Shortest and Straightest geodesics in Sub-Riemannian Geometry

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Abstract

There are several different, but equivalent definitions of geodesics in a Riemannian manifold, They are generalized to sub-Riemannian manifolds, but become non-equivalent.
H.R. Herz remarked that there are two main approaches for definition of geodesics: geodesics as **shortest curves** based on Mopertrui's principle of least action (variational approach) and geodesics as **straightest curves** based on d'Alembert's principle of virtual work (which leads to geometric descriptions based on the notion of parallel transport).
We shortly discuss different definitions of sub-Riemannian geodesics and interrelations between them.

A.M. Vershik and L.D. Faddeev showed that for a generic sub-Riemannian manifold Q all shortest geodesics (defined by a hamiltonian system) are different from straightest geodesics (defined by Schouten partial connection). They gave first example when shortest geodesics coincide with straightest Hamiltonian geodesics (with zero initial covector $\lambda \in T^*Q$) and stated the problem of characterisation of sub-Riemannian manifolds with such property.
We shove that this is true for Chaplygin transversally homogeneous systems, defined by the sub-Riemannian metric on the total space Q of a principal bundle $\pi : Q \to M = Q/G$ over a Riemannian manifold (M, g^M). associated with a principal connection. Hamiltonian geodesics of such system describe evolution of a charge particle in Yang-Mills field and straightest geodesics – motion of a classical mechanical system with non-holonomic constrain.
We describe some classes of homogeneous sub-Riemannian manifolds, where straightest geodesics coincides with shortest geodesics, including sub-Riemannian symmetric spaces.