Exotic Spheres

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Part III Essay submitted to the University of Cambridge 2016/2017

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1 Introduction

A smooth n-manifold is called an *exotic sphere* if it is homeomorphic but not diffeomorphic to S^n. Until 1953, when John Milnor discovered the first example of an exotic sphere in dimension 7, it was widely believed that no such object existed. This was firstly because of the subtleties underlying the definitions involved, though also due to the way in which dimension was believed to affect problems in topology. We begin by illustrating some of these subtleties and giving an overview of what was known at the time of this discovery.

1. Firstly, recall that the definition of a smooth manifold M is sufficiently generous that it allows us to choose any (possibly pathological) chart to define the smooth structure so long as it is smoothly compatible with the *other* charts chosen. For example $(\mathbb{R}, \varphi)$ is a smooth manifold for *any* homeomorphism $\varphi : \mathbb{R} \to \mathbb{R}$.

   This is rectified by choosing a similarly generous notion of equivalence. Namely a diffeomorphism between smooth manfolds M and N can be any homeomorphism that induces smooth maps on the coordinate patches. For example $(\mathbb{R}, \varphi)$ and $(\mathbb{R}, \psi)$ are diffeomorphic via the homeomorphism $\psi^{-1} \circ \varphi : \mathbb{R} \to \mathbb{R}$.

   We will use *smooth structure* (*on* M) to refer to an equivalence class of smooth manifolds homeomorphic to M up to diffeomorphism.

2. Secondly we remark that, before 1953, little was known about smooth structures on manifolds except in very small dimensions where direct proofs are often tractable.

   For example, to show S^1 has a unique smooth structure, note that a homeomorphism $\varphi : U \to \varphi(U) \subseteq \mathbb{R}$ for $U \subseteq S^1$ open gives a local diffeomorphism $\varphi^{-1} : (a, b) \to S^1$ for some $a, b \in \mathbb{R} \cup \{\infty\}$. The image of φ^{-1} is open and, by stitching it together with local diffeomorphisms around the boundary points of $\varphi^{-1}((a, b))$, we can also assume the image of φ^{-1} is closed. This shows that φ^{-1} is surjective and must have $(a, b) = \mathbb{R}$. It can then be checked that it descends to a diffeomorphism $\mathbb{R}/\mathbb{Z} \to S^1$.

   With more difficulty, it had also been shown that S^2 and S^3 had a unique smooth structure[1] and it was thought that the higher dimensional cases would be similar.

It was therefore considered very surprising when exotic spheres were shown to exist, and later even more surprising that they could be constructed in so many different ways and even classified in any given dimension. It was also shown in 1960 that not every topological manifold admitted a smooth structure (see [5]). Clearly these definitions have a lot more in them than was initially thought, and it is exploring this that will be the starting point for this exposition.

Suppose we wanted to prove that exotic n-spheres do not exist. We might attempt to use that every homeomorphism $f : M \to S^n$ can be uniformly approximated by a smooth map $M \to S^n$ though, whilst we get smooth maps $M \to S^n$ and $S^n \to M$, we have no way of amalgamating these approximations to give a diffeomorphism. Whilst this may be extremely counterintuitive, the reason for this comes from the fact that homeomorphisms can be far more pathological than our intuition usually suggests.

On the other hand, suppose we wanted to prove that exotic n-spheres do exist. Since no progress was (or ever could have been) made for $n \leq 3$, we would be interested in any result that is unique to higher dimensions. For example, in 1962 Stephen Smale proved

[1] In fact, it was proven later that every smooth n-manifold has a unique smooth structure for $n \leq 3$.

the *h-cobordism theorem* which implied the Poincaré conjecture[2] in dimensions $n \geq 5$. These ideas can be applied to the following construction which arose in the 1960s, known as *twisting*. Consider the smooth n-manifold

$$\Sigma_f = D^n \sqcup_f D^n,$$

where $f : S^{n-1} \to S^{n-1}$ is an orientation-preserving diffeomorphism, and assume $n \geq 5$. Since f must have degree 1, it is homotopic to the identity and so Σ_f is homotopic to $S^n = D^n \cup_{\mathrm{id}} D^n$. Hence Σ_f is homeomorphic to S^n by the Poincaré conjecture for $n \geq 5$. Furthermore, the h-cobordism theorem can be used to show that every smooth n-manifold homeomorphic to S^n can be represented in this form. It is then not difficult to see that smoothly isotopic diffeomorphisms $f : S^{n-1} \to S^{n-1}$ induce diffeomorphic manifolds Σ_f. In fact, it was shown by Jean Cerf in 1970 that the two are in correspondence.

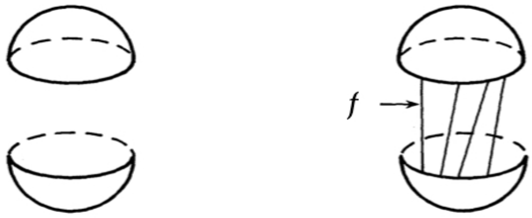


Figure 1: Discs $D^n \sqcup D^n$ (left) being attached by f to form Σ_f (right).

Whilst it may seem a good approach to study these smooth isotopy classes instead, explicit examples of diffeomorphisms not smoothly isotopic to the identity are difficult to work with[3]. Instead, what we should take away from this viewpoint is an understanding of how smooth structures can be inequivalent: whilst f can be “shrunken to a point” to give a homeomorphism to S^n, the derivatives of f may not behave nearly so well.

The primary focus of this essay will be to give an overview of the various constructions of exotic spheres as well as the techniques used for proving firstly that they are homeomorphic to S^n and secondly that they are not diffeomorphic to S^n. The first of these tasks usually amounts to a straightforward computation of the (co)homology groups and fundamental group of a space, with the heavy lifting being performed by the Poincaré conjecture for $n \geq 5$, though an alternate approach using Morse Theory will also be considered for completeness. With this is mind, the main obstacle lies is in developing invariants that are able to distinguish smooth structure. As a result, we will consider constructions which lend themselves particularly well to such an analysis.

The exotic spheres discovered by Milnor were constructed as S^3 bundles over S^4. In this case, the invariant λ we will develop that distinguish smooth structure will come from invariants of vector bundles, namely characteristic classes. We thus begin by giving an introduction to fibre bundles and characteristic classes before expanding the details of Milnor’s original paper. We then show how these ideas can only be made to work in dimensions 7 and 15.

We next take a detour to consider the theory of cobordism, an equivalence relation whereby two compact manifolds are equivalent if their disjoint union is the boundary

[2]This states that every oriented compact simply-conncted smooth n-manifold M with the homology of S^n is homeomorphic to S^n. In particular, every such homotopy n-sphere is homeomorphic to S^n

[3]Such maps can however be written down using already known exotic spheres. See [8] for an example.

of a compact manifold of one dimension higher. Here we prove Hirzebruch's signature theorem, the key ingredient in defining the invariant λ we previously used, and briefly discuss the h-cobordism theorem.

In search of more exotic spheres, and armed with various machinary, we then consider two further constructions known as *Plumbing* and *Brieskorn varieties*. These constructions will be restricted to the odd-dimensional cases and, common to both constructions of a suitible closed $(2n-1)$-manifold M, will be a bounding $2n$-manifold B. In fact, we can choose our constructions so that all the Ms we consider will be $(n-2)$-connected and the Bs will be $(n-1)$-connected. Thus showing M is homeomorphic to S^{2n-1} amounts to showing just one further connectivity condition, a condition we can characterise in a number of ways. We also find many interesting links to Lie groups and knot theory.

We end by discussing the classification of exotic spheres of a given dimension, showing that this is related to the problem in homotopy theory of computing the stable homotopy groups of spheres. The key step is to use the h-cobordism to allow us to swap diffeomorphism for the much easier to work with condition of being h-cobordant. The computations then rely on showing that the exotic n-spheres form a group Θ_n under connected sum and identifying the subgroup bP_{n+1} of manifolds which bound manifolds with trivial tangent bundle, the *bounded parallelisable manifolds*.

2 Characteristic Classes

In this chapter, we start by developing the basic theory of fibre bundles before describing a correspondence between fibre bundles over spheres and higher homotopy groups that we will use to construct exotic spheres in the following chapter. We next use the Euler class to construct further characteristic classes and give an overview of the various techniques we will need for computing them.

Here and elsewhere in the essay, we will rely on basic facts from Homotopy Theory. We will assume without further mention the definition of higher homotopy groups and the basic theory of vector bundles. Other facts will be stated briefly without proof, with the primary reference being the fourth chapter of [13].

2.1 Fibre Bundles

It is well known that any (real) vector bundle $\mathbb{R}^n \hookrightarrow E \to X$ has an associated sphere bundle $S^{n-1} \hookrightarrow \mathbb{S}(E) \to X$ and disc bundle $D^n \hookrightarrow \mathbb{D}(E) \to X$ by operating on the fibres. All of these bundles can be considered under the following heading:

Definition (Fibre bundle)**.** A *fibre bundle* structure on a space E is a pair of spaces F and X equipped with a projection map $\pi : E \to X$ such that there is an open cover $\{U_i\}$ of X and homeomorphisms $h_i : \pi^{-1}(U_i) \to U_i \times F$ coinciding with π in the first coordinate. We write:

$$F \hookrightarrow E \xrightarrow{\pi} X.$$

In particular, this means that each fibre $F_x = \pi^{-1}(x)$ maps homeomorphically to $\{x\} \times F$. We call E the *total space*, X the *base space*, F the *fibre* and h_i the (*local*) *trivialisations*.

This also generalises covering spaces which are fibre bundles with discrete fibres. For example, the covering map $S^n \to \mathbb{RP}^n$ gives a fibre bundle $\{\pm 1\} \hookrightarrow S^n \to \mathbb{RP}^n$. Note that, when $n = 1$, this specialises to the sphere bundle $S^0 \hookrightarrow S^1 \to S^1$. We may wonder if there are any other fibre bundles where all three spaces are spheres:

Example. (i) The quotient map $(\mathbb{C}^{n+1})^\times \to \mathbb{CP}^n$ has fibres $\mathbb{C}^\times$ and induces a fibre bundle $S^1 \hookrightarrow S^{2n+1} \to \mathbb{CP}^n$. When $n = 1$ this gives: $S^1 \hookrightarrow S^3 \to S^2$.

(ii) If $\mathbb{H}$ is the algebra of quaternions, we can similarly define $\mathbb{HP}^n = \mathbb{H}^{n+1}/\sim$ where $(q_0, \dots, q_n) \sim (cq_0, \dots, cq_n)$ for any $c \in \mathbb{H}^\times$. Since $\mathbb{H} = \mathbb{R}^4$ as spaces, we get a fibre bundle $S^3 \hookrightarrow S^{4n+3} \to \mathbb{HP}^n$. Now $\mathbb{HP}^1 = \{[q,1] : q \in \mathbb{H}\} \cup \{[1,0]\} \cong \mathbb{R}^4 \cup \{\infty\} \cong S^4$ is the compactification of $\mathbb{R}^4$. Hence when $n = 1$ this gives: $S^3 \hookrightarrow S^7 \to S^4$.

Remark. The only further example of this construction (and the only other real finite-dimensional division algebra) is the octonionic algebra $\mathbb{O}$ which gives $S^7 \hookrightarrow S^{15} \to S^8$. In fact, it can be shown that these are the only fibre bundles where F, E and X are spheres.

Note that, when defining vector bundles, each fibre comes equipped with the structure of a vector space that has to be respected by the trivialisations. More generally we have:

Definition (Structure group)**.** Let $F \hookrightarrow E \to X$ be fibre bundle and G a group acting on F by homeomorphisms such that any U_i, U_j in the trivialising open cover has

$$h_i \circ h_j^{-1} : (U_i \cap U_j) \times F \to (U_i \cap U_j) \times F, \quad (x, y) \mapsto (x, g_{ij}(x)y)$$

for some continuous $g_{ij} : U_i \cap U_j \to G$. We call G the *structure group* and such a fibre bundle a G-*bundle*.

Example. For a fibre bundle $\mathbb{R}^n \hookrightarrow E \to X$, giving a description of how $\mathrm{GL}(n, \mathbb{R})$ acts on the fibres is equivalent to giving a vector space structure on each fibre. Hence a real vector bundle is precisely a fibre bundle with fibre $\mathbb{R}^n$ and structure group $\mathrm{GL}(n, \mathbb{R})$.

Furthermore, note that giving an orientation on the vector bundle corresponds to a continuous choice of orientation on the fibres. This is equivalent to describing an action by $\mathrm{SL}(n, \mathbb{R})$ and so oriented vector bundles are fibre bundles with structure group $\mathrm{SL}(n, \mathbb{R})$.

Finally, this action on the correponding sphere bundle $S^{n-1} \hookrightarrow \mathbb{S}(E) \to X$ then corresponds to $O(n, \mathbb{R})$ and to $SO(n, \mathbb{R})$ for oriented vector bundles.

We can define a *morphism of fibre bundles* exactly as for vector bundles and can also make similar definitions in the category of smooth manifolds: if all spaces above are smooth manifolds and all maps are smooth, then we call such a fibre bundle a *smooth fibre bundle*.

In general, classifying fibre bundles up to isomorphism is a difficult task. However, in the case that the base space is a sphere, such fibre bundles are characterised by the following construction known as *clutching*.

First observe that, given a topological group G of diffeomorphisms of a space F, we can construct a class of smooth fibre bundles over S^n with fibre F and structure group G from a map $f : S^{n-1} \to G$ (the *clutching map*) as follows.

1. First let U_N and U_S be S^n without the north and south poles respectively and note that f induces a map $\bar{f} : U_N \cap U_S \to G$ by projecting to the equator.
2. Now let $E_f = (U_N \times F) \sqcup (U_S \times F)/\sim$, where $(u, x) \sim (u, \bar{f}(u)x)$.
3. Then $F \hookrightarrow E_f \to S^n$ is a fibre bundle with projection onto the first coordinate.

Remark. This may remind us of the twisting construction from the introduction, though the two are very different: whilst here the gluing is trivial along the zero section, the gluing done in the twisting construction can be non-trivial everywhere.

Indeed all such smooth fibre bundles arise in this way (for a proof, see [14]):

Theorem. Every smooth fibre bundle over S^n with fibre F and structure group G is isomorphic to one obtained by the construction above, and E_f, E_g are isomorphic iff $f, g : S^{n-1} \to G$ are homotopic, i.e. there is a correspondence

$$\pi_{n-1}(G) \longleftrightarrow \mathrm{Fib}_F(S^n) = \{G\text{-bundles over } S^n\}.$$

Hence we can use knowledge of $\pi_{n-1}(G)$ to give us a supply of smooth fibre bundles of a particular form. We now show how this can be used to construct S^3-bundles over S^4.

Example. We showed above that ensuring transition functions induce isometries on S^3 is the same as requiring that fibre bundle has structure group $SO(4)$. Hence constructing S^3-bundles over S^4 amounts to finding an explicit form for $\pi_3(SO(4))$.

Each $\phi \in SO(4)$ acts on by rotations on S^3, which we take to be the unit ball in the quaternionic space generated by $\{1, i, j, k\}$. If $SO(3)$ is the subgroup of rotations in the $\{i, j, k\}$-plane, then $\phi(1)^{-1}\phi \in SO(3)$ since it fixes 1. Hence we have a homeomorphism

$$SO(4) \cong S^3 \times SO(3)$$

given by $\phi \mapsto (\phi(1), \phi(1)^{-1}\phi)$ and with inverse $(u, \psi) \mapsto (v \mapsto u\psi(v))$. Now consider

$$\rho : S^3 \to SO(3), \quad u \mapsto (v \mapsto uvu^{-1}),$$

where multiplication is quaternionic. To show the image is in $SO(3)$, observe that it must be in $O(3)$ since it is an $\mathbb{R}$-linear, norm-preserving map fixing the identity. It contains the identity and is connected, since S^3 is connected, and so lies in $SO(3)$.

We claim this is a two-sheeting covering map. This can be shown by, for example, noting that an isomorphism $SO(3) \to \mathbb{RP}^3$ can be constructed that makes ρ into the quotient map from S^3 to $\mathbb{RP}^3$. We can now get an explicit form for $\pi_3(SO(4))$ as follows.

1. First note that $\pi_3(S^3) = \mathbb{Z}$ and its elements can be represented in the form

$$\phi_i : S^3 \to S^3, \quad u \mapsto u^i$$

   for $i \in \mathbb{Z}$. This is proven using the *Hurewicz theorem* which states that the first non trivial $H_k(S^3)$ for $k \geq 1$ coincides with (the abelianisation of) $\pi_k(S^3)$.
2. Next, we note that $\pi_3(SO(3)) \cong \mathbb{Z}$ and its elements can be represented in the form $\rho \circ \phi_j$ for $j \in \mathbb{Z}$. This is proven using that covering maps $\rho : S^3 \to SO(3)$ induce isomorphisms on higher homotopy groups.
3. Combining with the homeomorphism $S^3 \times SO(3) \to SO(4)$ established above, we have

$$\begin{aligned} S^3 &\longrightarrow S^3 \times SO(3) \longrightarrow SO(4) \\ u &\longmapsto (u^i, \rho(u^j)) \longmapsto (v \mapsto u^{i+j} v u^{-j}) \end{aligned}$$

   with the last term coming from $v \mapsto u^i \rho(u^j)(v)$ by noting that $\rho(u^j)(v) = u^j v u^{-j}$. There are no further elements since[4] $\pi_n(X \times Y) \cong \pi_n(X) \times \pi_n(Y)$ for any X, Y.

   Hence $\pi_3(SO(4)) \cong \mathbb{Z}^2$ and its elements can be represented in the form

$$\phi_{ij} : u \mapsto (v \mapsto u^i v u^j),$$

   where we substitute $(i, j) \mapsto (i + j, -j)$.

Write E_{ij} to refer to the smooth fibre bundle $S^3 \hookrightarrow E_{ij} \to S^4$ corresponding to $\phi_{ij} \in \pi_3(SO(4))$. We will return to this in the following chapter to show that the E_{ij} are homeomorphic to S^7 when $i + j = \pm 1$ and that many are in fact exotic spheres.

2.2 Chern Classes

Throughout this section, we will use $\mathbb{Z}$ coefficients and let $\xi = (E, X, \pi)$ denote a d-dimensional vector bundle with total space E, base space X and projection π. We also use $E^\#$ to denote $E \setminus s_0(X)$, where $s_0 : X \to E$ is the zero section.

In the case where ξ admits a $\mathbb{Z}$-orientation $\{\varepsilon_x\}_{x \in X}$, recall that there is a unique class $u_\xi \in H^d(E, E^\#)$ (the *Thom class*) which restricts to ε_x on each fibre. The *Euler class* $e(\xi) \in H^d(X)$ is then defined by the image of u_ξ under the composition

$$H^d(E, E^\#) \longrightarrow H^d(E) \xrightarrow[\sim]{s_0^*} H^d(X) \ .$$

This has the property that, if $\xi' = (E', X', \pi')$ is another oriented d-dimensional real vector bundle and $(F, f) : \xi \to \xi'$ is an orientation-preserving bundle morphism, then $f^* e(\xi') = e(\xi)$. We say that e satisfies the *naturality condition*.

[4]This is an immediate consequence of the fact that we can combine homotopies $h_X : S^n \times I \to X$ and $h_Y : S^n \times I \to Y$ to give homotopies $(h_X, h_Y) : S^n \times I \to X \times Y$.

That is, $e(\xi)$ is an invariant for the category of oriented real vector bundles. We call an invariant of this form for (some set of) vector bundles a *characteristic class*.

We will primarily be concerned with the following property of the Euler class:

Definition (Gysin sequence)**.** The long exact sequence for the pair $(E, E^\#)$ gives:

$$\begin{tikzcd}
H^{i+d}(E,E^\#) \arrow[r] & H^{i+d}(E) \arrow[r] \arrow[d, shift left, "s_0^*"] & H^{i+d}(E^\#) \arrow[r] \arrow[drr] & & H^{i+d+1}(E,E^\#) \\
H^i(X) \arrow[u, "\Phi", "\wr"'] \arrow[r, "\cdot\smile e(\xi)"] & H^{i+d}(X) \arrow[u, shift left, "\pi^*"] \arrow[ur, "\pi_0^*"'] & & & H^{i+1}(X) \arrow[u, "\Phi", "\wr"']
\end{tikzcd}$$

where Φ is the *Thom isomorphism*, given by cupping with the Thom class. The bottom sequence, with maps such that the diagram commutes, is the *Gysin sequence.*

Since the top row is exact, the Gysin sequence must be also. This shows:

Proposition. $\pi_0^* : H^i(X) \to H^i(E^\#)$ is an isomorphism for $i < d$.

We now consider the case where ξ is a complex d-dimensional vector bundle. This can be considered as a real vector bundle $\xi_{\mathbb{R}}$ by identifying $\mathbb{C}$ with $\mathbb{R}^2$. Note that bases for each fibre induce local orientations and that all such orientations are equivalent, a consequence of $\mathrm{GL}(n, \mathbb{C})$ being connected. That is, $\xi_{\mathbb{R}}$ has a canonical orientation.

Remark. In the real case, the determinant splits $\mathrm{GL}(n, \mathbb{R})$ into two connected components.

Hence we immediately get a top-dimensional invariant of complex vector bundles, namely $e(\xi_{\mathbb{R}}) \in H^{2d}(X)$. To find more invariants, observe that the above proposition gives a way to relate cohomology classes of vector bundles over X to those of vector bundles over $E^\#$. Fortunately, we have a natural way to construct a bundle over $E^\#$ as follows:

1. For each $v \in E_x^\# \subseteq E^\#$, let $\widetilde{E}_v = E_x/\mathbb{C}v$.
2. Now let $\widetilde{E} = \bigcup_{v \in E^\#} \widetilde{E}_v$ with the natural projection map $\widetilde{\pi} : \widetilde{E} \to E^\#$.
3. Then $\xi^\# = (\widetilde{E}, E^\#, \widetilde{\pi})$ is a $(d-1)$-dimensional complex vector bundle.

Since this is one dimension lower, it has an invariant $e(\xi_{\mathbb{R}}^\#) \in H^{2(d-1)}(E^\#)$ which indeed gives us an invariant $(\pi_0^*)^{-1} e(\xi_{\mathbb{R}}^\#) \in H^{2(d-1)}(X)$ of the original vector bundle. By considering a sequence ξ, $\xi^\#$, $(\xi^\#)^\#$, $\ldots$ of vector bundles, we get a family of invariants for ξ:

Definition. We define the i^{th} *Chern class* $c_i(\xi) \in H^{2i}(X)$ inductively on d by:

$$c_i(\xi) = \begin{cases} e(\xi_{\mathbb{R}}), & i = d. \\ (\pi_0^*)^{-1} c_i(\xi^\#), & i < d. \end{cases}$$

It is often useful to package all this information as a single element in the graded ring $H^*(X)$. In particular, the *total Chern class* is defined as

$$c(\xi) = 1 + c_1(\xi) + c_2(\xi) + \cdots + c_d(\xi) \in H^*(X).$$

In certain cases, as in the following example, computing the Chern classes is simply a consequence of noting that particular cohomology groups of the base space vanish.

Example. Recall that, similarly to $\mathbb{R}$ and $\mathbb{C}$, we can define the *tautological* (*line*) *bundle* over the quaternions $\mathbb{H}$

$$\mathbb{H} \hookrightarrow \gamma_{\mathbb{H}}^n = \{(x, v) \in \mathbb{HP}^n \times \mathbb{H}^{n+1} : [v] = x\} \to \mathbb{HP}^n$$

which we view an a 2-dimensional complex vector bundle by identifying $\mathbb{H}$ with $\mathbb{C}^2$. We will consider the case $n = 1$. We remarked earlier that $\mathbb{HP}^1$ is diffeomorphic to S^4 and so $c_1(\gamma_{\mathbb{H}}^1) \in H^2(\mathbb{HP}^1) = H^2(S^4) = 0$. If $e = e(\gamma_{\mathbb{H}}^1)$, then $c_2(\gamma_{\mathbb{H}}^1) = e$ and so $c(\gamma_{\mathbb{H}}^1) = 1 + e$.

Remark. The result is identical for $n > 1$ since $H^2(\mathbb{HP}^n) = 0$. To show this, we use the Gysin sequence to compute $H^*(\mathbb{HP}^n) = \mathbb{Z}[e]/(e^{n+1})$ just as for $\mathbb{RP}^n$ and $\mathbb{CP}^n$.

In general, computing the Chern classes usually requires understanding how the vector bundle relates to known cases like the one just considered. We now state a few basic properties of Chern classes, the proofs of which are standard and can be found in [15].

Proposition. Let $\xi = (E, X, \pi)$ and $\xi' = (E', X', \pi')$ be complex vector bundles. Then:

(i) If $(F, f) : \xi \to \xi'$ is a bundle map, then $f^*c_i(\xi') = c_i(\xi)$.

(ii) If $\varepsilon^k : X \times \mathbb{C}^k \to X$ is the trivial k-bundle, then $c(\xi \oplus \varepsilon^k) = c(\xi)$.

(iii) If $\xi \oplus \xi'$ denotes the *Whitney sum*, then $c(\xi \oplus \xi') = c(\xi)c(\xi')$.

(iv) If $\overline{\xi}$ denotes the *conjugate bundle*, formed by composing each trivialisation with the map that takes the complex conjugate on each copy of $\mathbb{C}$, then $c_i(\overline{\xi}) = (-1)^i c_i(\xi)$.

Example. To compute the Chern classes of the tangent bundle

$$\mathbb{C}^n \hookrightarrow T\mathbb{CP}^n \to \mathbb{CP}^n,$$

recall that $\mathbb{CP}^n = S^{2n+1}/\sim$ where $u \sim \lambda u$ for all $\lambda \in S^1$. Then

$$T\mathbb{CP}^n = \{[u, v] : (u, v) \in S^{2n+1} \times \mathbb{C}^{n+1}, u \cdot v = 0\},$$

where $[u, v]$ is the equivalence class of $(u, v) \in \mathbb{C}^{n+1} \times \mathbb{C}^{n+1}$ under $(u, v) \sim (\lambda u, \lambda v)$ for all $\lambda \in S^1$. We can write the trivial 1-bundle as $\varepsilon^1 = \{[u, \mu u] : u \in S^{2n+1}, \mu \in \mathbb{C}\}$. Since u, v are orthogonal in $T\mathbb{CP}^n$, we have

$$\begin{aligned} T\mathbb{CP}^n \oplus \varepsilon^1 &\cong \{[u, v + \mu u] : (u, v) \in S^{2n+1} \times \mathbb{C}^{n+1}, \mu \in \mathbb{C}, u \cdot (v + \mu u) = \mathrm{Re}(\mu)\} \\ &\cong \{[u, v + \mu u] : (u, v + \mu u) \in S^{2n+1} \times \mathbb{C}^{n+1}\} \cong \{[u, v] : (u, v) \in S^{2n+1} \times \mathbb{C}^{n+1}\} \end{aligned}$$

where the second isomorphism is since requiring that $u \cdot v' = \mathrm{Re}(\mu)$ for some $\mu \in \mathbb{C}$ is no restriction at all. Since $\gamma_{\mathbb{C}}^n = \{[u, v] : (u, v) \in S^{2n+1} \times \mathbb{C}\}$, this shows that

$$T\mathbb{CP}^n \oplus \varepsilon^1 \cong \gamma_{\mathbb{C}}^n \oplus \cdots \oplus \gamma_{\mathbb{C}}^n,$$

where there are $n + 1$ terms in the sum. It is easy to see that $c(\gamma_{\mathbb{C}}^n) = 1 + a$ where $a = e(\gamma_{\mathbb{C}}^n) \in H^2(\mathbb{CP}^n)$, and this is generator by the Gysin sequence for $\mathbb{C} \hookrightarrow \gamma_{\mathbb{C}}^n \to \mathbb{CP}^n$. Hence we get

$$c(T\mathbb{CP}^n) = c(T\mathbb{CP}^n \oplus \varepsilon^1) = c(\gamma_{\mathbb{C}}^n)^{n+1} = (1 + a)^{n+1}.$$

2.3 Pontryagin Classes

Recall that we found a characteristic class for oriented real vector bundles and then used it to construct an invariant for complex vector bundles. Interestingly we can now get an invariant for not-necessarily-oriented real vector bundles ξ as follows.

Definition. We define the i^{th} *Pontryagin class* $p_i(\xi) \in H^{4i}(X)$ by:

$$p_i(\xi) = (-1)^i c_{2i}(\xi \otimes_{\mathbb{R}} \mathbb{C}),$$

where $\xi \otimes_{\mathbb{R}} \mathbb{C}$ is the complex vector bundle formed by tensoring each fibre with $\mathbb{C}$.

Remark. We only consider the even Chern classes since $\overline{\xi \otimes_{\mathbb{R}} \mathbb{C}} \cong \xi \otimes_{\mathbb{R}} \mathbb{C}$ implies that $c_{2i+1}(\xi \otimes_{\mathbb{R}} \mathbb{C}) = (-1)^{2i+1} c_{2i+1}(\xi \otimes_{\mathbb{R}} \mathbb{C})$ and so $2c_{2i+1}(\xi \otimes_{\mathbb{R}} \mathbb{C}) = 0$.

We can similarly define the *total Pontryagin class*

$$p(\xi) = 1 + p_1(\xi) + p_2(\xi) + \cdots p_n(\xi) \in H^*(X),$$

where n is maximal so that $4n \leq d$, the dimension of ξ. Most of the basic properties of of Chern classes carry over immediately, with one small change:

Proposition. Let $\xi = (E, X, \pi)$ and $\xi' = (E', X', \pi')$ be complex vector bundles. Then $p(\xi \oplus \xi') = p(\xi)p(\xi')$ modulo 2-torsion elements.

Proof: The above remark shows that $c(\xi \otimes_{\mathbb{R}} \mathbb{C}) \equiv \sum (-1)^i p_i(\xi)$ modulo 2-torsion. Since $(\xi \oplus \xi') \otimes_{\mathbb{R}} \mathbb{C} \cong (\xi \otimes_{\mathbb{R}} \mathbb{C}) \oplus (\xi' \otimes_{\mathbb{R}} \mathbb{C})$, the product formula for Chern classes gives

$$\begin{aligned}\sum_k (-1)^k p_k(\xi \oplus \xi') &\equiv \sum_i (-1)^i p_i(\xi) \cdot \sum_j (-1)^j p_j(\xi') \\ &\equiv \sum_k \sum_{i+j=k} (-1)^k p_i(\xi) p_j(\xi') \quad \text{modulo 2-torsion.}\end{aligned}$$

We can then cancel the $(-1)^k$s by noting this holds at each grade in $H^*(X)$. □

We now introduce a useful computational tool that allows us to compute Pontryagin classes from Chern classes.

Proposition. Let ξ be a complex vector bundle. Then

$$1 - p_1(\xi_{\mathbb{R}}) + p_2(\xi_{\mathbb{R}}) - \cdots = (1 - c_1(\xi) + c_2(\xi) - \cdots)(1 + c_1(\xi) + c_2(\xi) + \cdots).$$

Proof: This amounts to noting that $\xi_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ is isomorphic to $\xi \oplus \overline{\xi}$, which can be checked on the fibres. It then follows directly from $c(\xi_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}) = c(\xi)c(\overline{\xi})$ and the fact that the odd terms vanish completely: $c_{2i+1}(\xi_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}) = \sum_k (-1)^k c_k(\xi) c_{2i+1-k}(\xi) = 0$. □

Example. We can then use this to continue the examples we established before.

(i) For the line bundle $\gamma^1_{\mathbb{H}}$ over $\mathbb{HP}^1$, we now have:

$$1 - p_1(\gamma^1_{\mathbb{H}}) + p_2(\gamma^1_{\mathbb{H}}) - \cdots = (1 + e)^2 = 1 + 2e + e^2,$$

where $e = e(\gamma^1_{\mathbb{H}})$. Hence $p(\gamma^1_{\mathbb{H}}) = 1 - 2e + e^2$, i.e. $p_1(\gamma^1_{\mathbb{H}}) = -2e$ and $p_2(\gamma^1_{\mathbb{H}}) = e^2$.

(ii) For the tangent bundle of $\mathbb{CP}^n$ we now have:

$$1 - p_1(T\mathbb{CP}^n) + p_2(T\mathbb{CP}^n) - \cdots = (1 - a)^n (1 + a)^{n+1} = (1 - a^2)^{n+1},$$

where $a = e(\gamma^n_{\mathbb{C}})$. Hence $p(T\mathbb{CP}^n) = (1 + a^2)^{n+1}$, i.e. $p_i(T\mathbb{CP}^n) = \binom{n+1}{i} a^{2i}$.

Example. The n-sphere has tangent and normal bundle:

$$T_u S^n = \{v \in \mathbb{R}^{n+1} : u \cdot v = 0\}, \quad (\mathcal{V}_{S^n \subseteq \mathbb{R}^{n+1}})_u = \mathbb{R}u.$$

for any $u \in S^n$. Now $TS^n \oplus \mathcal{V}$ is the trivial bundle since $T_u S^n \oplus \mathcal{V}_u = \mathbb{R}^{n+1}$. Since $\mathcal{V}$ is also trivial, we get $p(TS^n) = p(TS^n \oplus \mathcal{V}) = 1$, i.e. $p_i(TS^n) = 0$ for all i.

We will now conclude this chapter by considering how the Pontryagin classes can be used to give invariants for smooth manifolds. Consider a compact connected oriented smooth manifold M and the Pontryagin classes $p_i(TM) \in H^{4i}(M)$ of its tangent bundle. Whilst previously we wanted to compare different vector bundles with the same base space, we now want to compare the tangent bundles of different manifolds M and N. To do this, we have to get around the issue that the Pontryagin classes lie inside $H^*(M)$ and $H^*(N)$ which do not necessarily have a canonical identification. The solution is to extract an integer-valued invariant, and this can be done as follows when M is $4d$-dimensional.

First recall that we have a canonical element in $H_{4d}(X)$ given by the fundamental class $[M]$ (corresponding to an orientation on M). Taking the cap product of an element in $H^{4d}(M)$ with $[M]$ then induces a map

$$H^{4d}(M) \to H_0(M) \cong \mathbb{Z}$$

where we fix an identification with $\mathbb{Z}$ (this is of course one the *Poincaré duality* maps). Furthermore, for any partition $(i_1, \cdots, i_r)$ of d, we get an element of $H^{4d}(M)$ given by $p_{i_1}(TM) \smile \cdots \smile p_{i_r}(TM)$ since this has grading $4i_1 + \cdots + 4i_r = 4d$. This motivates the following definition.

Definition. Let M be a compact connected oriented manifold of dimension $4d$ and $I = (i_1, \cdots, i_r)$ a partition of d. The I^{th} *Pontryagin number* is

$$p_I[M] = p_{i_1} \cdots p_{i_r}[M] = (p_{i_1}(TM) \cdots p_{i_r}(TM))[M] \in \mathbb{Z}.$$

Example. Consider the compact connected $4n$-dimensional smooth manifold $\mathbb{CP}^{2n}$. Note that this is orientable since, as before, its underlying real manifold has a canonical orientation. We showed that

$$p_i(T\mathbb{CP}^{2n}) = \binom{2n+1}{i} a^{2i},$$

where $a \in H^2(\mathbb{CP}^{2n})$ is a generator, so we need to compute $a^{2n}[\mathbb{CP}^{2n}]$. To do this, note that

$$e(T\mathbb{CP}^{2n})[\mathbb{CP}^{2n}] = c_{2n}(T\mathbb{CP}^{2n})[\mathbb{CP}^{2n}] = (2n+1)a^{2n}[\mathbb{CP}^{2n}].$$

A standard fact[5] about the Euler class, and the reason for its name, is that $e(TM)[M] = \chi(M)$ where χ is the *Euler characteristic* of M. We compute

$$\chi(\mathbb{CP}^{2n}) = \sum_i (-1)^i \operatorname{rank} H^i(\mathbb{CP}^{2n}) = 2n+1$$

and so we must conclude that $a^{2n}[\mathbb{CP}^{2n}] = 1$. Hence

$$p_I[\mathbb{CP}^{2n}] = \binom{2n+1}{i_1} \cdots \binom{2n+1}{i_r}.$$

Remark. We can similarly construct *Chern numbers* to get invariants for complex manifolds, though will not consider them in this essay.

[5] For a proof, see chapter 11 of [15].

3 Milnor's Exotic Spheres

The goal of this chapter will be to construct the exotic spheres discovered by Milnor in 1953, expanding the details of the arguments given in [1]. The spaces Milnor considered were the sphere bundles

$$S^3 \hookrightarrow E_{ij} \to S^4$$

we constructed in the previous chapter, for each $i, j \in \mathbb{Z}$. More specifically, let M_k be the total space E_{ij} when $i + j = 1$ and $i - j = k$.

We will start by showing that the M_k are homeomorphic to S^7 using the tools that were available to Milnor at the time, namely Morse theory[6]. We next use Pontryagin classes to develop an invariant λ that can distinguish smooth structure; this will rely heavily on Hirzebruch's signature theorem, the proof of which we delay to the following chapter. This chapter then concludes by showing how these ideas can also be used to construct exotic 15-spheres but do not work in other dimensions.

3.1 Morse Theory

The observation of Morse theory is that topological information about a manifold M can be inferred from knowledge of the existence of a smooth map $f : M \to \mathbb{R}$ with certain properties. In particular, we would like such functions to be of the following form.

Definition. Let M be a smooth compact manifold. We say a smooth map $f : M \to \mathbb{R}$ is a *Morse function* if every critical point is non-degenerate, i.e. if $p \in M$ has local coordinates $x_1, \cdots, x_n$ around p with $\left.\frac{\partial f}{\partial x_i}\right|_p = 0$ for all i, then $\det\left(\left.\frac{\partial^2 f}{\partial x_i \partial x_j}\right|_p\right) \neq 0$.

This is well-defined since it can be shown that non-degeneracy of a critical points is a coordinate-independent phenomenon. We have the following key lemma.

Lemma (Morse's Lemma)**.** Let M be a smooth compact n-manifold and $f : M \to \mathbb{R}$ a Morse function. If $p \in M$ is a critical point, then there are local coordinates $v_1, \cdots, v_n$ around p such that

$$f(v_1, \cdots, v_n) = f(p) - v_1^2 - \cdots - v_k^2 + v_{k+1}^2 + \cdots + v_n^2$$

for some $1 \leq k \leq n - 1$ which we call the *index* of f.

Intuitively this says that we can pick coordinates on a neighbourhood U around a critical point p so that each coordinate axis cuts through U in such a way that its image under f gives the shape of a "$\pm x^2$-graph". To prove this, note that we can expand f near p to get

$$f(x_1, \cdots, x_n) = f(p) + \sum_{i,j} x_i x_j H_{ij}(x_1, \cdots, x_n)$$

for smooth functions H_{ij}. We can assume H is symmetric replacing H_{ij} with $(H_{ij} + H_{ji})/2$. The rest is a straightforward exercise in diagonalisation using the inverse function theorem.

We can then use this to deduce the following important theorem. We will assume basic knowledge of Differential Geometry.

[6] As we will see in the following section, this can alternatively be done by computing the homology groups using the Gysin sequence by the Poincaré conjecture for $n \geq 5$.

Theorem (Reeb's Theorem)**.** Let M be a smooth compact manifold and $f : M \to \mathbb{R}$ a Morse function with exactly two critical points. Then M is homeomorphic to S^n.

Proof: By compactness, f has a minimum x_0 and a maximum x_1 and so these are the only critical points and, by linear rescaling, we can assume that $f(x_0) = 0$, $f(x_1) = 1$. The Morse lemma gives us a neighbourhood U around x_0 and coordinates such that

$$f(v_1, \cdots, v_n) = v_1^2 + \cdots + v_n^2,$$

since x_0 is a minimum point. Thus $dv_1^2 + \cdots + dv_n^2$ is a Riemannian metric $\langle \cdot, \cdot \rangle$ on U, which can be extended to M by partitions of unity. We can now let ∇f be the vector field defined by $\langle X, \nabla f \rangle = X(f)$ for all $f : M \to \mathbb{R}$ and vector fields X, noting that this is singular precicely at x_0 and x_1.

Conisder the vector field $\nabla f / \|\nabla f\|^2$ defined on $M \setminus \{x_0, x_1\}$ and let $\phi_t : M \to M$ be the associated flow, a 1-parameter family of diffeomorphisms. For any $q \in M$, note that

$$\frac{df(\phi_t(q))}{dt} = \left\langle \frac{d\phi_t(q)}{dt}, \nabla f \right\rangle = \left\langle \frac{\nabla f}{\|\nabla f\|^2}, \nabla f \right\rangle = 1$$

and so $f(\phi_t(q)) = f(q) + t$. Hence for each $0 \leq \varepsilon < 1 - t$, the flow map ϕ_ε induces a diffeomorphism $f^{-1}[0, t] \to f^{-1}[0, t + \varepsilon]$. Now $D^n = f^{-1}[0, \varepsilon]$ is a disc for sufficiently small ε, and we define

$$\Phi : D^n \to M, \quad q \mapsto \phi_{f(q)/\varepsilon}(q)$$

which retricts to a diffeomorphism $\mathring{D}^n \to M \setminus \{x_1\}$. Since $\partial D^n \mapsto \{x_1\}$, this then descends to a homeomorphism $S^n \to M^n$. □

To apply this to the M_k recall that it was defined via the clutching construction to be

$$M_k = (\mathbb{H} \times S^3) \sqcup (\mathbb{H}' \times S^3) / \sim$$

where $\mathbb{H} = S^4 \setminus \{N\}$ and $\mathbb{H}' = S^4 \setminus \{S\}$ are each identified with the quaternions for ease of notation. Note that, to identify $\mathbb{H}$ and $\mathbb{H}'$, we have to "flip" the algebraic structure in that $u \in \mathbb{H}^\times$ corresponds to $u^{-1} \in (\mathbb{H}')^\times$. More generally

$$(u, v) \sim (u', v') = \left(u^{-1}, \frac{u^i v u^j}{\|u\|^{i+j}} \right) = \left(u^{-1}, \frac{u^i (vu) u^{-i}}{\|u\|} \right)$$

since $i + j = 1$. Define a function $f : \mathbb{H}' \times S^3 \to \mathbb{R}$ by

$$f(u', v') = \frac{\mathrm{Re}(v')}{(1 + \|u'\|^2)^{1/2}}$$

where $\mathrm{Re}(v')$ denotes the real part of the quaternion $v' \in S^3 \subseteq \mathbb{H}'$. Since the real part is the kernel of the action by $SO(3)$, it is fixed by conjugation and so

$$f(u', v') = \frac{\mathrm{Re}(u^i (vu) u^{-i})}{(1 + \|u\|^2)^{1/2}} = \frac{\mathrm{Re}(vu)}{(1 + \|u\|^2)^{1/2}}$$

which we take as our definition for $f(u, v)$ for $(u, v) \in \mathbb{H} \times S^3$, thus extending f to a smooth map $f : M_k \to \mathbb{R}$. We can then check by writing f out explicitly using coordinates that f has two critical points $(u', v') = (0, \pm 1)$ in the first chart and none in the second. Hence M_k is homeomorphic to S^7 by Reeb's theorem.

3.2 Milnor's λ Invariant

Here we use Pontryagin numbers to construct an invariant λ for manifolds homeomorphic to S^7 which is able to distinguish smooth structure. From this point onwards, we will take *n-manifold* to mean a connected compact orientable smooth n-dimensional manifold-with-boundary and *closed* to in addition mean that a manifold has no boundary.

Let M be an closed 7-manifold with the homology of S^7 and which bounds[7] an 8-manifold B. Note that, since $H^3(M) = H^4(M) = 0$, the inclusion map $i : M \hookrightarrow B$ gives rise to an isomorphism $i : H^4(B, M) \to H^4(B)$ by the long exact sequence for (B, M).

Note that, similarly to the case of closed manifolds, an orientation on B is determined by its fundamental class $[B] \in H^8(B, M)$. The relative cap product then gives a symmetric bilinear form $\langle \cdot\, , \cdot \rangle : H^4(B, M) \times H^4(B, M)/(\text{torsion}) \to \mathbb{Z}$ given by

$$\langle x, y \rangle = (x \smile y)[B].$$

Since this form is symmetric, it has a well-defined signature which we denote $\sigma(B)$. Observe that there is a formula for the signature in the case of closed manifolds. The proof uses *cobordism theory* and will be postponed until the following chapter.

Theorem (Hirzebruch's Signature Theorem)**.** If M is a closed $4n$-manifold as above, then there is a polynomial $L_n(x_1, \cdots, x_n) \in \mathbb{Q}[x_1, \cdots, x_n]$ such that

$$\sigma(M) = L_n(p_1, \cdots, p_n)[M],$$

where $p_i = p_i(TM) \in H^{4i}(M)$ and each monomial in $L_n(p_1, \cdots, p_n)$ lies in $H^{4n}(M)$. In particular $L_1 = \frac{1}{3}p_1$ and $L_2 = \frac{1}{45}(7p_2 - p_1^2)$.

Whilst B is not a closed manifold, we might wonder if expression was still true in a certain sense. Consider the Pontryagin classes $p_1 = p_1(TB) \in H^4(B)$ and $p_2 = p_2(TB) \in H^8(B)$. Now p_1 can be pulled back along i to get $i^{-1}p_1 \in H^4(B, M)$ and hence an invariant

$$(i^{-1}p_1)^2[B] \in \mathbb{Z}.$$

However, there is no analogous quantity for p_2 since $i : H^8(B, M) \to H^8(B)$ isn't necessarily an isomorphism. So since "$\sigma(B) - \frac{1}{45}(7(i^{-1}p_2) - (i^{-1}p_1)^2)[B]$" is not properly defined (let alone equal to $\sigma(B)$), the best we can hope for is that they are equal when we kill the $i^{-1}p_2$ term by dropping to a quotient ring. In particular, multiplying through by 45 and reducing mod 7 gives $3\sigma(B) + (i^{-1}p_1)^2[B] \mod 7$.

Definition (Milnor Invariant)**.** Let M be a closed 7-manifold as above. Then define

$$\lambda(M) = 2q(B) - \sigma(B) = 2(q(B) + 3\sigma(B)) \mod 7$$

for any 8-manifold B with boundary M, where $q(B) = (i^{-1}p_1)^2[B]$.

Example. Let $M = S^7$ be the standard 7-sphere and choose $B = D^8$. Since $H^4(B, M) = H^4(B) = 0$, the quadratic form $H^4(B, M) \otimes H^4(B, M) \to \mathbb{Z}$ is zero and so $\lambda(M) = 0$.

The difficulty comes in showing that $\lambda(M)$ is independent of the choice of B. If so then λ being a diffeomorphism invariant follows from noting that a diffeomorphism $f : M \to M'$ can be extended[8] to a diffeomorphism between B and a suitible B' with boundary M'.

[7]In fact, as will we see in the following section, every such closed 7-manifold bounds such an 8-manifold.

[8]This uses that *oriented cobordism* is an equivalence relation on the category of manifolds, which we prove in the following section.

To show that this is independent of the choice of B, i.e. that the defect to the signature theorem holding for B_1 and B_2 are the same, we form a closed 8-manifold $C = B_1 \sqcup_M B_2$ by gluing along M and give it orientation compatible with that of B_1 and $-B_2$ (i.e. the opposite orientation to the one on B_2). Since C is closed, the signature theorem gives that $\lambda(C) = 0$ (noting that $i_* = \mathrm{id}$ and so $q(C) = p_1^2[C]$) and so it remains to show that

$$\lambda(C) = \lambda(B_1) - \lambda(B_2).$$

This follows immediately from the following lemma.

Lemma. (i) $\sigma(C) = \sigma(B_1) - \sigma(B_2)$, and (ii) $q(C) = q(B_1) - q(B_2)$.

Proof: We start by considering all the relevant cohomology groups and the maps between them. We get the following diagram of isomorphisms:

$$\begin{tikzcd} H^4(C,M) \arrow[r, "h", "\sim"'] \arrow[d, "j", "\wr"'] & H^4(B_1,M) \oplus H^4(B_2,M) \arrow[d, "i_1 \oplus i_2", "\wr"'] \\ H^4(C) \arrow[r, "k", "\sim"'] & H^4(B_1) \oplus H^4(B_2) \end{tikzcd}$$

where h and k come from the Mayer-Vietoris sequence for relative cohomology and i_1, i_2 and j are the inclusions of M into B_1, B_2 and C respectively. This is commutative by naturality of the Mayer-Vietoris sequence.

(i) Let $\alpha_i \in H^4(B_i, M)$ and the corresponding $\alpha = jh^{-1}(\alpha_1 \oplus \alpha_2) \in H^4(C)$. Then, since $[C] = (jh^{-1})([B_1] \oplus [-B_2])$ by construction, we get:

$$\alpha^2[C] = jh^{-1}(\alpha_1^2 \oplus \alpha_2^2)[C] = (\alpha_1^2 \oplus \alpha_2^2)([B_1] \oplus (-[B_2])) = \alpha_1^2[B_1] - \alpha_2^2[B_2].$$

Hence the quadratic form for C splits as a direct sum of the quadratic forms for B_1 and $-B_2$, i.e. the matrix for C is block diagonal consisting of B_1 and $-B_2$. The result then follows.

(ii) Note that inclusion $f_i : B_i \hookrightarrow C$ is an embedding and so $(Df_i, f_i) : TB_i \to TC$ is a bundle map. By naturality, this gives that $(f_i)^* p_1(TC) = p_1(TB_i)$ for $i = 1, 2$. By definition of k, this says that

$$k(p_1(TC)) = p_1(TB_1) \oplus p_1(TB_2).$$

By commutativity, this is equivalent to

$$p_1(TC) = jh^{-1}(i_1^{-1}p_1(TB_1) \oplus i_2^{-1}p_1(TB_2)).$$

The formula dervied above then gives that what we want:

$$p_1(TC)^2[C] = (i_1^{-1}p_1(TB_1))^2[B_1] - (i_2^{-1}p_1(TB_2))^2[B_2].$$

□

This completes the proof that λ is a well-defined diffeomorphism invariant. Hence by our previous example if M is diffeomorphic to S^7 then $\lambda(M) = 0$. The rest of this subsection will be devoted to computing $\lambda(M_k)$. We start by considering a special case.

Example. To compute E_{01} we note that it is diffeomorphic to $\gamma^1_{\mathbb{H}}$, the total space of the tautological bundle over $\mathbb{HP}^1$. This can be shown by writing down the obvious maps between the pairs of charts used to define each manifold and checking they agree.

We showed previously that $p_1(\gamma^1_{\mathbb{H}}) = -2e$ and $p_2(\gamma^1_{\mathbb{H}}) = e^2$ where $e = e(\gamma^1_{\mathbb{H}}) \in H^4(\mathbb{HP}^1)$. To write this is a more universal form, fix an isomorphism $H^4(\mathbb{HP}^1) \to H^4(S^4)$ and let $\nu \in H^4(S^4)$ be the image of e under this map. Hence $p_1(E_{01}) = -2\nu$ and $p_2(E_{01}) = \nu^2$.

Recall the *Grassmannian* $\mathrm{Gr}_k(\mathbb{R}^n) = \{k\text{-dimensional linear subspaces of } \mathbb{R}^n\}$ and γ^k its tautological bundle. Then it can be shown that every n-dimensional vector bundle over S^k is a pullback of γ^k. In particular, this construction gives a group homomorphism

$$\begin{tikzcd} & \mathrm{Fib}_{S^3}(S^4) \arrow[dr, "E_{ij}\mapsto(u\mapsto(\mathbb{R}(E_{ij}))_u)/"] & \\ \pi_3(SO(4)) \arrow[ur, "\phi_{ij}\mapsto E_{ij}"] \arrow[rr, dashed] & & \pi_4(G_4(\mathbb{R}^8)) \end{tikzcd}$$

where $\mathbb{R}^4 \hookrightarrow \mathbb{R}(E_{ij}) \mapsto S^4$ is the vector bundle corresponding to the sphere bundle E_{ij}. We use this to establish the following.

Proposition. $p_1(E_{ij}) = 2(i-j)\nu$.

Proof: We start by showing that $p_1(E_{ij})$ is linear in i and j, i.e. that $(i,j) \to p_1(E_{ij})$ is a group homomorphism. This is done by exhibiting the map as the composition:

$$\mathbb{Z}^2 \to \pi_3(SO(4)) \to \pi_4(G_4(\mathbb{R}^8)) \to H^4(S^4),$$

where the third map is $[f] \mapsto p_1(f^*(\gamma^4))$. To show this is a group homomorphism, note that $p_1(f^*(\gamma^4)) = f^*(p_1(\gamma^4))$ by naturality and so

$$(f\cdot g)^*(p_1(\gamma^4)) = f^*(p_1(\gamma^4))g^*(p_1(\gamma^4))$$

for $f, g \in \pi_4(G_4(\mathbb{R}^8))$. Hence $p_1(E_{ij})$ is linear.

Next note that if $\overline{E_{ij}}$ is formed by taking quaternionic conjugate on each fibre, then $p_1(E_{ij}) = p_1(\overline{E_{ij}})$ by earlier results. The transitions function between the two charts is $(u,v) \mapsto (u^{-1}, \frac{u^i v u^j}{\|u\|^{i+j}})$ and conjugating both fibres gives

$$(u, \bar{v}) \mapsto \frac{\bar{u}^j \bar{v} \bar{u}^i}{\|u\|^{i+j}} = \frac{u^{-j}\bar{v}u^{-i}}{\|u\|^{-j-i}},$$

so $\overline{E_{ij}}$ has transition function $(u,v) \mapsto \frac{u^{-j}vu^{-i}}{\|u\|^{-j-i}}$, which shows[9] that $\overline{E_{ij}} \cong E_{-j,-i}$. Hence $p_1(E_{ij}) = p_1(E_{-j,-i})$. By linearity, we can now write $p_1(E_{ij}) = a(i-j)\nu$ for some $a \in \mathbb{Z}$ and generator $\nu \in H^4(S^4)$. Finally $a = 2$ since $p_1(E_{01}) = -2\nu$. $\square$

Remark. We can also show that $e(E_{ij}) = (i+j)\nu$ using $e(E_{ij}) = e(E_{-i,-j})$ and $e(E_{01}) = \nu$.

Note that M_k has a corresponding disc bundle $D^4 \hookrightarrow B_k \to S^4$ where the total space B_k has boundary M_k, and so M_k is indeed of the form we have considered in this chapter. So far we know that $p_1(M_k) = 2k\nu$ and we want to compute $p_1(TB_k)$. This is done using the following trick:

Lemma. If $\pi : E \to X$ is a smooth fibre bundle over a closed manifold X equipped with a Riemannian metric on each fibre, then

$$TE \cong \pi^*(TX) \oplus \pi^*(E).$$

This can be proven by noting that the surjection $D\pi : TE \to \pi^*(TX)$ induces a short exact sequence

$$0 \longrightarrow T_vE \longrightarrow TE \xrightarrow{D\pi} \pi^*(TX) \longrightarrow 0.$$

It can then be checked that this splits and that $T_vE = \pi^*(E)$.

[9] This is why we ignored the cases where $i + j = -1$ even though E_{ij} is homeomorphic to S^7.

Theorem. $\lambda(M_k) \equiv k^2 - 1 \mod 7$.

Proof: Since B_k is a vector bundle, the zero section gives a homotopy equivalence $B_k \simeq S^4$. Hence $H^4(B_k) \cong \mathbb{Z}$ and we can pick our orientation so that $\sigma(B_k) = 1$.

The embedding $i : M_k \hookrightarrow B_k$ induces a bundle map $(Di, i) : TM_k \to TB_k$ and so $i^*(p_1(TB_k)) = p_1(TM_k)$. By the lemma above,

$$TM_k \cong \pi^*(TS^4) \oplus \pi^*(M_k)$$

where $\pi : M_k \to S^4$ is projection. Since $H^*(S^4) \cong \mathbb{Z}$ has no 2-torsion, this gives

$$p_1(TM_k) = \pi^*(p_1(TS^4)) \oplus \pi^*(p_1(M_k)) = \pi^*(0) \oplus \pi^*(2k\nu) = 2k\pi^*(\nu),$$

where we showed that $p_1(TS^4) = 0$ in a previous example. This can be rewritten as $p_1(TB_k) = 2k\alpha$, where $\alpha = (i^*)^{-1}(\pi^*(\nu))$ is a generator of $H^4(B_k)$. Hence

$$\lambda(M_k) = 2q(B_k) - \sigma(B_k) = 2(i^{-1}(2k\alpha))^2[B_k] - 1 = 8k^2 - 1 \equiv k^2 - 1 \mod 7$$

where $(i^{-1}\alpha)^2[B_k] = 1$ since α corresponds to our choice of orientation. $\square$

Hence, if $k^2 \not\equiv 1 \mod 7$, then M_k cannot be diffeomorphic to S^7. So we have shown that S^7 has at least 4 distinct differentiable structures S^7, M_0, M_2 and M_3, corresponding to the quadratic residues mod 7. This concludes our construction of exotic 7-spheres.

3.3 Higher Dimensions

We will now more generally consider the case of closed $(4k-1)$-manifolds M homeomorphic to S^{4k-1}. To generalise our λ invariant, let B be a $4k$-manifold with boundary M. Note from before that inclusion $M \hookrightarrow B$ induces an isomorphism

$$i : H^n(B, M) \to H^n(B)$$

when $H^n(M) = H^{n+1}(M) = 0$, i.e. whenever $2 \leq n \leq 4k-3$. Hence, if $p_1, \cdots, p_k$ denote the Pontryagin classes of the tangent bundle of M, then $p_1, \cdots, p_{k-1}$ can all be pulled back to give invariants in $H^n(B, M)$. As before, we must eliminate the p_k terms:

Definition (Milnor Invariant)**.** Let M be a closed $(4k-1)$-manifold as above. Then define

$$\lambda(M) = \frac{1}{s_k}(\sigma(B) - L_k(i^{-1}p_1, \cdots, i^{-1}p_{k-1}, 0)[B]) \in \mathbb{Q}/\mathbb{Z}$$

for any $4k$-manifold B with boundary M, where s_k is the coefficent of p_k in $L_k(p_1, \cdots, p_k)$.

Similarly to before, we could also clear denominators and reduce modulo the numerator of s_k; we write this as $\widetilde{\lambda}$. The proof that this is an invariant is the same as before.

We will now try and generalise the above construction as much as possible to other values of k. In particular, we want to find fibre bundles of the form

$$S^{2k} \hookrightarrow E \to S^{2k-1}.$$

where E is homeomorphic to S^{4k-1}. As mentioned before, such fibre bundles only exist when $1 \leq k \leq 3$, with structure groups $SO(2)$, $SO(4)$ and $SO(8)$ respectively.

We will first consider the case $k = 1$. Since $SO(2)$ is the group of rotations of $\mathbb{R}^2$, it inherits the topology of S^1 and so $\pi_1(SO(2)) = \mathbb{Z}$. Thus the elements $n \in \mathbb{Z}$ precisely correspond to fibre bundles $S^1 \hookrightarrow E_n \to S^2$. Alternatively, for $n \neq 0$ the rotation group $G_n = \{\exp(2\pi i/n) : i \in \mathbb{Z}\} \leq S^1$ induces a fibre bundle

$$S^1/G_n \cong S^1 \hookrightarrow S^3/G_n \to S^3/S^1 \cong S^2.$$

Now S^3/G_n has to equal E_m for some m, and it can be checked that in fact $S^3/G_n \cong E_n$ (see [14]). These are examples of *Lens spaces* which have $\pi_1(S^3/G_n) \cong G_n \cong \mathbb{Z}_n$ and so are not homeomorphic to S^3 except when $n = \pm 1$. It then follows that $S^1 \hookrightarrow S^3 \to S^2$ is the only smooth fibre bundle with total space homeomorphic to S^3. This agrees with the fact that there are no exotic 3-spheres.

When $k = 3$, we can follow a similar process as in the $k = 2$ case. Indeed a similar proof shows that $\pi_7(SO(8)) \cong \mathbb{Z} \oplus \mathbb{Z}$ generated by $u \mapsto \phi_{ij}(u) : v \mapsto u^i v u^j$ where multiplication is octonionic. We thus get a set of 15-manifolds E_{ij}.

Since $L_4 = \frac{1}{14175}(381p_4 - 71p_1p_3 - 19p_2^2 + 22p_1^2p_2 - 3p_1^4)$, the λ invariant is

$$\lambda(M) = \frac{14175}{381}\sigma(B) - \left(-\frac{71}{381}p_1p_3 - \frac{19}{381}p_2^2 + \frac{22}{381}p_1^2p_2 - \frac{3}{381}p_1^4\right)$$

where, by abuse of notation, we write p_i to denote $i^{-1}p_i$ and omit the $[B]$ from the end of each term. For convenience, we multiply through by 381 and work mod 381 (rather than mod 1):

$$\widetilde{\lambda}(M) = 78\sigma(B) + 71p_1p_3 + 19p_2^2 - 22p_1^2p_2 + 3p_1^4 \mod 381.$$

We then similarly let M_k be E_{ij} with $i + j = 1$ and $i - j = k$. To compute p_1, p_2 and p_3, let B_k be the corresponding disc bundle $D^8 \hookrightarrow B_k \to S^8$. By the long exact sequence on homotopy groups, this gives that $\pi_8(B_k) = \pi_8(S^8) \cong \mathbb{Z}$ is the first non-vanishing homotopy group and so $H^8(B_k) \cong \mathbb{Z}$ is the first non-vanishing homology group by the Hurewicz theorem. Poincaré Duality then gives that the only non-trivial homology groups are $H^i(B_k) \cong \mathbb{Z}$ when $i = 0, 8, 16$.

This implies $p_1 = p_3 = 0$, and it can be calculated similarly to above that $p_2 = 6k\alpha$ where $\alpha \in H^8(B, M)$ is a generator. That $H^8(B, M) \cong \mathbb{Z}$ also gives that $\sigma(B) = 1$ with appropriate choice of orientation. Hence

$$\widetilde{\lambda}(M) = 78 + 19(i^{-1}(6k\alpha))^2[B_k] = 78(1 - k^2) \mod 381.$$

Since $(78, 381) = 3$ and $381/3 = 127$, we get that $\widetilde{\lambda}(M) = 0$ iff $k^2 \equiv 1 \mod 127$. Choosing suitable Morse functions also shows that M_k is homeomorphic to S^{15}.

Hence, if $k^2 \not\equiv 1 \mod 127$, then M_k is homeomorphic but not diffeomorphic to S^7. This gives $\frac{127-1}{2} = 63$ distinct exotic 15-spheres, each corresponding to the quadratic residues not equal to 1 (using that 127 is prime).

We have thus shown that there exist ≥ 4 distinct smooth structures on S^7 and ≥ 64 distinct smooth structures on S^{15}. As we shall see in the final chapter, there are many more distinct smooth structures both on S^7 and S^{15}, as well as in other dimensions.

4 Cobordism

The goal of this section will be to give an introduction to *cobordism*, the study of manifolds up to the equivalence relation where two manifolds are equivalent if their disjoint union is the boundary of a manifold of one dimension higher[10]. Next we repay the debt created in the last chapter by proving Hirzebruch's Signature Theorem and finally we state the h-cobordism theorem, showing how it can be used to deduce the Poincaré conjecture for $n \geq 5$. Throughout this section, $-M$ will denote the oriented manifold M with opposite orientation and $+$ will denote the disjoint union.

Definition. An (*oriented*) *cobordism* between closed oriented n-manifolds M and N is a compact oriented manifold X such that

$$\partial X \cong M + (-N)$$

is an orientation-preserving diffeomorphism, where ∂X has the induced orientation. We say M and N are (*oriented*) *cobordant* or belong to the same (*oriented*) *cobordism class*.

Note that we need compactness since otherwise $M = \partial(M \times [0, \infty))$ and so all cobordism classes would be trivial. Following the conventions from earlier, we now drop the words oriented and compact. We will need the following result from Differential Geometry:

Theorem (Collar Neighbourhood Theorem)**.** If X is a smooth manifold, then there is an open set $\partial X \subseteq U \subseteq X$ such that $U \cong \partial X \times [0, 1)$ is a diffeomorphism.

This can be proven similarly to Reeb's theorem by picking a Riemannian metric (using partitions of unity) and then considering the flow from an appropriate vector field.

Lemma. Oriented cobordism is an equivalence relation on the category of manifolds.

Proof: For reflexivity, suppose $f : M \to M'$ is a diffeomorphism. Then $X = [0, 1] \times M \sqcup_{(\mathrm{id}, f)} [0, 1] \times M'$ has boundary $M + (-M')$, as required. Symmetry is obvious.

For transitivity, suppose $M \sim N$ and $N \sim R$, so that $M + (-N) \cong \partial X_1$ and $N + (-R) \cong \partial X_2$ for $(n+1)$-manifolds X_1 and X_2. By applying the theorem above and restricting to components of the boundary, we can find open sets $\partial X_i \subseteq U_i$ with diffeomorphisms $f_i : U_i \to N \times [0, 1)$. We can check that $X_3 = X_1 \cup_{f_1(U_1) \sim f_2(U_2)} X_2$ is a smooth compact manifold with $M + (-R) \cong \partial X_3$, and so $M \sim R$. □

Let Ω_n denote the oriented cobordism classes of n- manifolds. Note that Ω_n forms an abelian group under $+$ with $0_{\Omega_n} = [M]$ for any closed manifold M. We also have a bilinear product

$$\Omega_m \times \Omega_n \to \Omega_{m \times n}, \quad ([M], [N]) \mapsto [M \times N]$$

as $M \times N$ is an $(n + m)$-manifold. Since $(M \sqcup N) \times R \cong (M \times R) \sqcup (N \times R)$, this gives Ω_* the structure of a graded ring with 2-sided identity $* \in \Omega_0$. In fact, Ω_* is graded commutative since $M^m \times N^n \cong (-1)^{mn} N^n \times M^m$ is orientation-preserving.

Example. (i) $\Omega_0 \cong \mathbb{Z}$. This is since the boundaries of 1-manifolds are either $\emptyset$ or $P + (-Q)$, and so the sum of the signs of points is a complete cobordism invariant.

(ii) $\Omega_1, \Omega_2 = 0$ since $S^1 = \partial D^2$ and Σ_g can be filled in by a connected sum of solid tori.

[10] With the operation of passing to the boundary, this is a *generalised homology theory* in that it satisfies all but one of the axioms of homology.

To compute these groups, we need to develop a cobordism invariant. Recall that for a $4n$- manifold M and a partition $I = (i_1, \cdots, i_r)$ of n, the Ith Pontryagin number is

$$p_I[M] = p_{i_1 \cdots i_r}[M] = (p_{i_1}(TM) \cdots p_{i_r}(TM))[M].$$

Note that changing the orientation of M does not change the Pontryagin classes, but it does change the sign of the fundamental class and so changes the sign of $p_I[M]$. Hence if $p_I[M] \neq 0$, then there doesn't exist an orientation-reversing diffeomorphism. In fact, it can be shown that, if M is the boundary of a $(4n+1)$-dimensional manifold, then $p_I[M] = 0$ for all partitions I of n. We can now to show that p_I is a cobordism invariant:

Proposition. For any partition I of n, we have a well-defined homomorphism $\Omega_{4n} \to \mathbb{Z}$ given by $[M] \mapsto p_I[M]$.

Proof: It can be easily checked that $p_I[M+N] = p_I[M] + p_I[N]$. To show well-defined, suppose $[M] = [N]$. Then there is a compact oriented $(4n+1)$-manifold X such that $M + (-N) \cong \partial X$. The above identity and lemma give that

$$p_I[M] + p_I[-N] = p_I[M + (-N)] = p_I[\partial X] = 0$$

for all partitions I. Noting $p_I[-N] = -p_I[N]$ then gives that $p_I[M] = p_I[N]$. □

This now gives us a way of proving that $\Omega_n \neq 0$ for various n.

Example. We can show $\Omega_{4n} \neq 0$ by exhibiting a closed $4n$-manifold with a non-vanishing Pontryagin number. In particular, as computed previously, we have that

$$p_I[\mathbb{CP}^{2n}] = \binom{2n+1}{i_1} \cdots \binom{2n+1}{i_r} \neq 0.$$

So for each partition $I = (i_1, \cdots, i_r)$ of n, the products $\mathbb{CP}^{2i_1} \times \cdots \times \mathbb{CP}^{2i_r}$ are non-zero in Ω_{4n}. This gives $p(n)$ elements in Ω_{4n} and $p(n)$ invariants to distinguish them, where $p(n)$ is the number of partitions of n. By ordering the set of partitions appropriately, the corresponding matrix can be show to be upper triangular with non-zero entries on the diagonal and thus non-singular. Hence the $\mathbb{CP}^{2i_1} \times \cdots \times \mathbb{CP}^{2i_r}$ represent linearly independent elements in Ω_{4n}, and so $\operatorname{rank}(\Omega_{4n}) \geq p(n)$.

Remarkably, Thom showed the following using the *Thom space* construction (see [15]):

Theorem (Thom)**.** The products $\mathbb{CP}^{2i_1} \times \cdots \times \mathbb{CP}^{2i_r}$, where $i_1, \cdots, i_r$ ranges over the partitions of n, are a basis for Ω_{4n}. In particular, $\operatorname{rank}(\Omega_{4n}) = p(n)$.

Example. (i) The torsion-free part of $\Omega_4 \cong \mathbb{Z}$ is generated by $\mathbb{CP}^2$.

(ii) The torsion-free part of $\Omega_8 \cong \mathbb{Z} \oplus \mathbb{Z}$ is generated by $\mathbb{CP}^2 \times \mathbb{CP}^2$ and $\mathbb{CP}^4$.

Furthermore, Terence Wall showed in 1960 that Ω_n is completely determined by the Pontryagin numbers and the Stiefel-Whitney numbers, which come from corresponding characteristic classes with $\mathbb{Z}_2$ coefficients. This in turn shows that the Ω_n were the direct sum of copies of $\mathbb{Z}_2$ and, if n is a multiple of 4, copies of $\mathbb{Z}$:

n	0	1	2	3	4	5	6	7	8	9	10	11
Ω_n	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	0	0	$\mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$

4.1 Hirzebruch's Signature Theorem

Here we will use $\text{rank}(\Omega_{4n}) = p(n)$ to deduce Hirzebruch's Signature Theorem. We start by introducing the notion of a multiplicative sequence.

Let R be a commutative ring with unity and let $A^* = (A^0, A^1, A^2, \cdots)$ be a commutative graded R-algebra with unity. Also let A^π be the formal sums $a_0 + a_1 + \cdots$ with $a_i \in A^i$ and let A_1^π be the subgroup consisting of those sums with $a_0 = 1$. For example, we can take $R = \mathbb{Q}$ and $A^n = H^{4n}(B; \mathbb{Q})$. Now consider a sequence of polynomials over R:

$$K_1(x_1), K_2(x_1, x_2), K_3(x_1, x_2, x_3), \cdots$$

where we let $\deg(x_i) = i$, and with the property that each $K_n(x_1, \cdots, x_n)$ is homogeneous of degree n. Given $a = 1 + a_1 + a_2 + \cdots \in A_1^\pi$, define $K(a) \in A_1^\pi$ by:

$$K(a) = 1 + K_1(a_1) + K_2(a_1, a_2) + \cdots$$

Definition. The K_n form a *multiplicative sequence* of polynomials if $K(ab) = K(a)K(b)$ for all R-algebras A^* and for all $a, b \in A_1^\pi$.

Example. Some examples of multiplicative sequences for general rings R are as follows.

(i) Given $\lambda \in \Lambda$, let $K_n(x_1, \cdots, x_n) = \lambda^n x_n$. This is a multiplicative sequence with

$$K(1 + a_1 + a_2 + \cdots) = 1 + \lambda a_1 + \lambda^2 a_2 + \cdots$$

i.e. K simply replaces a_i with $\lambda^i a_i$.

(ii) Insisting that $K(a) = a^{-1}$ defines a multiplicative sequence:

$$-x_1, x_1^2 - x_2, -x_1^3 + 2x_1x_2 - x_3, \cdots$$

which, in general, takes the form

$$K_n(x_1, \cdots, x_n) = \sum_{i_1 + 2i_2 + \cdots + ni_n = n} \frac{(i_1 + \cdots i_n)!}{i_1! \cdots i_n!} (-x_1)^{i_1} \cdots (-x_n)^{i_n}.$$

Now consider any $t \in A^*$ of degree 1. Then we have

$$K(1 + t) = 1 + K_1(t) + K_2(t, 0) + \cdots = 1 + \lambda_1 t + \lambda_2 t^2 + \cdots$$

where λ_n is the coefficient of x_1^n in $K_n(x_1, \cdots, x_n)$. The following lemma says that this relation is enough to determine the entire multiplicative sequence..

Lemma (Hirzebruch)**.** Given a formal power series $f(t) = 1 + \lambda_1 t + \lambda_2 t^2 + \cdots$ with $\lambda_i \in R$, there is a unique multiplicative sequence $\{K_n\}$ with coefficients in Λ such that

$$K(1 + t) = f(t).$$

We say $\{K_n\}$ *belongs* to $f(t)$.

Proof: For uniqueness, let $A^* = \Lambda[t_1, \cdots, t_n]$ where $\deg(t_i) = 1$ and let

$$\sigma = (1 + t_1) \cdots (1 + t_n) \in A_1^\pi.$$

Since K is multiplicative, we have

$$K(\sigma) = K(1 + t_1) \cdots K(1 + t_n) = f(t_1) \cdots f(t_n).$$

Now $\sigma = 1 + \sigma_1 + \cdots \sigma_n$, where σ_i are the elementary symmetric polynomials, so $K_n(\sigma_1, \cdots, \sigma_n)$ is determined by f. Hence f determines K_n since it is a basic result from Galois Theory that $\sigma_1, \cdots, \sigma_n$ are algebraically independent.

For existence, let $I = i_1, \cdots, i_r$ be a partition of n and let

$$s_I(\sigma_1, \cdots, \sigma_n) = \sum t_1^{i_1} \cdots t_r^{i_r},$$

where the sum over all the distinct monomials that are produced by permuting the t_is. It can be shown that the s_I form an additive basis for the set of symmetric polynomials. Now let

$$K_n(\sigma_1, \cdots, \sigma_n) = \sum_I \lambda_I s_I(\sigma_1, \cdots, \sigma_n),$$

which gives a well-defined sequence since the σ_i are algebraically independent, as above. To show the K_n are multiplicative, we can easily check that

$$s_I(ab) = \sum_{HJ=I} s_H(a) s_J(b),$$

where HJ denotes adjoining of partitions. Hence

$$K(ab) = \sum_I \lambda_I s_I(ab) = \sum_I \sum_{HJ=I} s_H(a) s_J(b) = \sum_{H,J} \lambda_H s_H(a) \lambda_J s_J(b) = K(a)K(b).$$

□

Example. Applying the above lemma to our examples from before, we get:

(i) $(\lambda x_1, \lambda x_2, \lambda x_3, \cdots) \leftrightarrow 1 + \lambda t$, since $K(1 + a_1) = 1 + \lambda a_1$.

(ii) $(-x_1, x_1^2 - x_2, \cdots) \leftrightarrow 1 - t + t^2 - \cdots$, since $K(1 + a_1) = (1 + a_1)^{-1}$.

Consider now a multiplicative sequence K with rational coefficients and a closed $4n$-manifold M.

Definition. The K-*genus* of a $4n$-manifold M is given by:

$$K[M] = K_n(p_1, \cdots, p_n)[M],$$

where $p_i = p_i(TM) \in H^{4i}(M; \mathbb{Q})$ is the Pontryagin class and $[M]$ is the fundamental class. If 4 doesn't divide the dimension of M, we define $K[M] = 0$.

In turns out that the K-genus is a cobordism invariant which respects the disjoint union and cartesian product of manifolds. This is summarised in the following lemma.

Lemma. If K_n is a multiplicative sequence with rational coeffients, then $[M] \mapsto K[M]$ defines a ring homomorphism $\Omega_* \to \mathbb{Q}$ and hence an algebra homomorphism $\Omega_* \otimes \mathbb{Q} \to \mathbb{Q}$.

Proof: We previous had that $p_I[M + N] = p_I[M] + p_I[N]$ and $p_I[\partial X] = 0$ and so we need only show that $K[M \times N] = K[M]K[N]$. Indeed note that

$$p(T(M \times N)) = p(TM \oplus TN) \equiv p(TM)p(TN) \quad \text{modulo 2-torsion},$$

so are equal with $\mathbb{Q}$ coefficients. Since K is multiplicative, we have

$$K(p(T(M \times N))) = K(p(TM))K(p(TN)).$$

Comparing top dimenional terms and noting that $[M \times N] = [M][N]$ then gives that $K[M \times N] = K[M]K[N]$, as required. □

We note that the signature of a manifold satisfies the same properties.

Lemma. The map $[M] \mapsto \sigma(M)$ defines a ring homomorphism $\sigma : \Omega_* \to \mathbb{Z}$ and hence an algebra homomorphism $\Omega_* \otimes \mathbb{Q} \to \mathbb{Q}$.

Proving this amounts the showing the following for manifolds M and N.

1. $\sigma(M + N) = \sigma(M) + \sigma(N)$.
2. $\sigma(M \times N) = \sigma(M)\sigma(N)$.
3. If M bounds, then $\sigma(M) = 0$.

The first part is immediate from noting that the matrix for $M + N$ is block diagonal consisting of M and N. The second part can be proven using the Künneth isomorphism and the final part using Poincaré duality. Whilst illuminating, we will omit the arguments for brevity and direct the reader to [17].

Our goal is now to find a multiplicative sequence L such that $\sigma(M) = L[M]$ for any $4n$-manifold M or equivalently to find the corresponding power series $f(t)$. Once we find such an f, showing that it works is an exercise in finding the nth term in a Taylor Series[11]. Here however we take special care to show how such an f can be derived without any previous knowledge.

First note that, since $M \mapsto \sigma(M)$ and any $M \mapsto L[M]$ both give rise to algebra homomorphisms $\Omega_* \otimes \mathbb{Q} \to \mathbb{Q}$, we need only check equality on a set of generators for $\Omega_* \otimes \mathbb{Q}$, i.e. on the complex projective spaces $\mathbb{CP}^{2k}$ for $1 \leq k \leq n$.

Using $a \in e(\gamma^n_{\mathbb{C}}) \in H^2(\mathbb{CP}^{2n})$ as a generator gives that $\sigma(\mathbb{CP}^{2k}) = 1$ since the corresponding matrix sends $1 \mapsto a^{2n}[\mathbb{CP}^{2n}] = 1$, by earlier calculation. So we require

$$L_k(p_1, \cdots, p_k)[\mathbb{CP}^{2k}] = L[\mathbb{CP}^{2k}] = 1$$

for all $k \geq 1$, where $p_i = p_i(T\mathbb{CP}^{2k})$. Since $p(T\mathbb{CP}^{2k}) = (1 + a^2)^{2k+1}$ we know that if $f(t) = L(1 + t)$, then

$$(f(a^2))^{2k+1} = L(1+a^2)^{2k+1} = L(1+p_1+p_2+\cdots) = 1+L_1(p_1)+\cdots+L_k(p_1,\cdots,p_k)+\cdots$$

and so $L_k(p_1, \cdots, p_k) = \lambda a^{2k}$ where λ is the coefficient of t^{2k} in the Taylor expansion of $(f(t^2))^{2k+1}$. Hence the above gives that $\lambda a^{2k}[\mathbb{CP}^{2k}] = 1$ and so $\lambda = 1$ by earlier calculations. Hence we need a formal power series $f(t)$ such that

$$(f(t^2))^{2k+1} = a_0 + a_2 t^2 + \cdots + t^{2k} + \cdots$$

for all $k \geq 1$. To find such a power series, we appeal to a classical formula which relates the Taylor series of a power of a function f to the Taylor series of the inverse of $x/f(x)$:

Theorem (Lagrange-Bürmann formula)**.** Let f be a formal power series with non-zero constant term and $\varphi(t) = t/f(t)$. If $f(t)^n = a_o + a_1 t + \cdots$, then

$$\varphi^{-1}(t) = b_0 + b_1 t + \cdots + \frac{a_{n-1}}{n} t^n + \cdots$$

The proof is an exercise in computing residues and so is omited for brevity. Given our f above, the corresponding φ must be an odd function and consist of only odd terms.

[11] This can be achieved by computing residues and such a treatment can be found in [15].

Furthermore, the coefficient of t^{2k+1} must be $\frac{1}{2n+1}$, so formal integration gives

$$\varphi^{-1}(t) = t + \frac{1}{3}t^3 + \frac{1}{5}t^5 + \cdots = \int (1 + t^2 + t^4 + \cdots)\,\mathrm{d}t = \int \frac{1}{1-t^2}\,\mathrm{d}t$$
$$= \frac{1}{2}\int \left(\frac{1}{1+t} + \frac{1}{1-t}\right)\mathrm{d}t = \frac{1}{2}\log\left(\frac{1+t}{1-t}\right)$$

and finally

$$\frac{1}{2}\log\left(\frac{1+\varphi(t)}{1-\varphi(t)}\right) = t \Rightarrow \frac{1+\varphi(t)}{1-\varphi(t)} = e^{2t} \Rightarrow \varphi(t) = \frac{e^{2t}-1}{e^{2t}+1} = \tanh(t).$$

Hence, by the formula above, we get $f(t^2) = t/\tanh(t)$ and so $f(t) = \sqrt{t}/\tanh\sqrt{t}$. Hence, by using general formulae to find the nth term of this series, we have shown:

Theorem (Hirzebruch's Signature theorem)**.** If $\{L_n\}$ be the multiplicative sequence of polynomials belonging to the power series

$$\frac{\sqrt{t}}{\tanh\sqrt{t}} = 1 + \frac{1}{3}t - \frac{1}{45}t^2 + \frac{2}{945}t^3 - \cdots + \frac{2^{2n}B_{2n}}{(2n)!}t^n + \cdots$$

where the B_n are the Bernoulli numbers, then the signature $\sigma(M)$ of any $4n$-manifold M is equal to the L-genus $L[M]$.

This in particular shows that the L-genus is an integer-valued homotopy invariant.

Example. Note that each term of the sequence gives a constraint on the Pontryagin numbers $p_i = p_i(TM)$. To compute these polynomials, we will use the formula

$$L_n(\sigma_1, \cdots, \sigma_n) = \textstyle\sum_I \lambda_I s_I(\sigma_1, \cdots, \sigma_n)$$

from the proof that multiplicative sequences correspond to formal power series, where $\lambda_1 = 1/3$, $\lambda_2 = -1/45$, $\lambda_3 = 2/945$, $\cdots$ are the coefficients in the Taylor series above.

(i) That $L_1 = \lambda_1 p_1 = \frac{1}{3}p_1$ follows immediately.

(ii) The partitions of 2 are $(1,1)$, (2), and $\sigma_1 = p_1 + p_2$, $\sigma_2 = p_1p_2$, so

$$L_2(\sigma_1, \sigma_2) = \lambda_1^2 p_1p_2 + \lambda_2(p_1^2 + p_2^2) = \frac{1}{9}\sigma_2 - \frac{1}{45}(\sigma_1^2 - 2\sigma_2) = \frac{1}{45}(7\sigma_2 - \sigma_1^2)$$

Hence $L_2 = \frac{1}{45}(7p_2 - p_1^2)$.

(iii) We have $\sigma_1 = p_1 + p_2 + p_3$, $\sigma_2 = p_1p_2 + p_2p_3 + p_3p_1$ and $\sigma_3 = p_1p_2p_3$, so

$$L_3(\sigma_1, \sigma_2, \sigma_3) = \lambda_1^3 p_1p_2p_3 + \lambda_1\lambda_2 \sum_i (p_ip_{i+1}^2 + p_i^2p_{i+1}) + \lambda_3(p_1^3 + p_2^3 + p_3^3)$$
$$= \frac{1}{27}\sigma_3 - \frac{1}{135}(\sigma_1\sigma_2 - 3\sigma_3) + \frac{2}{945}(\sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3).$$

Hence $L_3 = \frac{1}{945}(62p_3 - 13p_1p_2 + 2p_1^3)$.

(iv) Similarly we can compute that $L_4 = \frac{1}{14175}(381p_4 - 71p_1p_3 - 19p_2^2 + 22p_1^2p_2 - 3p_1^4)$.

As above, computing L_n amounts to finding expressions for the each s_I in terms of the σ_i. The coefficient of p_1^n in each L_n is $\lambda_n = 2^{2n}B_{2n}/(2n)!$ by construction, and using such formulae we can also get that the coefficient of p_n is

$$2^{2n}(2^{2n-1} - 1)B_{2n}/(2n)!$$

However, surprisingly little is own about the other coefficients. For example, it is not even known whether or not every monomial has non-zero coefficient.

4.2 h-Cobordism Theorem

We end this section by outlining an important result in high-dimensional topology, namely the h-cobordism theorem. This will be a key ingredient in the classification of exotic spheres, though for now our aim will be to show that this implies the Poincaré conjecture for $n \geq 5$. We end by showing how this will allow us to develop tools for proving manifolds are homeomorphic to spheres, which we use in the following sections. We start by refining the notion of cobordism that we considered previously.

Definition. An (oriented) cobordism X between closed n-manifolds M and N is a *h-cobordism* if the inclusion maps $M \hookrightarrow X$ and $N \hookrightarrow X$ are homotopy equivalences.
We say a h-cobordism is *simply-connected* if M, N and X are all simply-connected.

It is easy to see that h-cobordism between closed manifolds and simply-connected h-cobordism between simply-connected closed manifolds are both equivalence relations. In 1962, Stephen Smale proved[12] the following.

Theorem (h-Cobordism Theorem)**.** Let M and N be simply-connected closed n-manifolds and X a simply-connected h-cobordism between them. If $n \geq 5$, then X is diffeomorphic to $M \times [0, 1]$. In particular, M and N are diffeomorphic.

This is done by finding a Morse function on X and showing that this decomposes X in to a number of "handles" of different types. This handle decomposition can then be put into a canonical form where we can then attempt to "cancel" handles. A careful analysis using the *Whitney trick* then reveals that having sufficiently high dimensions is necessary to ensure that the resulting manifold remains simply-connected.

Hence to show two simply connected n-manifolds M and N for $n \geq 5$ are diffeomorphic, it suffices to show that they are (simply-connectedly) h-cobordant. Indeed, this led to many of the key questions in simply-connected topology being solved over the coming years[13]. For convenience (but not necessity), we record the following results which allow us to avoid working with maps $M, N \hookrightarrow X$

Theorem (Poincaré Duality for Cobordisms)**.** If X is a cobordism between n-manifolds M and N, then $H_i(X, M) \cong H^{n+1-i}(X, N)$.

This can be proven using handles, as in the proof on the theorem above.

Lemma. If X is a simply-connected cobordism between simply-connected manifolds M and N, then X is a h-cobordism iff $H_*(X, M) = 0$.

Proof: By the relative version of the Hurewicz theorem, we have that $\pi_*(X, M) = 0$. By noting that the relative homotopy groups fit into a long exact sequence (as do the homology groups), this gives that the inclusion $M \hookrightarrow X$ induces isomorphisms $\pi_*(M) \to \pi_*(X)$. The *Whitehead theorem* tells us that a map which induces isomorphisms on all homotopy groups must be a homotopy equivalence, and so $M \hookrightarrow X$ is a homotopy equivalence.

By the theorem above, we have that $H^*(X, N) = 0$ and so $H_*(X, N) = 0$ also. Hence $N \hookrightarrow X$ is also a homotopy equivalence. The converse follows easily by reversing the above argument. □

[12] Actually he originally proved this for $n \geq 7$, though the result was later expanded. For this, he was awarded the Fields Medal 1966.

[13] For example, Terence Wall completed a classification of $(n-1)$-connected $(2n-1)$-manifolds.

We can use this to give the above application.

Theorem. Let M be a simply-connected n-manifold with simply-connected boundary, for $n \geq 5$. Then M is diffeomorphic to D^n iff M has the homology of a point.

Proof: Suppose M has the homology of a point and let $D_0 \subseteq M \setminus \partial M$ be an n-disc embedded in a chart of M. Then $M \setminus \mathring{D}_0$ is still simply-connected since $\partial D_0 \cong S^{n-1}$ and $n > 2$, and so is a simply-connected cobordism between ∂M and ∂D_0. To show this is a h-cobordism, we need that $H_*(M \setminus \mathring{D}_0, \partial D_0) = 0$ by the above.

Since M and D_0 both have the homology of a point, it follows that $H_*(M, D_0) = 0$. To apply excision we have $\partial D_0 \simeq D_0' \setminus \mathring{D}_0$ for a larger disc $D_0 \subseteq D_0'$. Hence

$$H_*(M \setminus \mathring{D}_0, \partial D_0) = H_*(M \setminus \mathring{D}_0, D_0' \setminus \mathring{D}_0) = H_*(M, D_0') = 0.$$

By the h-cobordism theorem, $M \setminus \mathring{D}_0$ is diffeomorphic to $\partial D_0 \times [0, 1]$. There are many ways to show this can be extended to a diffeomorphisc between M and D^n. For example, note that M and D^n are both the composition of the cobordism from $\emptyset$ to ∂D_0 (given by D_0) and the cobordism from ∂D_0 to ∂M (given by $M \setminus \mathring{D}_0$). Since there is a unique smooth structure on this composition of two cobordisms, we must have that M is diffeomorphic to D. □

Hence we have shown that any n-manifold homeomorphic to D^n must be diffeomorphic to D^n for any $n \geq 5$. This is the statement that there are no exotic n-discs for $n \geq 5$.

Even better, this is then powerful enough to resolve all but a handful of cases of one of the longest standing theorems in topology.

Theorem (Poincaré Conjecture)**.** Let M be a closed simply-connected n-manifold with the homology of S^n. If $n \geq 5$, then M is homeomorphic to S^n.

Proof: Let $D \subseteq M$ be an n-disc embedded in a chart of M. Now, by Poincaré Duality (for manifolds-with-boundary) as well as using the excision argument from the proof above, we have

$$H_i(M \setminus \mathring{D}) \cong H^{n-i}(M \setminus \mathring{D}, \partial D) \cong H^{n-i}(M, D).$$

The long exact sequence on relative homology gives that $H^{n-i}(M, D) \cong H^{n-i}(M) \cong H^{n-i}(S^n)$ when $i \neq n$, and $H^0(M, D) \cong 0$. So $M \setminus \mathring{D}$ has the homology of a point and, by the result above, $M \setminus \mathring{D}$ is diffeomorphic to the n-disc. Hence M is a the union of two n-discs glued along a diffeomorphism of their boundaries, i.e. M is a twisted sphere, which we previously showed is homeomorphic to S^n. □

This is often stated in the following slightly weaker form.

Corollary. Let M be a closed n-manifold. If $n \geq 5$, then M is homotopic to S^n iff M is homeomorphic to S^n.

We will now give a few applications of this result. We start by giving an alternate proof that the spheres M_k constructed in the previous section are indeed homeomorphic to S^7.

Proposition. If E_{ij} is the sphere bundle $S^3 \hookrightarrow E_{ij} \to S^4$ constructed in the previous section, then E_{ij} is homeomorphic to S^7 iff $i + j = \pm 1$. In particular M_k is homeomorphic to S^7.

Proof: Firstly, the long exact sequence associated to the fibre bundle gives easily that $\pi_1(E_{ij}) = 0$. Secondly, the Gysin sequence for the corresponding 4-dimensional vector bundle then gives

$$0 \longrightarrow H^3(M_k) \longrightarrow H^0(S^4) \xrightarrow{\cdot \smile e(E_{ij})} H^4(S^4) \longrightarrow H^4(M_k) \longrightarrow 0$$

since $H^1(S^4) = H^3(S^4) = 0$, and also gives that $H^i(M_k) = H^i(S^7)$ for all $i \neq 0, 3, 4$. Since E_{ij} is simply-connected, the result above gives it is homeomorphic to S^7 iff $H^3(M_k) = H^4(M_k) = 0$ iff the middle map is an isomorphism. We previously calculated that $e(E_{ij}) = (i + j)\nu$, where $\nu \in H^4(S^4)$ is a generator, and so the middle map is an isomorphism iff $i + j = \pm 1$. □

Remark. This actually follows from the Whitehead theorem which we used earlier in this section. In particular, Hurewicz implies $\pi_7(M_k) = H_7(M_k) = \mathbb{Z}$ and the generator gives a map $S^7 \to M_k$ which we can show is an isomorphism on all homotopy groups.

We conclude this section by extracting the key ingredients of the above proof. In particular, consider a closed $(n - 2)$-connected $(2n - 1)$-manifold M. Then the Hurewicz theorem implies that $H_i(M) = 0$ for all $1 \leq i \leq n - 2$. Since all these homology groups are free, Poincaré duality then tells us this holds for $n + 1 \leq i \leq 2n - 2$ and also that $H_{2n-1}(M) = H_0(M) = \mathbb{Z}$. Hence we need only worry about the $(n-1)$th homology group in this case. Explicitly, we have:

Corollary. If M is a closed $(n-2)$-connected $(2n-1)$-manifold, then M is homeomorphic to S^{2n-1} iff $H_{n-1}(M) = 0$.

In the following two sections, we give two different constructions of $(n - 2)$-connected $(2n - 1)$-manifolds M which bound $(n - 1)$-connected manifolds B and which we claim are exotic spheres. To show they are homeomorphic to S^{2n-1}, consider the long exact sequence for the pair (B, M). Since M is $(2n - 1)$-connected, this reduces to

$$0 \longrightarrow H_n(M) \longrightarrow H_n(B) \xrightarrow{i_*} H_n(B, M) \longrightarrow H_{n-1}(M) \longrightarrow 0,$$

where we use Poincaré duality and the universal coefficients formula to deduce that $H_{n+1}(B, M) \cong H^{n-1}(B) = 0$. By the condition above, we know that M is homeomorphic to S^{2n-1} iff the middle map i_* is an isomorphism. We get the same situation for cohomology by dualising to f^*.

Now in the example of M_k above, we used that the bounding manifold B was the total space of a fibre bundle $\pi : B \to X$ to show that $i^* : H^n(B, M) \to H^n(B)$ induces a map

$$\begin{tikzcd} H^n(B, M) \arrow[r, "i^*"] & H^n(B) \\ H^{n-d}(X) \arrow[u, "\Phi", "\wr"'] \arrow[r, dashed] & H^n(X) \arrow[u, "\pi^*", "\wr"'] \end{tikzcd}$$

where d is the dimension of the bundle and Φ is the Thom isomorphism. We then identified the bottom map as cupping with the Euler class of the fibre bundle and thus were able to show that M was homeomorphic to asphere iff $e(M) = \pm 1$.

The examples we will consider in the following two sections will not be sphere bundles, but in both cases we alter i to give a linear map $\mathbb{Z}^m \to \mathbb{Z}^m$ for some m and note that this is an isomorphism if the corresponding matrix M is unimodular, i.e. $\det(M) = \pm 1$.

5 Plumbing

Here we give a brief overview of a different method of constructing exotic spheres, also introduced by Milnor[14]. We introduce an operation on disc bundles over manifolds known as *plumbing*, and show that this operation can be used to construct manifolds with a given intersection form. The basic construction is as follows.

5.1 Plumbing Disc Bundles

For $i = 1, 2$, consider disc bundles

$$D_i^n \hookrightarrow E_i \xrightarrow{\pi_i} N_i,$$

where the N_i are closed n-manifolds and the E_i are oriented compatibly with the N_i. Let $x_i \in N_i$ and pick an n-disc $x_i \in D_i' \subseteq N_i$ that is contained is a trivialising open set of the disc bundle. Then $\pi_i^{-1}(x_i) \subseteq D_i' \times D_i$ under the trivialisation around D_i', for $i = 1, 2$.

Now pick canonical diffeomorphisms $h_\pm : D_1' \to D_2$ and $k_\pm : D_1 \to D_2'$ such that h_+, k_+ and h_-, k_- are orientation-preserving and orientation-reversing respectively.

Definition (Plumbing)**.** The result of *plumbing together* E_1 *and* E_2 *with sign* ± 1 is

$$E_1 \square E_2 = E_1 \sqcup E_2 / \sim,$$

where $(x, y) \in D_1' \times D_1$ is identified with $(k_\pm(y), h_\pm(x)) \in D_2' \times D_2$, for some sign $\pm$.

More generally, if $m \geq 1$, the result of *plumbing together* m *points with sign* ± 1 is given by repeating the above for m distinct pairs of points. In particular, pick points $p_i^j \in D_i$ for $j = 1, \ldots, m$, disjoint discs $D_i'^j$ around each p_i^j, maps $h_\pm^j : D_1'^j \to D_2$, $k_\pm^j : D_1 \to D_2'^j$ and then identify $(x, y) \in D_1'^j \times D_1$ with $(k_\pm^j(y), h_\pm^j(x)) \in D_2'^j \times D_2$ for each $j = 1, \ldots, m$.

Example. We will consider the where case $n = 1$, $N_1 = N_2 = S^1$ and $E_1 = E_2 = S^1 \times [0, 1]$. Here the neighbourhoods D_i' correponds to arcs of S^1 and so the identification of the squares $[0, 1] \times D_i'$ corresponds to the diagram below. The picture below is in general a good way to picture plumbing disc bundles in higher dimensions.

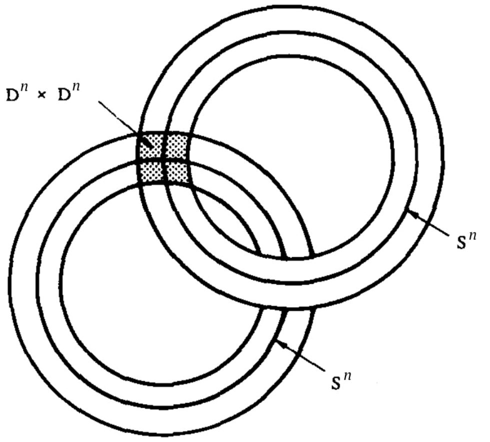


Figure 2: Two disc bundles E_1 and E_2 attached to form $E_1 \square E_2$.

[14] This construction goes back at least to [3], a set of particularly illuminating unpublished notes written by Milnor in 1959. A more recent exposition of this material can be found in [19].

This can be made into a smooth manifold by "straightening out the angles" using bump functions at the intersection points, and it can be given an orientation compatible with N_1 and also either N_2 if n is even or $-N_2$ if n is odd.

It can be shown that $E_1 \square E_2$ is an n-disc bundle over an appropriate n-manifold. Also observe the effect of plumbing on the boundary.

Proposition. If ∂E_1 and ∂E_2 are $(n-2)$-connected, then $\partial(E_1 \square E_2)$ is the union of two $(n-2)$-connected sets whose intersection is $(n-2)$-connected.

Proof: We start by noting that noting that

$$\partial(E_1 \square E_2) = (\partial E_1 \setminus D_1' \times \partial D_1) \cup (\partial E_2 \setminus D_2' \times \partial D_2)$$

which is clear from the picture above. Now $\partial E_i \setminus D_i' \times \partial D_i \simeq \partial E_i \setminus \partial D_i$ and, since $S^{n-1} \cong \partial D_i \hookrightarrow \partial E_i$ is a codimension n embedding, we know that

$$\pi_j(\partial E_i \setminus D_i' \times \partial D_i) \to \pi_j(\partial E_i) = 0$$

is an isomorphism for $j \leq n-2$, and so $(\partial E_i \setminus D_i' \times \partial D_i)$ is $(n-2)$-connected.

To see that intersection is $(n-2)$-connected since, the picture above gives that

$$(\partial E_1 \setminus D_1' \times \partial D_1) \cap (\partial E_2 \setminus D_2' \times \partial D_2) = \partial D_1' \times \partial D_1 \cong S^{n-1} \times S^{n-1}$$

and we have that $\pi_j(S^{n-1} \times S^{n-1}) \cong \pi_j(S^{n-1}) \times \pi_j(S^{n-1}) = 0$ for $j \leq n-2$. $\square$

Now note that we can consider N_i as lying in $E_1 \square E_2$ by mapping into E_i along the zero section followed by inclusion. If E_1 and E_2 are plumbed together at m points $p_1, \cdots, p_n$, then N_1 and N_2 have intersection $N_1 \cap N_2 = \{p_1, \cdots, p_n\}$. Since $E_1 \square E_2$ has dimension $2n$, this is transverse with $\operatorname{sgn}(p_i) = \pm 1$ for each i, depending on the sign of the plumbing.

Now let $W = E_1 \square E_2$ and consider the intersection product

$$H_n(W) \times H_n(W) \to H_0(W) \cong \mathbb{Z}, \quad (x, y) \mapsto x \cdot y = D_W(D_W^{-1}(x) \smile D_W^{-1}(y))$$

where $D_W : H^n(W, \partial W) \to H_n(W)$ is the Poincaré duality map. To compute the intersection product of N_1 and N_2, let $i_1 : N_1 \hookrightarrow W$ and $i_2 : N_2 \hookrightarrow W$ be as above and note that push forward the fundamental classes $[N_1]$ and $[N_2]$ givest elements in $H_n(W)$. The following standard result then gives this in terms of the signs of the intersection points.

Lemma.
$$(i_1)_*[N_1] \cdot (i_2)_*[N_2] = \sum_{x \in N_1 \cap N_2} \operatorname{sgn}(x).$$

Since $E_1 \square E_2$ is still a disc bundle of the same form, we can carry out this process for k disc bundles $E_i \to N_i$ for $i = 1, \cdots, k$. In particular, for any $m_{ij} \in \mathbb{Z}$ for $i \neq j$, we can build a $2n$-manifold W with submanifolds $N_1, \cdots, N_m$ each with intersection number $(i_1)_*[N_1] \cdot (i_2)_*[N_2] = n_{ij}$, i.e. plumb together $|m_{ij}|$ points with sign $\operatorname{sgn} m_{ij}$. The case when $i = j$ is described the the following result.

Lemma.
$$(i_1)_*[N] \cdot (i_2)_*[N] = e(\mathcal{V}_{N \subseteq W})[N].$$

This is proven by considering a tubular neighbourhood of N in W and using the Hopf index theorem. The proofs of these lemmas can be found in [19]. We can now use this to find manifolds which attain any intersection form possible.

Theorem. Let M be an $k \times k$ symmetric matrix with integer entries and even entries on the diagonal. The for every $n \geq 2$, there is a $4n$-manifold W with boundary such that

(i) The matrix corresponding to the intersection form $H_{2n}(W) \times H_{2n}(W) \to \mathbb{Z}$ is M.

(ii) W is $(2n-1)$-connected, ∂W is $(2n-2)$-connected and $H_{2n}(W)$ is free abelian.

Proof: Let $M = (m_{ij})$, where $m_{ij} = m_{ji}$ and $m_{ii} = 2\lambda_i$. Consider spheres $S_1^{2n}, \cdots, S_k^{2n}$ their tangent bundles

$$\mathbb{R}^{2n} \hookrightarrow TS_i^{2n} \to S_i^{2n}$$

which have associated disc and sphere bundles $\mathbb{D}(TS_i^{2n})$ and $\mathbb{S}(TS_i^{2n})$ respectively. Since $\mathbb{S}(TS_i^{2n})$ can be build from the clutching construction, it must correspond to an element in $\pi_{2n-1}(SO(2n))$ from our earlier work. Let $\lambda_i\mathbb{S}(TS_i^{2n})$ denote the sphere bundle corresponding to λ_i times this element in $\pi_{2n-1}(SO(2n))$, and let $E_i = \lambda_i\mathbb{D}(TS_i^{2n})$ be the corresponding disc bundle. Now note that, by the relation between the Euler class and Euler characteristic given earlier, we have

$$e(TS_i^{2n})[S_i^{2n}] = \chi(S_i^{2n}) = 1 + (-1)^{2n} = 2$$

and this can be extended to show that $e(\mathcal{V}_{S_i^{2n} \subseteq E_i})[S_i^{2n}] = e(\lambda_i TS_i^{2n})[S_i^{2n}] = 2\lambda_i$.

Now let $U = E_1 \square \cdots \square E_k$, where each E_i and E_j are plumbed together at $|m_{ij}|$ points with sign $\operatorname{sgn} m_{ij}$, when $i \neq j$. This gives the right matrix off the diagonal and on the diagonal gives $2\lambda_i = m_{ii}$ by our calculation above. To show there are no other entries inthe matrix, i.e. that $H_{2n}(U)$ has basis $(i_j)_*[S_j^{2n}]$ where $i_j : S_j^{2n} \hookrightarrow U$, it suffices to note that U deformation retracts onto $\bigcup_i S_i^{2n}/\sim$ where $\sim$ attaches points on the spheres in such a way that $S_i^{2n} \cap S_j^{2n} = |m_{ij}|$.

By our earlier remark about the effect of Plumbing on the boundary of the manifold, using that the S_i^{2n} are $(2n-1)$-connected, we have that ∂U is the union of k different$(2n-2)$-connected sets with a number of $(2n-2)$-connected intersections. Mayer-Vietoris then gives that $H_i(X) = H_i(\partial X) = 0$ for all $2 \leq i \leq 2n-2$ for each connected component of U, where we note that $n \geq 2$.

It can then be shown, by a series of attachments on the boundary, that U can be extended to a manifold W which is simply connected and has $H_i(W) = H_i(U)$ for all i and $H_i(\partial W) = H_i(\partial U)$ for all $i \leq 2n-2$. The still has intersection matrix M and, by the Hurewicz theorem, W is $(2n-1)$-connected, has $(2n-2)$-connected boundary and has free $2n$-th homology. □

Since ∂W is a $(2n-2)$-connected $(4n-1)$-manifold and bounds a $(2n-1)$-connected $4n$-manifold W, we follow the approach outlined in the previous section to characterise when is homeomorphic to S^{4n-1}. In particular, we have that the long exact sequence for the pair $(W, \partial W)$ gives

$$0 \longrightarrow H_{2n}(\partial W) \longrightarrow H_{2n}(W) \xrightarrow{i_*} H_{2n}(W, \partial W) \xrightarrow{\partial} H_{2n-1}(\partial W) \longrightarrow 0,$$

where $i : \partial W \hookrightarrow W$ is inclusion. Now note that Poincaré duality gives an isomorphism $D_W^{-1} : H_{2n}(W, \partial W) \to H^{2n}(W)$ and the universal coefficients formula gives an isomorphism $H^{2n}(W) \to \operatorname{Hom}(H_{2n}(W), \mathbb{Z})$.

We can thus characterise whether or not this map is an isomorphism as follows.

Theorem. If W is as above, then ∂W is homeomorphic to S^{4n-1} iff $\det(M) = \pm 1$.

Proof: Consider the map $\alpha : H_{2n}(W) \to \mathrm{Hom}(H_{2n}(W), \mathbb{Z})$ defined by the diagram below.

$$\begin{tikzcd}
H_{2n}(W) \arrow[r, "i_*"] \arrow[ddr, dashed, "\alpha"] & H_{2n}(W, \partial W) \arrow[d, "D_W^{-1}", "\wr"'] \\
& H^{2n}(W) \arrow[d, "\mathrm{UCT}", "\wr"'] \\
& \mathrm{Hom}(H_{2n}(W), \mathbb{Z})
\end{tikzcd}$$

Now α sends $x \in H_{2n}(W)$ to $D_W^{-1}(x) \in H^{2n}(W)$ and then to the map $y \mapsto D_W(D_W^{-1}(x) \smile D_W^{-1}(y))$ in $\mathrm{Hom}(H_{2n}(W), \mathbb{Z})$, which can be checked by running through the proof of the universal coefficients theorem and showing this definition works. So α corresponds to the intersection product on $H_{2n}(W)$ and so is represented by the matrix $M : \mathbb{Z}^{2n} \to \mathbb{Z}^{2n}$. Hence ∂W is homeomorphic to S^{4n-1} iff i_* an isomorphism iff M is unimodular, i.e. $\det(M) = \pm 1$. $\square$

5.2 Examples of Exotic Spheres

Our aim now is to find unimodular matrices M of the above form. We would then know that the corresponding smooth $(4n-1)$-manifold ∂W would be homotopic to S^{4n-1}. Since this comes with a bounding manifold W, we would then have some hope of showing that ∂W is an exotic sphere using the characteristic class invariants established earlier.

Consider the following 8×8 matrix M (left) which is of the required form. By successive row and column operations, we get the diagonal matrix D (right).

$$\begin{pmatrix}
2 & 1 & & & & & & \\
1 & 2 & 1 & & & & & \\
& 1 & 2 & 1 & & & & \\
& & 1 & 2 & 1 & & & \\
& & & 1 & 2 & 1 & 0 & 1 \\
& & & & 1 & 2 & 1 & 0 \\
& & & & 0 & 1 & 2 & 0 \\
& & & & 1 & 0 & 0 & 2
\end{pmatrix}
\to
\begin{pmatrix}
2 & 0 & & & & & & \\
0 & \frac{3}{2} & 0 & & & & & \\
& 0 & \frac{4}{3} & 0 & & & & \\
& & 0 & \frac{5}{4} & 0 & & & \\
& & & 0 & \frac{6}{5} & 1 & 0 & 1 \\
& & & & 1 & 2 & 1 & 0 \\
& & & & 0 & 1 & 2 & 0 \\
& & & & 1 & 0 & 0 & 2
\end{pmatrix}
\to
\begin{pmatrix}
2 & & & & & & & \\
& \frac{3}{2} & & & & & & \\
& & \frac{4}{3} & & & & & \\
& & & \frac{5}{4} & & & & \\
& & & & \frac{7}{10} & & & \\
& & & & & \frac{4}{7} & & \\
& & & & & & \frac{1}{4} & \\
& & & & & & & 2
\end{pmatrix}$$

This shows that M has signature 8 and determinant 1. Hence, for any $n > 1$, we get a corresponding $4n$-manifold W with boundary ∂W homeomorphic to S^{4n-1}.
In the notation of the section on higher dimensions at the end of the third chapter, we now want to compute

$$\lambda(\partial W) = \frac{1}{s_n}(\sigma(W) - L_n(i^{-1}p_1, \cdots, i^{-1}p_{n-1}, 0)[W]) \in \mathbb{Q}/\mathbb{Z},$$

where s_n is the coefficent of p_n in $L_n(p_1, \cdots, p_n)$. The important thing here is that, since W is $(2n-1)$-connected, we have that $H^i(W, \partial W) \cong H_{4n-i}(W) = 0$ for $i \neq 0, 2n, 4n$. In particular

$$i^{-1}p_i \in H^{4i}(W, \partial W) = 0$$

when $4i \neq 0, 2n, 4n$ and so all Pontryagin classes in the expression above, except possibly $i^{-1}p_{n/2}$ in the case where n is even, are zero. In fact $i^{-1}p_{n/2} = 0$ by noting that the

Pontyagin classes of TS_i^{2n} are all zero, by previous calculations, and we can check that the result of Plumbing then restricting to the boundary doesn't change this fact. Combining with the fact the signature is 8 from the matrix above, we then have that

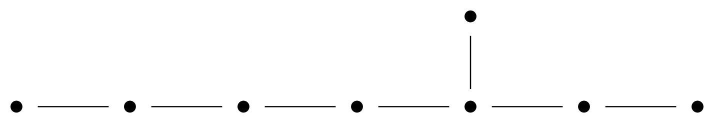
$$\lambda(\partial W) = 8/s_n \in \mathbb{Q}/\mathbb{Z}.$$

Finally, this can be computed by the formula

$$s_n = 2^{2n}(2^{2n-1} - 1)B_{2n}/(2n)!$$

which we stating in this previous section. The denominator of $\lambda(\partial W^{4n-1})$ is then 7, 31, 127 and 73 when $n = 2$, 3, 4 and 5 respectively. It turns out that the denominator is always > 1 and so ∂W is always an exotic sphere for any $n \geq 2$. We call this the *Milnor sphere* and denote it by Σ_M. In particular, this shows that there are exotic spheres of arbitarirly large dimension of the form S^{4n-1}.

An interesting observation can be made by representing the construction of W by a graph, namely by letting the vertices be the spheres $S_1^{2n}, \cdots, S_k^{2n}$ and drawing $|m|$ appropriately signed edges between S_i^{2n} and S_j^{2n} whenever they are plumbed together at $|m|$ points. In this case all attachments are made by single points with sign 1 and give the graph



which can be identified as the dynkin diagram of the Lie group E_8. These links can be explained by considering the theory of symmetric matrices with integer entries and even entries on the diagonal.

These constructions can be taken much further and can in fact be used to construct exotic spheres of the form S^{4n+1} also. In particular the *Kervaire sphere*, which we write as Σ_K, can be constructed by plumbing together two copies of $\mathbb{D}(TS^{2n+1})$ in such a way that the associated intersection form is

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

which has determinant -1. This gives a $(4n+2)$-manifold K and ∂K is indeed an exotic sphere for some values of n. In fact, determining which n have ∂K^{4n+2} an exotic sphere has turned out to be closely related to many other problems in algebraic topology. This is known as the *Kervaire invariant problem* and very recent breakthroughs have shown that it is true precisely when $n = 2$, 6, 14, 30, 62 and possibly 162 (the only unknown case). This gives us exotic spheres in dimensions 9, 25, 57, 121, 249 and possibly 649.

An interesting application of this comes from resticting our attention to the 9-dimensional Kervaire sphere Σ_K. Let W be the 10-manifold bounding Σ_K in the above construction. If $f : \Sigma_K \to S^9$ is a homeomorphism, then let

$$\overline{W} = W \cup_f D^{10}$$

be the topological 10-manifold form by gluing W and D^{10} along f. Amazingly, the fact that Σ_K is an exotic sphere can be used to show that this topological manifold does not admit any smooth structure. For a proof, the reader is directed to [5].

We will return to both the Milnor spheres and Kervaire spheres in the following chapters.

6 Brieskorn Varieties

Here we discuss a construction of Exotic Spheres which can be build out of taking a complex hypersurface in $\mathbb{C}^{n+1}$ and intersecting it with a very small $(2n+1)$-sphere around a particular point in V.

Explicitly, let $f(z_1, \cdots, z_{n+1})$ be a polynomial in complex variables and let

$$V = V(f) = f^{-1}(0)$$

be the zero set of f, equipped with the subspace topology from $\mathbb{C}^{n+1}$. Pick $z_0 \in V$, $\varepsilon > 0$ and define the *link at* z_0 to be

$$K = V \cap S_\varepsilon,$$

where $S_\varepsilon = \{z \in \mathbb{C}^{n+1} : \|z - z_0\| = \varepsilon\}$ is the small sphere of radius ε around z_0.

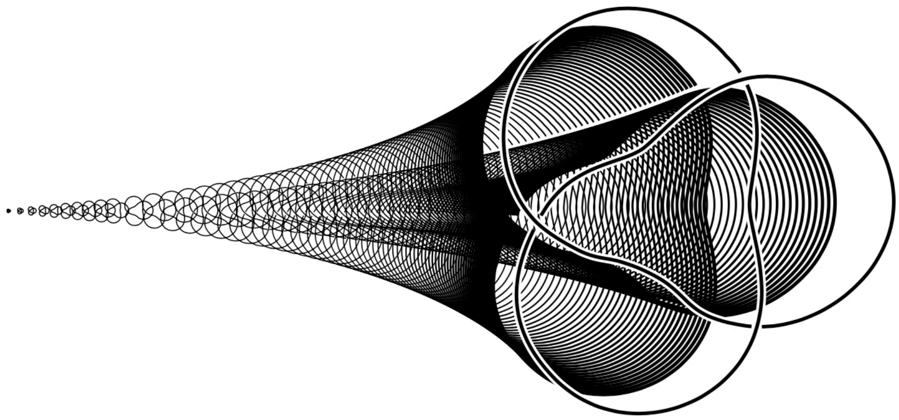


Figure 3: An illustration of the link around a singularity K.

From now on, we will assume $\deg f > 1$. This ensures we can find i for which $\partial f/\partial z_i$ is non-constant. Hence any critical point of f is a root of $\partial f/\partial z_i$ which shows that the set $\Sigma(V)$ of critical points is finite.

Proposition. K is an $(2n-1)$-manifold for all $\varepsilon > 0$ sufficiently small.

Proof: Since $\Sigma(V)$ is finite, we know that $\mathbb{C}^{n+1} \setminus \Sigma(V)$ is an open $(2n+2)$-manifold. Now $0 \in \mathbb{C}$ is a regular value for

$$f|_{\mathbb{C}^{n+1}\setminus\Sigma(V)} : \mathbb{C}^{n+1} \setminus \Sigma(V) \to \mathbb{C}$$

and so $V \setminus \Sigma(V)$ is a smooth $2n$-manifold, by the preimage theorem. Now consider the radius map

$$r : V \setminus \Sigma(V) \to \mathbb{R}, \quad z \mapsto \|z - z_0\|^2 .$$

Since r is a polynomial, it has finitely many critical points (as above) and so finitely many critical values. Hence we can find $\varepsilon^2 > 0$ smaller than all critical values in $\mathbb{R}$ (except possibly 0). The preimage theorem then gives that

$$r^{-1}(\varepsilon^2) = (V \setminus \Sigma(V)) \cap S_\varepsilon$$

is a smooth $(2n-1)$-manifold. Since the critical points of f are isolated, we can pick ε sufficently small so that $S_\varepsilon \cap \Sigma(V) = \emptyset$, even if z_0 is itself a critical point. $\square$

Remark. This also works in the real case and the case of more general algebraic varieties.

Inclusion $i : K \hookrightarrow S_\varepsilon$ in then a codimension 2 embedding. If K was homeomorphic to S^{2n-1}, then this would be a higher dimensional knot. As we shall soon see, this K often has an exotic differentiable structure and invariants that distinguish differentiable structure are closely related to invariants for knots.

We start with the less interesting case when $z_0 \in V$ is a regular point.

Example. If $z_0 \in V$ is a regular point of f, then $z_0 \in V \setminus \Sigma(V)$ and so $r : z \mapsto \|z - z_0\|^2$ now has a critical point at z_0. In fact, z_0 is non-degenerate since $\left.\frac{\partial^2 f}{\partial z_i \partial z_j}\right|_p = 2\delta_{ij}$.
Since $\Sigma(V)$ is finite and does not contain z_0, we can pick ε sufficiently small so that $B_\varepsilon \cap \Sigma(V) = \emptyset$, i.e. so that $r : V \cap B_\varepsilon \to \mathbb{R}$. Now r is a Morse function mapping from a compact $2n$-manifold so, by Morse's lemma and the fact that $r(z) \geq 0$, we can find local real coordinates $u_1, \cdots, u_{2n}$ for $V \cap B_\varepsilon$ near z_0 such that

$$r(u_1, \cdots, u_{2n}) = u_1^2 + \cdots + u_{2n}^2.$$

Rechoosing ε so that $V \cap B_\varepsilon$ corresponds to this neighbourhood, we now have a diffeomorphism

$$K = V \cap S_\varepsilon \to \{(u_1, \cdots, u_{2n}) \in \mathbb{R}^{2n} : u_1^2 + \cdots + u_{2n}^2 = \varepsilon^2\} = S^{2n-1}.$$

As mentioned before, $K \hookrightarrow S_\varepsilon$ is an embedding $S^{2n-1} \hookrightarrow S^{2n+1}$ and so is a higher dimensional knot. However, K is not knotted. Indeed, the above argument can be extended so that $r : \mathbb{C}^{n+1} \cap B_\varepsilon \to \mathbb{R}$ is given local real coordinates $u_1, \cdots, u_{2n+2}$ and so that $V \cap B_\varepsilon$ is retrieved by the slice coodinates $u_{2n+1} = u_{2n+2} = 0$.

So if z_0 is a regular point, the corresponding K is both diffeomorphic to S^{2n-1} and is unknotted when embedded into S^{2n+1}. In search of more interesting examples, we now consider two cases where z_0 is a critical point.

Example. Let $f(z_1, z_2) = z_1^2 + z_2^3$, which has a single critical point at the origin. Since K is a smooth 1-manifold, we know that K is diffeomorphic to S^1. However, by finding K explicitly, it is possible to show that $K \hookrightarrow S^3$ is knotted.

Pick $\varepsilon > 0$ and let $(z_1, z_2) \in K = V(f) \cap S_\varepsilon$ so that $z_1^2 = -z_2^3$ and $|z_1|^2 + |z_2|^2 = \varepsilon^2$. Combining gives that

$$|z_2|^3 + |z_2|^2 = \varepsilon^2.$$

Since the left side is strictly increasing, there is a unique $\eta > 0$ such that $|z_2| = \eta$ (by, for example, the intermediate value theorem) and so also a unique $\xi > 0$ such that $|z_1| = \xi$. Hence (z_1, z_2) lies in a torus

$$T = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| = \xi, |z_2| = \eta\}.$$

If $z_1 = \xi e^{3i\theta}$, then $z_2 = \eta e^{ki\pi/3} z_1^{2/3} = \eta e^{2i\theta + ki\pi/3}$ for some $1 \leq k \leq 3$ corresponding to the three roots of unity. By swapping θ for $\theta + 2\pi/3$ successively changes the value of k, so we can assume $k = 3$. Hence

$$K = \{(\xi e^{3i\theta}, -\eta e^{2i\theta}) \in T : 0 \leq \theta < 2\pi\}.$$

which is the $(2, 3)$ torus knot, i.e. the right-handed trefoil knot. It can easily be shown, using the standard tools of knot theory, that this is in fact knotted. Note that this always works for 2 and 3 replaced by any coprime p and q, giving a (p, q) torus knot.

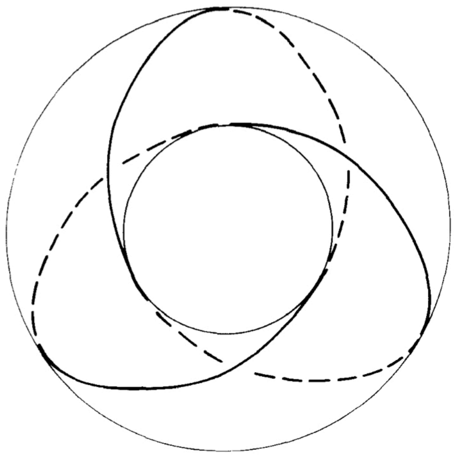


Figure 4: The $(2,3)$ torus knot, i.e. the right-handed trefoil.

To have any hope of finding an exotic sphere, we must of course consider higher-dimensional varieties. In particular, the above example can be generalised by considering

$$f(z_1, \cdots, z_{n+1}) = z_1^2 + \cdots + z_n^2 + z_{n+1}^3.$$

Here $K \hookrightarrow S^{2n+1}$ turns out to give what is refered to as a *generalised trefoil knot.* As we shall soon see, K is homeomorphic to S^{2n-1} when n is odd. When $n = 1$ (above) or $n = 3$, this is diffeomorphic to S^{2n-1} however when $n = 5$, we can show that this is an exotic 9-sphere. The focus of the rest of this section will be to outline to tools needed to establish this fact.

For the sake of introducing a piece of standard terminology, we note that this is an example of a *Brieskorn variety*, which are those links K around the origin which come from polynomials of the form $z_1^{a_1} + \cdots + z_{n+1}^{a_{n+1}}$ for integers $a_i \geq 2$.

6.1 The Alexander Polynomial of K

We now construct an invariant for K which generalises the Alexander polynomial for knots. For a modern construction of the Alexander polynomial for knots, see [21].

As in our discussion above, let $\deg f > 1$, $V = f^{-1}(0)$ and $z_0 \in V$ be a critical point (which we showed must be isolated). Let S_ε be a sphere centred at z_0 with ε sufficiently small so that $K = V \cap S_\varepsilon$ is a smooth $(2n-1)$-manifold.

Consider the knot complement $S_\varepsilon \setminus K = \{z \in S_\varepsilon : f(z) \neq 0\}$ and the well-defined map

$$\phi : S_\varepsilon \setminus K \to S^1, \quad z \mapsto f(z)/\left\|f(z)\right\|.$$

Let $F_\theta = \phi^{-1}(e^{i\theta}) = \{z \in S_\varepsilon \setminus K : \arg(f(z)) = \theta\}$ be the fibres of this map.

We will assume the following fact, for which the reader is directed to [20] for a proof.

Theorem (Milnor's Fibration Theorem)**.** The radial projection map ϕ induces a smooth fibre bundle

$$F_\theta \hookrightarrow S_\varepsilon \setminus K \xrightarrow{\phi} S^1.$$

Each fibre F_θ is a smooth $2n$-manifold which is homotopic to the wedge of $\mu \geq 1$ copies of S^n and has $\overline{F_\theta} = F_\theta \cup K$, where the common boundary K is $(n-2)$-connected.

To define our invariant Δ, we first define an action of $\pi_1(S^1)$ on the fibre $F_0 = \phi^{-1}(1)$. If $z \in F_0$ and $\gamma \in \pi_1(S^1)$ is a loop based at 1, then the fact that ϕ above is a fibre bundle and hence a fibration (see [13]) means we can lift γ to a unique path $\widetilde{\gamma} : [0, 2\pi] \to S_\varepsilon \setminus K$ starting at z. Then action[15] is then defined as $\gamma : z \mapsto \widetilde{\gamma}(2\pi)$.

Let $h : F_0 \to F_0$ denote the action of the generator id $\in \pi_1(S^1)$ on F_0, which induces an automorphism

$$h_* : H_*(F_0) \to H_*(F_0).$$

Now note that the above theorem gives that $H_i(F_\theta) = \mathbb{Z}^\mu$ when $i = n$ and is trivial for all other $1 \leq i \leq 2n$, so the only non-trivial map is $h_n : \mathbb{Z}^\mu \to \mathbb{Z}^\mu$. Our invariant then comes from the characteristic polynomial of this linear map.

Definition (Alexander polynomial)**.** The *Alexander polynomial* of a link K is

$$\Delta_K(t) = \Delta(t) = \det(tI_n - h_n),$$

where I is the identity map on F_0.

Since h_n is an isomorphism, we must have that $\det(h_n) = \pm 1$. Therefore $\Delta(t)$ is a polynomial with integer coefficients of the form

$$t^m + a_1 t^{m-1} + \cdots + a_{m-1} t \pm 1.$$

To take things further, we note that the monodromy action of id $\in \pi_1(S^1)$ induces a mapping $F_0 \times [0, 2\pi] \to S_\varepsilon \setminus K$ given by $(z, t) \mapsto \widetilde{\gamma}(t) \in F_t$. We can then check this gives an isomorphism

$$H_i(F_0 \times [0, 2\pi], F_0 \times \{0\} \cup F_0 \times \{2\pi\}) \to H_i(S_\varepsilon \setminus K, F_0).$$

Furthermore, the long exact sequence for relative homology can be used to show that the group on the left is $H_{i-1}(F_0)$. Replacing $H_i(S_\varepsilon \setminus K, F_0)$ with $H_{i-1}(F_0)$ in the long exact sequence for the pair $(S_\varepsilon \setminus K, F_0)$ then gives us the following.

Lemma (Wang sequence)**.** The fibre bundle $F_0 \hookrightarrow S_\varepsilon \setminus K \to S^1$ induces a long exact sequence

$$\cdots \longrightarrow H_{i+1}(S_\varepsilon \setminus K) \longrightarrow H_i(F_0) \xrightarrow{h_* - I_*} H_i(F_0) \longrightarrow H_i(S_\varepsilon \setminus K) \longrightarrow \cdots$$

Since K is an $(n-2)$-connected $(2n-1)$-manifold and bounds the $(n-1)$-connected $2n$-manifold F_0, we can follow a similar approach to the one outlined at the end of the section on cobordism to characterise when K is homeomorphic to S^{2n-1}.

Indeed, since $S_\varepsilon \setminus K$ is a compact subspace of a $(2n+1)$-sphere, Alexander duality (see [13]) gives that $H_i(S_\varepsilon \setminus K) \cong H^{2n-i}(K)$ and $H^{2n-i}(K) \cong H_{i-1}(K)$ by Poincaré duality. The Wang sequence is then

$$0 \longrightarrow H_n(K) \longrightarrow H_n(F_0) \xrightarrow{h_n - I_n} H_n(F_0) \longrightarrow H_{n-1}(K) \longrightarrow 0.$$

We know from our comments of highly-connected manifolds that K is homeomorphic to S^{2n-1} iff $H_{n-1}(K) = 0$ iff the map $h_n - I_n$ is an isomorphism which happens precicely when $\Delta(1) = \det(h_n - I_n) = \pm 1$. Hence we have shown the following.

[15] This is the same construction we have already seen in covering space theory, sometimes known as the *monodromy action*. Also note that this corresponds to the construction of the *infinite cyclic cover* found in [21], and so indeed can be seen to generalise the Alexander polynomial.

Theorem. If $n \geq 3$, then K is homeomorphic to S^{2n-1} iff $\Delta(1) = \det(I_n - h_n) = \pm 1$.

Now assume K is homeomorphic to S^{2n-1}. To characterise the smooth structure, we can of course use the characteristic class invariants we have already established. However, complete invariants can also be derived. There are two distinct cases.

1. When n is odd, i.e. $\dim K \equiv 1 \mod 4$, the smooth structure is completely determined by the Kervaire invariant $c(F_0) \in \mathbb{Z}/2\mathbb{Z}$ (a definition can be found in [4]).
2. When n is even, i.e. $\dim K \equiv 3 \mod 4$, the smooth structure is completely determined by the signature of the intersection product

$$s : H_n(F_0) \times H_n(F_0) \to \mathbb{Z}.$$

Whilst proofs of these facts are beyond the scope of this essay, we will state a remarkable result of Levine (see [11]). Namely, if n is odd, then

$$c(F_0) = \begin{cases} 0, & \text{if } \Delta(-1) \equiv \pm 1 \mod 8 \\ 1, & \text{if } \Delta(-1) \equiv \pm 3 \mod 8 \end{cases}$$

Hence to verify that a particular K of dimension $(4n+1)$ is an exotic sphere amounts to computing $\Delta(t)$.

6.2 Examples of Exotic Spheres

Here we will present two constructions of exotic spheres and illustrate the two approachs mentioned above for classifying their smooth structure. In both cases, we must compute the Alexander polynomial (or at least the sum of its coefficients). Here is where it is helpful to restrict to the case of Brieskorn varieties, where the Alexander polynomials can be classified as follows.

Theorem (Brieskorn-Pham)**.** Let F_0 be the fibre associated with the Brieskorn variety $z_1^{a_1} + \cdots + z_{n+1}^{a_{n+1}}$. Then $H_n(F_0)$ has rank $\mu = (a_1 - 1) \cdots (a_{n+1} - 1)$ and

$$\Delta(t) = \prod (t - \omega_1 \cdots \omega_{n+1}),$$

where the product is over all the non-trivial a_i-th roots of unity w_i for each i.

For a proof, the reader is directed to [20]. We now consider the two examples.

1. Consider, as mentioned before, the $(2n-1)$-dimensional link K corresponding to

$$f(z_1, \cdots, z_{n+1}) = z_1^2 + \cdots + z_n^2 + z_{n+1}^3.$$

To compute $\Delta(t)$, note that $w_1 = \cdots = w_n = -1$ and $w_{n+1} = \mu^{\pm 1}$, where μ is a non-trivial third root of unity. Hence

$$\Delta(t) = (t - (-1)^n \mu)(t - (-1)^n \mu^{-1}) = t^2 - (-1)^n (\mu + \mu^{-1}) t + 1 = t^2 + (-1)^n t + 1,$$

which gives that $\Delta(1) = 2 + (-1)^n$ and $\Delta(-1) = 2 + (-1)^{n+1}$.

If $n = 2m + 1$ is odd and $n \geq 3$, this shows that K is homeomorphic to S^{4m+1}. We can also compute that $\Delta(-1) = 3$ and so

$$c(F_0) = 1.$$

It remains to detemine which value of $c(F_0)$ corresponds to the standard sphere in each dimension. When $n = 3$, we must have the standard smooth structure since there are no exotic 5-spheres. However, when $n = 4$, this turns out to give an exotic 9-sphere. In fact, this is the Kervaire 9-sphere we considered in the previous section.

2. Consider the 7-dimensional link K corresponding to

$$f(z_1, z_2, z_3, z_4, z_5) = z_1^2 + z_2^2 + z_3^2 + z_4^5 + z_5^3.$$

Now $\Delta(t)$ has $w_1 = w_2 = w_3 = -1$, $w_4 = \nu^{\pm 1}, \nu^{\pm 2}$ and $w_5 = \mu^{\pm 1}$, where ν is a non-trivial fifth root of unity and μ is as above. We compute $\Delta(1)$ directly:

$$\Delta(1) = \prod_{i=\pm 1, \pm 2} (1 + \mu\nu^i)(1 + \mu^{-1}\nu^i) = \prod_{i=\pm 1, \pm 2} (1 - \nu^i + \nu^{2i}).$$

Now note that $(1 - \nu + \nu^2)(1 - \nu^2 + \nu^{-1})$ can be expanded out to give

$$(1 - \nu^2 + \nu^{-1}) + (-\nu + \nu^{-2} - 1) + (\nu^2 - \nu^{-1} + \nu) = \nu^{-2}$$

and so $(1 - \nu^{-1} + \nu^{-2})(1 - \nu^{-2} + \nu) = \nu^2$, by replacing ν with ν^{-1} in the formula. Hence $\Delta(1) = \nu^2 \cdot \nu^{-2} = 1$ and so K is homeomorphic to S^7.

We can show this is an exotic 7-sphere by showing that the signature of the bounding manifold $N = F_0$ is $\sigma(N) = -8$, and then computing the Pontryagin classes using the methods we have used throughout this essay.

Generalising the last example, Hirzebruch established that the link around the origin of

$$z_1^2 + \cdots + z_{2n-1}^2 + z_{2n}^{6k-1} + z_{2n+1}^3$$

was homeomorphic to S^{4n-1} and characterised the k for which this gives an exotic sphere. For example, when $n = 2$ this gives distinct exotic 7-spheres for $k = 1, \ldots, 28$. As we shall see in the following section, these are all the possible exotic 7-spheres.

We conclude by mentioning a generalisation of this result, proven by Brieskorn:

Theorem (Brieskorn)**.** Every homotopy $(2n-1)$-sphere for $n \geq 4$ that bounds a manifold with trivial tangent bundle is the link K around the origin of a hypersurface $V = f^{-1}(0)$ where

$$f(z_1, \cdots, z_{n+1}) = z_1^{a_1} + \cdots + z_{n+1}^{a_{n+1}},$$

for some integers $a_i \geq 2$.

We will return to these results in the following section, showing how they fit in amongst the bigger picture of the classification.

7 Classification of Exotic Spheres

The goal of this section will be to bring together many of the idea presented in this essay and to talk about the classification of exotic spheres that exist in any given dimension. For $n \geq 5$, it was reduced to solving classical problems in Algebraic Topology by Milnor and Kervaire in 1963 (see [4]). We will give an overview of the methods used to do this and, finally, will make a few brief remarks about how the various constructions we have presented fit in to this picture.

The idea is to find a way to put an algebraic structure on the diffeomorphism classes of manifolds homeomorphic to S^n. In particular, the operation we will use is connected sum $M\#N$. This can be defined in a smooth setting and we can check that it is unique up to orientation-preserving diffeomorphisms.

Let Θ_n be the set of oriented n-manifolds homotopic to S^n up to h-cobordism. If $n \geq 5$, we know that being h-cobordant is the same as being diffeomorphic and being homotopic to S^n is then same as being homeomorphic to S^n. Thus Θ_n also corresponds to the set of distinct smooth structures on S^n.

Proposition. Θ_n is an abelian group under connected sum.

Proof: First note that if $M \in \Theta_n$, then $S^n\#M$ is diffeomorphic to M, so $[S^n]$ acts as the identity. That # is associative and commutative is immediate. To show closure, simply note that the connected sum of two copies of S^n is a homotopy sphere.

Finally, let $M \in \Theta_n$ be a homotopy n-sphere. To show that inverses exist, we claim that $M\#(-M) = S^n$. To prove this, note that $M\#(-M)$ bounds the cylinder $B = (M \setminus D^n) \times [0, 1]$ for a small n-disc $D^n \subseteq M$. Since B is a compact contractible manifold with simply-connected boundary, it must be diffeomorphic to D^n by our first corollary of the h-cobordism theorem. In particular $M\#(-M) = S^n$, which completes the proof. □

Thus the problem of finding the number of exotic n-spheres is reduced to finding the order of the group of h-cobordism classes of homotopy n-spheres.

It can be shown that Θ_n is finite for every $n \geq 5$. The proof of this fact, which we will not give, relies on identifying a finite-index subgroup and then showing that the subgroup is finite.

Definition. A smooth manifold M is said to be *parallelisable* if its tangent bundle is trivial and *almost-parallelisable* if $M \setminus F$ is parallelisable for some finite set F.

We write bP_{n+1} for the set of h-cobordism classes of n-manifolds which are boundaries of parallelisable manifolds.

Note that bP_{n+1} is a subgroup of Θ_n since, if $M_1, M_2 \in bP_{n+1}$ bound parallelisable manifolds B_1 and B_2 respectively, then $M_1\#M_2$ bounds the parallelisable manifold formed by attaching B_1 and B_2 using an approriate operation.

We will now briefly review a few basic facts and definitions from homotopy theory. Firstly, note that the suspension homomorphism induces a map

$$\pi_{n+k}(S^k) \to \pi_{n+k+1}(S(S^k)) = \pi_{n+k+1}(S^{k+1})$$

and this map is an isomorphism for $n > k+1$. We can thus define the *nth stable homotopy group of spheres* to be the group this stabilises to, i.e. $\pi_n^s = \lim_{k\to\infty} \pi_{n+k}(S^k)$. Some examples of these groups can be found in the table below. By noting that the homotopy groups of spheres are finite except for a class of known examples, it can be shown that the π_n^s are finite for all n.

Secondly, note that a homomorphism $J_n : \pi_n(SO(k)) \to \pi_{n+k}(S^k)$ can be constructed as follows. An element of $\pi_n(SO(k))$ has the form $S^n \times S^{k-1} \to S^{k-1}$ and so, by the *Hopf construction* (see [13]), this induces a map

$$S^{n+k} = S^n * S^{k-1} \to S(S^{k-1}) = S^k,$$

where $*$ denotes the join of two spaces, and so gives an element of $\pi_{n+k}(S^k)$. This in turn induces a map $J : \pi_n(SO) \to \pi_n^s$, where SO denotes the direct limit of the special orthogonal groups, known as the *J-homomorphism.* It can be shown that the image, which we denote by J, is a cyclic subgroup of π_n^s. These are related to our groups as follows.

Theorem. There is an injective homomorphism

$$\Theta_n/bP_{n+1} \to \pi_n^s/J.$$

In particular, since π_n^s is finite, this shows that Θ_n/bP_{n+1} is also finite. This relationship can be seen in the examples tabulated below.

n	1	2	3	4	5	6	7	8	9	10
π_n^s	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_{240}$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_6$
π_n^s/J	0	$\mathbb{Z}_2$	0	0	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_6$
Θ_n/bP_{n+1}	0	0	0	0	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_6$

Interestingly, this map turns out to either be an isomorphism of have index 2. The above table shows this it is index 2 when $n = 2$ or 6 and is an isomorphism otherwise. It can be shown that it is always an isomorphism when $n \equiv 3 \bmod 4$.

Remark. Classifying the cases for which the index is 2 is equivalent to the Kervaire invariant problem which we discussed at the end of the previous section. Recall that recent results have deduced that this is only the case in dimensions $n = 2, 6, 14, 30, 62$ and possibly 162.

We now review a few basic facts about bP_{n+1}. Firstly, it is finite cyclic group which is trivial when n is even and has order 1 or 2 when $n \equiv 1 \bmod 4$. The complicated case is when $n = 4m - 1$, which we construct below.

Let σ_m be the smallest positive signature obtained by an almost-parallelizable $4m$-manifold without boundary. If $M \in bP_{4m}$ bounds the parallelisable $4m$-manifold B, then let

$$M \mapsto \sigma(B) \mod \sigma_m.$$

This can be shown to be a well-defined homomorphism $bP_{4m} \to \mathbb{Z}_{\sigma_m}$ using the facts we previously established about the signature. It can also be shown to be injective and have image of size $\sigma_m/8$, hence showing that bP_{4m} is cyclic. Furthermore, we have the following result.

Theorem. Let B_{2m} be the Bernoulli numbers, $j_m = |J(\pi_{4m-1}(SO))|$ and a_m is 1 when m is even and 2 when m is odd. Then

$$\sigma_m = 2^{2m-1}(2^{2m-1} - 1)\frac{|B_{2m}|a_m j_m}{m}.$$

Hence, since the map $\Theta_n/bP_{n+1} \to \pi_n^s/J$ is an isomorphism for $n = 4m-1$, we get that

$$|\Theta_{4m-1}| = |\pi_n^s/J| \cdot \sigma_m/8 = 2^{2m-4}(2^{2m-1}-1)\frac{|B_{2m}||\pi_{4m-1}^s|a_m}{m}.$$

Subject to computing whether or not bP_{n+1} has order 1 or 2 when $n \equiv 1 \bmod 4$, we can extends our above table to get:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\lvert\Theta_n\rvert$	1	1	1	1	1	1	28	2	8	6	992	1	3	2	16256	2
$\lvert bP_{n+1}\rvert$	1	1	1	1	1	1	28	1	2	1	992	1	1	1	8128	1

Note that Θ_n only gives us the number of exotic n-spheres for $n \geq 5$. In particular, this shows that we found 4 of the 28 exotic 7-spheres and 64 of the 16256 exotic 15-spheres by the sphere bundles construction. Of course, in many cases we can also determine Θ_n as a group from this information alone. For instance, whenever $|\Theta_n| = |bP_{n+1}|$, we know that Θ_n is cyclic. So the 28 exotic 7-spheres form a cyclic group under connected sum. We now consider how this classification relates to the constructions from the last two sections.

Firstly, recall the Milnor spheres Σ_M of dimensions $4n-1$ and the Kervaire spheres Σ_K of dimensions $4n+1$ that we constructed in the section on Plumbing. We can easily verify that they both bound parallelisable manifolds. In fact, it can be shown that they each generate bP_{4n} and bP_{4n+2} respectively. The latter case shows the equivalence of the two forms of the Kervaire invariant problem that we have mentioned in this essay.

We might also ask how the exotic spheres we found as Brieskorn varieties fit into this picture. Recall that the sphere K we constructed was the boundary of the fibre F_0. It can be shown that F_0 is always parallelisable and so each K lies in bP_{n+1}. Hence, since $|bP_{10}| = 2$, the exotic 9-sphere we found was the only bounded parallelisable sphere other than S^9. This proves that it is diffeomorphic to the corresponding Kervaire sphere. Furthermore, the Brieskorn varieties which come from polynomials of the form

$$z_1^2 + \cdots + z_{2n-1}^2 + z_{2n}^{6k-1} + z_{2n+1}^3$$

are homotopy $(4n-1)$-spheres and can be shown to represent the class of $(-1)^n k \in bP_{4n}$ (where we take 1 to be the generator). For example, when $n = 2$, the exotic 7-spheres are all bounded parallelisable and so are precisely these varieties for $k = 1, \ldots, 28$, as mentioned in the previous section. More generally, recall the result proven by Brieskorn that said that every homotopy $(2n-1)$-sphere for $n \geq 4$ that bounds a parallelisable manifold corresponds to a Brieskorn variety. This shows the Brieskorn varieties generate bP_{2n} for every $n \geq 4$. We now conclude by making a number of final remarks.

Firstly, and perhaps rather unexpectedly, we comment on the real world applications of exotic spheres. Recall that spacetime can be meaningfully modelled as a smooth manifold in such a way that changes in reference frames corresponds to diffeomorphisms of spacetime. We therefore would expect exotic spheres to have at least some physical significance, and in fact many believe they do. In particular it is conjectured by Edward Witten that *gravitational instantons* and/or *solitons* take the form of very exotic spheres, i.e. those that do not bound parallelisable manifolds. Such ideas can be found in a paper written by Witten in 1985 (see [12]) where he argues that the existence of exotic spheres are needed to explain certain global gravitiational anomolies. For a more recent exposition, see [22].

Secondly, we may wonder whether or not the results we have proven about smooth (i.e. C^∞) structures would be completely different had we considered C^k structures for $k \geq 1$. However this is not the case by the delightful result that any C^k structure can be extended to a C^∞ structure uniquely up to diffeomorphisms. In fact, the corresponding map from C^k structures to C^∞ structures is a bijection on the corresponding equivalence classes. Hence the classification of C^k structures are completely identical for any $k \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$.

Finally we remark that these ideas developed in a number of different directions once the h-cobordism theorem was proven and the classification of exotic spheres was completed. On one hand, note that the 4th dimension which remains characteristically absent from any of the results we have established: it is still not known whether or not there exists an exotic 4-sphere. In fact, it is not even known whether or not the number of smooth structures is finite. The 4th dimension tends to be a wild and unusual place; for example, it can be shown that $\mathbb{R}^n$ has a unique smooth structure for all $n \neq 4$, though infinitely many when $n = 4$. Alternatively, one might consider leaving spheres and perhaps even simply-connected spaces altogether. In this direction, we note that the h-cobordism theorem was later generalised to the *s-cobordism theorem* which states that a (general) h-cobordism X between N and M is trivial if and only if an invariant $\tau(X, M)$ known as the *Whitehead torsion* vanishes. These ideas live on today in the field of *Algebraic K-Theory*, where many of Milnor's next great accomplishments can be found.

8 References

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