees of freedom, and rational components

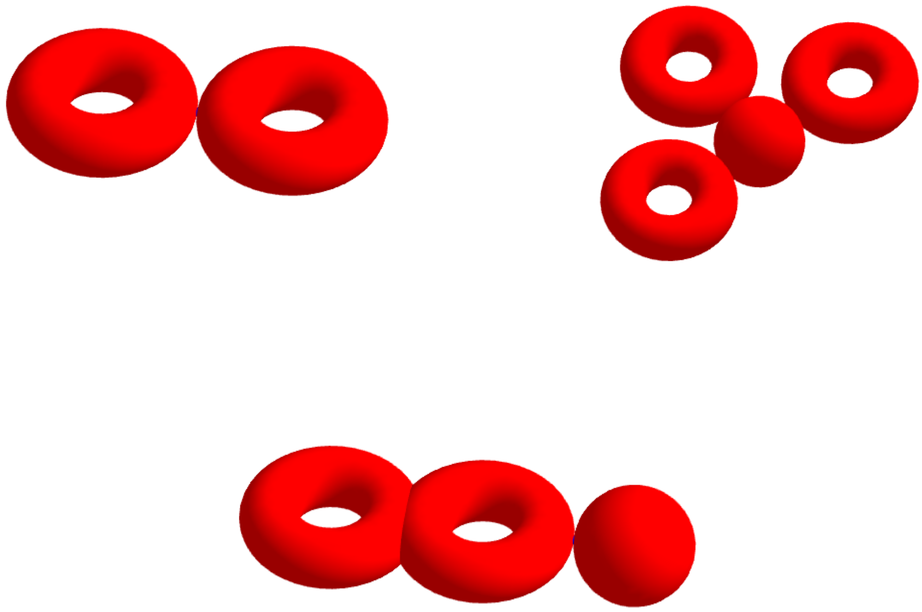

Figure B.1: Degenerations of stable curves.

can only permute their three or more marked points which defines finitely many elements in PGL(2).) For the n-pointed version, we see that we can include the cases $g = 1, n \geq 1$ and $g = 0, n \geq 3$, which amounts to requiring $2g - 2 + n > 0$.

Lemma B.1. *A stable curve C over an algebraically closed field has no non-trivial vector fields.*

Proof. The proof in [DM69, 1.4] is combinatorial. After remarking that vector fields on C correspond to vector fields on its normalisation $\tilde{C}$[30] that vanish on the points lying over double points, one can exclude all irreducible components E of genus greater 1. The remaining five cases E non-singular, $\tilde{E}$ rational and E one double point, $\tilde{E}$ rational and $E \geq 2$ double points, E non-singular elliptic, $\tilde{E}$ elliptic and $E \geq 1$ double points, have in common: On rational $\tilde{E}$ the vector field has at least three zeros and on elliptic $\tilde{E}$ at least one. These are too many to be cancelled out by $2 \cdot 0 - 2$ respectively $2 \cdot 1 - 2$ poles, the degree of the canonical sheaf. □

Finiteness and reducedness of the automorphism group are needed to establish that the diagonal of the moduli stack of curves is unramified.

[30]This is the disjoint union of the normalised irreducible components.

The dualising sheaf $\omega_{C/S}$ has the following description for a nodal curve over algebraically closed $S = \operatorname{Spec} k$. If $z_1, \ldots, z_n$ are the double points of C and $x_1, y_1, \ldots, x_n, y_n$ the points in $\tilde{C}$ lying over, then $\omega_{C/S}$ is the sheaf of 1-forms on $\tilde{C}$ regular everywhere except for the x_i, y_i's where they may have poles of order 1 such that the residues at each pair (x_i, y_i) sum up to 0. For a demonstration that this is dualising at least for invertible sheaves over $\mathbb{C}$ and further literature, see [HM98, 3.6-8].

Using this explicit description and applying duality to the well-known separation of points and tangent vectors conditions, one can show

Lemma B.2. *For a stable n-pointed curve $\pi : C \to S$ of genus $g > 1$, $\omega_{C/S}^{\otimes\nu}$ is relatively very ample on C/S if $\nu \geq 3$ and $\pi_*(\omega_{C/S}^{\otimes\nu})$ is locally free of rank*

$$h^0(\omega_{C/S}^{\otimes\nu} \otimes \mathcal{O}_{C_s})$$

Proof. see [DM69, Corollary of 1.2] or [arb11, 10.6.1]. □

The rank in the theorem computes as $N = (2\nu - 1)(g - 1) + \nu n$ by Riemann-Roch, thus we have an embedding of C into $\mathbb{P}_S^N$. Also by Riemann-Roch and the fact that higher cohomology vanishes for high enough powers, we see that the Hilbert polynomial of the embedding is

$$P_\nu(t) = (2\nu t - 1)(g - 1) + \nu n t$$

The moduli problem of (proper[31]) flat families over S that are subschemes of $\mathbb{P}_S^N$ has a fine moduli space solution, the *Hilbert scheme*. This Hilbert scheme has to decompose as a disjoint union of Hilb_N^P, parametrising subschemes with Hilbert polynomial P, since flatness implies locally constant Hilbert polynomials of fibres.

Following [arb11, 11.5.2], there exists a subscheme $\bar{H}_{g,n,\nu}$ of $\mathrm{Hilb}_N^{P_\nu}$ representing the functor of ν-canonically embedded stable n-pointed curves, i.e.

$$\mathrm{Hom}(S, \bar{H}_{g,n,\nu}) = \{\text{stable curves } C/S \text{ together with } \mathbb{P}(\pi_*(\omega_{C/S}^{\otimes\nu})) \cong \mathbb{P}_S^N\}/\cong$$

$\bar{H}_{g,n,\nu}$ is smooth over $\operatorname{Spec} \mathbb{Z}$ ([DM69, 1.7]). The subscheme $H_{g,n,\nu} \subset \bar{H}_{g,n,\nu}$ parametrising smooth curves is open and dense because its complement is a divisor with normal crossings ([DM69, 1.9]).

Via Geometric Invariant Theory as in [Edi00, 4.2], it is possible to construct quotient schemes $H_{g,n,\nu}/\mathrm{PGL}(N+1)$ and $\bar{H}_{g,n,\nu}/\mathrm{PGL}(N+1)$ which automatically are coarse moduli spaces for smooth/stable n-pointed curves.

Finally, we need two propositions used as valuative criteria for the separateness and properness of the moduli stack of stable curves. Especially the second one is highly non-trivial.

[31] holds automatically

Proposition B.3. *If C, C' are stable curves over a discrete valuation ring with algebraically closed residue field with smooth generic fibres, then any isomorphism between their generic fibres extends to an isomorphism $C \cong C'$.*

Proof. see [DM69, 1.12]. $\square$

Proposition B.4 (Stable reduction theorem)**.** *Take a discrete valuation ring R with quotient field K and a smooth, geometrically irreducible curve C of genus $g > 1$ defined over K. We can find a finite extension K'/K and a curve C' defined over the algebraic closure R' of R in K' such that the generic fibre of C' is isomorphic to $C \times_{\operatorname{Spec} K} \operatorname{Spec} K'$.*

Proof. see [DM69, 2.7]. $\square$

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