Algebra III: Rings and Modules
Problem Sheet 1, Autumn Term 2022-23

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1. For sets X and Y, let $\mathrm{Fun}(X, Y)$ denote the set of all functions $f : X \to Y$.

   (i) Let X be a set and R a ring. Given $f, g \in \mathrm{Fun}(X, R)$, we can define $f + g, f \cdot g \in \mathrm{Fun}(X, R)$ via $(f + g)(x) = f(x) +_R g(x)$ and $(f \cdot g)(x) = f(x) \cdot_R g(x)$ for $x \in X$. Given $a \in R$, we can consider the constant function $c_a : x \mapsto a$ for all $x \in X$. Show that $\mathrm{Fun}(X, R)$ is a ring with $0 = c_0$ and $1 = c_1$.

   (ii) Let $X = [0, 1] \subseteq \mathbb{R}$, the interval, and let $R = \mathbb{R}$. Show that the subset of $\mathrm{Fun}([0, 1], \mathbb{R})$ of continuous functions is a subring, and that the subset of differentiable functions is a further subring.

   (iii) Show that, if X has at least two elements, then $\mathrm{Fun}(X, R)$ is not an integral domain (regardless of what R is). [Hint: For a complete solution you will have to consider separately the trivial case $R = \{0\}$.]

2. For a ring R, an element $a \in R$ is *nilpotent* if $a^n = 0$ for some $n \geq 1$. Let $\mathrm{nil}(R) \subseteq R$ be the subset of nilpotent elements.

   (i) Let R be a commutative ring. Show that $\mathrm{nil}(R)$ is an ideal. [Hint: Prove that the binomial formula for the expansion of $(x + y)^n$ holds in arbitrary commutative rings.]

   (ii) Give an example of a non-commutative ring where $\mathrm{nil}(R)$ does not form an ideal.

   (iii) Let $x \in R$ be nilpotent (and do not assume R is commutative). Show that $1 + x \in R^\times$.

   (iv) Find all the nilpotent elements in the ring $R = \mathbb{Z}/p^r\mathbb{Z}$ for every prime p and $r \geq 1$. [Optional: extend this to $\mathbb{Z}/n\mathbb{Z}$ for every $n \in \mathbb{Z}$.]

3. Show the following:

   (i) If $a > 0$, then $\mathbb{R}[X]/(X^2 - a) \cong \mathbb{R} \times \mathbb{R}$.

   (ii) Show that $(\mathbb{Z}/3)[X]/(X^2 + 1)$ is a field with nine elements.

   (iii) Show that, for any $n \geq 1$, then $\mathbb{Z}[i]/(n) \cong (\mathbb{Z}/n)[X]/(X^2 + 1)$.

   (iv) Show that $\mathbb{Z}[i]/(2) \cong (\mathbb{Z}/2)[X]/(X^2)$. In particular observe that this is not a field.

4. Let S be a subset of the nonnegative integers, and let $\mathbb{C}[S]$ be the subset of $\mathbb{C}[X]$ consisting of polynomials $P(X) = \sum_{i=0}^{d} a_i X^i$ such that $a_i = 0$ for $i \notin S$. For which S is $\mathbb{C}[S]$ a subring of $\mathbb{C}[X]$?

5. Let F be a field and $f, g \in F[X]$. Prove that there exists $r, q \in F[X]$ such that

$$f = gq + r,$$

   with $\deg r < \deg g$. [This shows that $F[X]$ is a Euclidean domain with Euclidean function $\deg : F[X] \setminus \{0\} \to \mathbb{Z}_{\geq 0}$.]

6. Let R be the ring of continuous functions on the unit interval $[0,1]$, where addition and multiplication of functions is defined pointwise.

   (i) Show that for any $c \in [0,1]$, the subset $\{f \in R : f(c) = 0\}$ is a maximal ideal M_c of R.

   (ii) Show that if $b \neq c$, then $M_b \neq M_c$.

   (iii) Show that if M is any maximal ideal of R, then $M = M_c$ for some c.

   (iv) Show that M_c is not generated by the element $f(x) = x - c$ of R. Show further that M_c is not even finitely generated.

7. Let R be a ring, let $f : G \to H$ be a surjective group homomorphism and let $f_* : R[G] \to R[H]$ be the ring homomorphism induced by f. Let $N = \ker(f)$. Show that $\ker(f_*) = (N - 1)$, i.e. the ideal generated by the set $N - 1 = \{x - 1 : x \in N\} \subseteq R[G]$. [Hint: Start by considering the case where H is the trivial group and $f_* : R[G] \to R$.]

8. For each commutative ring R and ideal $I \subseteq R$ below, determine (with proof) whether or not I is prime and whether or not I is maximal. [You may assume any results from the course.]

   (i) $R = \mathbb{Z}, \quad I = (6)$.

   (ii) $R = \mathbb{Z}, \quad I = (8, 12)$.

   (iii) $R = \mathbb{Z}[X], \quad I = (X + 1)$.

   (iv) $R = \mathbb{R}[X], \quad I = (X^2 - 5)$.

   (v) $R = \mathbb{C}[X], \quad I = (X^2 + 3, X^3 - 1)$.

   (vi) $R = (\mathbb{Z}/13\mathbb{Z})[X], \quad I = (X^2 + 1)$.

   (vii) $R = \mathbb{Q}[X, Y, Z], \quad I = (X - Y^2)$.

9. (i) Show that every finite integral domain is a field.

   (ii) Let R be a commutative ring. Show that an ideal $I \subseteq R$ is prime if and only if R/I is an integral domain. Deduce that every maximal ideal is a prime ideal.

10. Show that $\mathbb{Z}[\sqrt{-2}]$ and $\mathbb{Z}[\sqrt{2}]$ are Euclidean domains.

+11. Show that $\mathbb{Z}[\frac{1+\sqrt{-19}}{2}]$ is a principal ideal domain but not a Euclidean domain.