Introduction to Algebraic Topology

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Lecture 1. Spaces and equivalences

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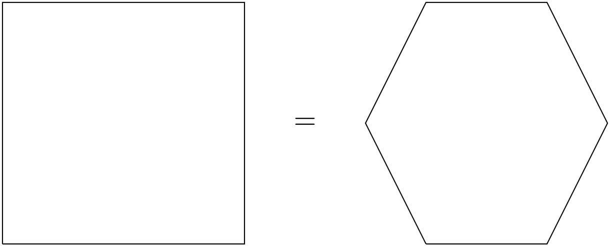
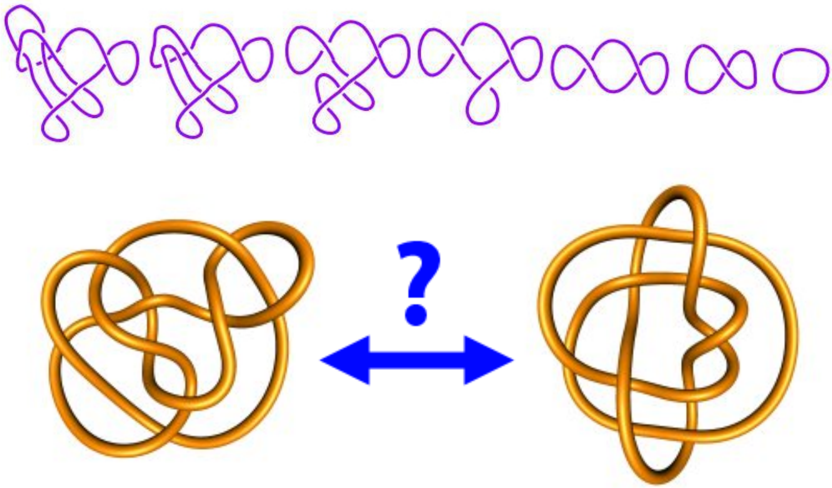
What is Topology?

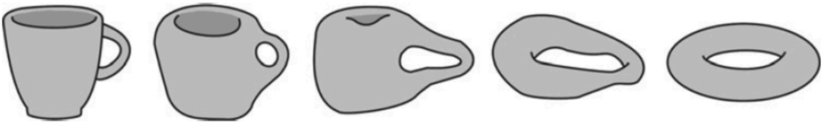
Topology consists of the following game:

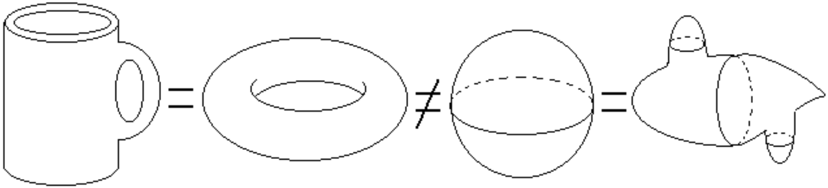
- Choose a class of "spaces" (e.g. 2-dimensional polyhedra)
- Choose a notion of "equivalence" (e.g. equivalent if you can bend one shape into the other without pinching or tearing)
- When are two spaces equivalent? What are the equivalence classes?

The idea is to choose *the right notions of space and equivalence* so that something fundamental about spaces is captured in this game









The goal of this lecture will be to formalise some of these pictures

Spaces: subsets of $\mathbb{R}^n$

The first type of space will consider will be subsets $X \subseteq \mathbb{R}^n$

Examples:

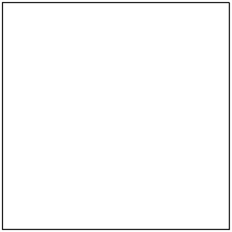
- The simplest example is n-dimensional space $\mathbb{R}^n$
- The n-sphere
  $S^n = \left\{ (x_0, \ldots, x_n) \;\subseteq \mathbb{R}^{n+1} \;\middle|\; \sum_{i=1}^{n} x_i^2 = 1 \right\} \subseteq \mathbb{R}^{n+1}$
- S^2 is a sphere and S^1 is a circle (a one-dimensional sphere)
- The unit interval $I = [0, 1] \subseteq \mathbb{R}$
- The point space $* = \{0\} \subseteq \mathbb{R}$

We can build new spaces from old ones in all the usual ways. For example, if $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$, then:

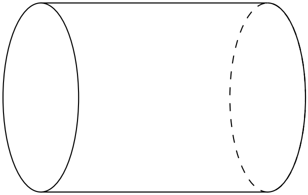
$$X \times Y = \{(x, y) : x \in X, y \in Y\} \subseteq \mathbb{R}^{n+m}.$$

- The square I^2
- The (hollow) cylinder $S^1 \times I$

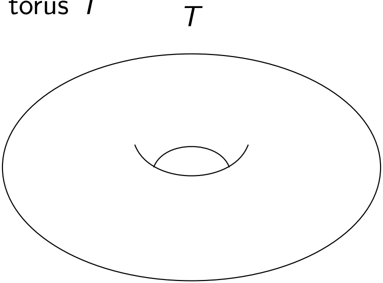
I^2



$S^1 \times I$



- The (hollow) torus T



After staring at it for a while, one can eventually come up with the following explicit formula:

$$T = \left\{((2+\cos\theta)\cos\varphi, (2+\sin\theta)\cos\varphi, \sin\varphi) \in \mathbb{R}^3 \;\middle|\; 0 \leqslant \theta, \varphi < 2\pi\right\}$$

Exercise: find a formula like this for the coffee mug

Just kidding (definitely do not do this...)

Is there any easier way to define a space?

Spaces: cell complexes

We will now define the notion of an *n-dimensional cell complex*

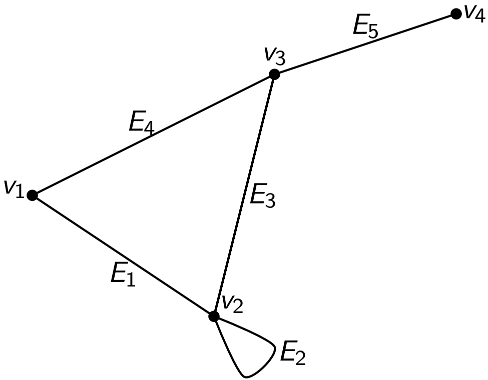
Definition

A 1-*dimensional cell complex* (or 1-complex) is a graph (V, E) where $V = \{v_1, \ldots, v_n\}$ is the vertex set and $E = \{E_1, \ldots, E_m\}$ is a collection of edges $E_i \in V \times V$

- We will consider these graphs to be unoriented, but each edge $E = (v_1, v_2)$ comes with a natural orientation $v_1 \to v_2$
- We will take 'collection' to mean that repeats are allowed, so $E = \{(v_1, v_2)\}$ is not the same as $E = \{(v_1, v_2), (v_1, v_2)\}$

Example: Let $V = \{v_1, v_2, v_3, v_4\}$ and

$$E = \{(v_1, v_2), (v_2, v_2), (v_2, v_3), (v_1, v_3), (v_3, v_4)\}$$



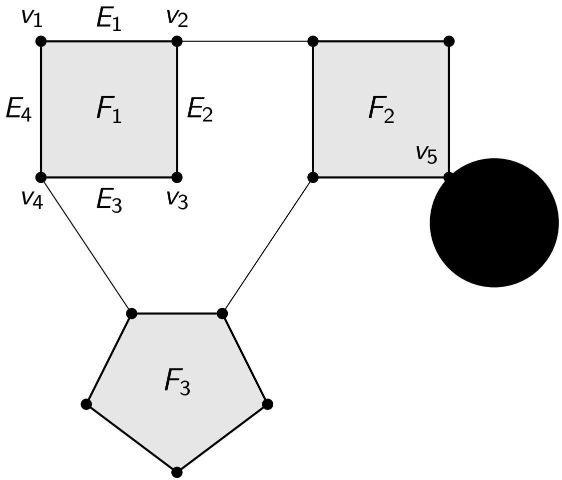
If we want to talk about orientations, we will write $E_1^{+1} = (v_1, v_2)$ and $E_1^{-1} = (v_2, v_1)$ to denote the edge in the opposite direction

Definition

A *2-dimensional cell complex* (or 2-complex) is a triple (V, E, F) where (V, E) is a graph and $F = \{F_1, \ldots, F_r\}$ is a collection of *faces* which are defined in either of the following ways:

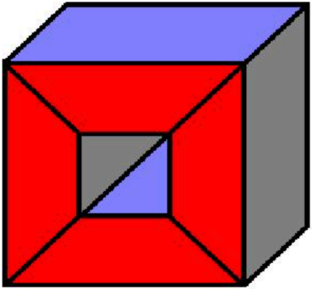
- As sequences of edges $F_i = (E_{i_1}^{\pm 1}, \ldots, E_{i_k}^{\pm 1})$ where $E_{i_t}^{\pm 1}$ and $E_{i_{t+1}}^{\pm 1}$ are adjacent (taking $i_{k+1} = i_1$). This is the face attached along the path $(E_{i_1}, \ldots, E_{i_k})$
- As a vertex $F_i = (v)$. This is the face attached at a single vertex v by wrapping it up into the shape of a sphere

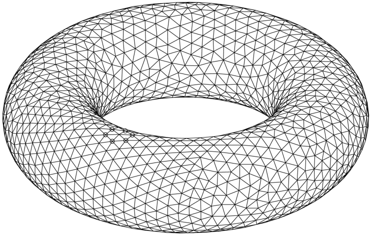
Example:



$E_1 = (v_1, v_2), E_2 = (v_2, v_3), E_3 = (v_3, v_4), E_2 = (v_4, v_1),$
$F_1 = (E_1^{+1}, E_2^{+1}, E_3^{+1}, E_4^{+1}), \cdots, F_4 = (v_5)$

How can we define the torus T as a 2-dimensional cell complex?





- One cell complex has 16 faces, but the other has 100s of faces
- Neither cell complex is ‘smooth’ like our previous definition

Later, we will choose our notion of equivalence so that all three definitions give equivalent spaces

Spaces: metric spaces

Definition

For a set X, a *metric on* X is a map $d : X \times X \to \mathbb{R}_{\geq 0}$ such that:

(i) $d(x, y) = 0 \iff x = y$

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$

(iii) $d(x, z) \leqslant d(x, y) + d(y, z)$ for all $x, y, z \in X$.

A *metric space* is a pair (X, d) where X is a set and d is a metric on X.

Definition

If (X, d_X) and (Y, d_Y) are metric spaces, then a function $f : X \to Y$ is *continuous* if, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $d_X(x, y) < \delta$ implies $d_Y(f(x), f(y)) < \varepsilon$.

Example: $\mathbb{R}$ is a metric space with metric $d(x, y) = |x - y|$

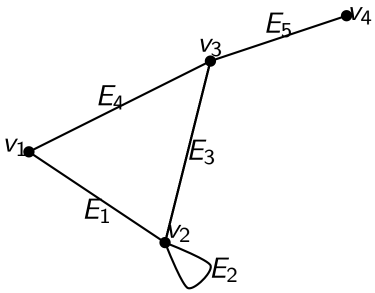
- $f : \mathbb{R} \to \mathbb{R}$ is continuous if and only if, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$
- This matches the usual definition for continuous

Subsets of $\mathbb{R}^n$ are metric spaces:

If $X \subseteq \mathbb{R}^n$, then (X, d) is a metric space with

$$d((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

1-dimensional cell complexes are metric spaces:

$X :=$ 

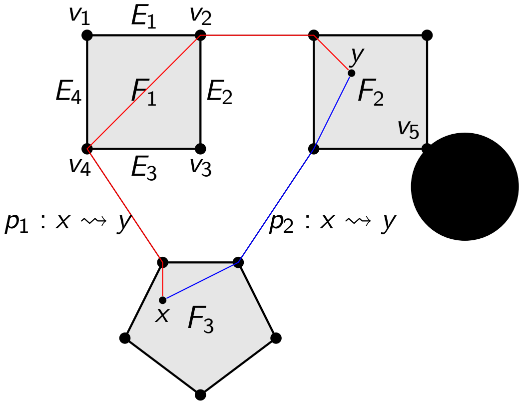
- Every edge has length 1, i.e. we take $E_i = [0, 1]$
- A path p is a sequence of segments of edges

For $x, y \in X$, we can define:

$$d(x, y) = \min\{\text{length}(p) \mid p \text{ a path from } x \text{ to } y\}$$

Example: $d(v_1, v_4) = 2, d(v_1, v_3) = 1$

2-dimensional cell complexes are metric spaces:



Suppose a face F_i has n sides

- If $n \geq 3$, distances are that of the 'standard n-gon'
- If $n = 1, 2$, distances can be taken from curved surfaces

For $x, y \in X$, we can define:

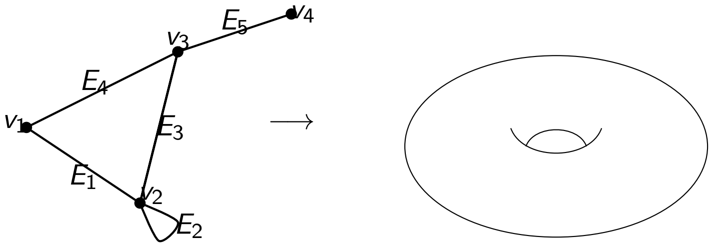
$$d(x, y) = \min\{\text{length}(p) \mid p \text{ a path from } x \text{ to } y\}$$

Since subsets of $\mathbb{R}^n$ and cell complexes are metric spaces, we now have a notion of continuous function $f : X \to Y$

This notion **matches our intuition for continuous**

Example: The function

$f :$



which 'draws the graph on the torus' is a continuous function

There is a more general notion of *topological space*

This definition is the answer to the question:
"what is the least amount of information we need to add to a set in order to have a meaningful notion of continuous function?"

Every metric space is a topological spaces but there are topological spaces which are not metric spaces

We will only consider metric spaces in this course

Equivalences: homeomorphism

From now on, will use the *map* to mean continuous function

Definition

We say that metric spaces X and Y are *homeomorphic* ($X \cong Y$) if there are maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$

We say that such a map f is a *homeomorphism* from X to Y.

Exercise: Prove that $\cong$ is an equivalence relation

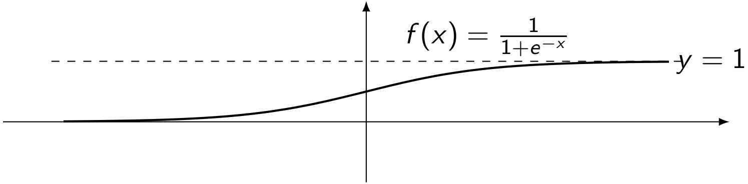
Example: $X = [0,1]$ and $Y = [0,2]$

Let $f(x) = 2x$ and $g(x) = \frac{1}{2}x$

Then $f(g(x)) = x$ and $g(f(x)) = x$ and so $X \cong Y$

Example: $X = \mathbb{R}$ and $Y = (0,1)$

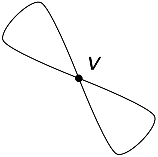
Let $f(x) = \frac{1}{1+e^{-x}}$, the sigmoid function, and $g(x) = -\log\left(\frac{1}{x} - 1\right)$



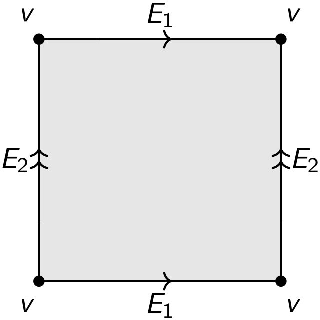
Then $f(g(x)) = x$ and $g(f(x)) = x$ and so $X \cong Y$

Example: $V = \{v\}, E = \{(v,v),(v,v)\}, F = \{(E_2, E_1, E_2^{-1}, E_1^{-1})\}$

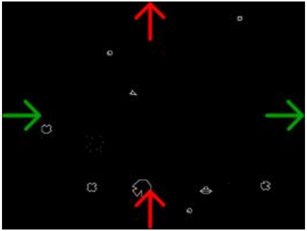
$(V, E) =$



$X :=$



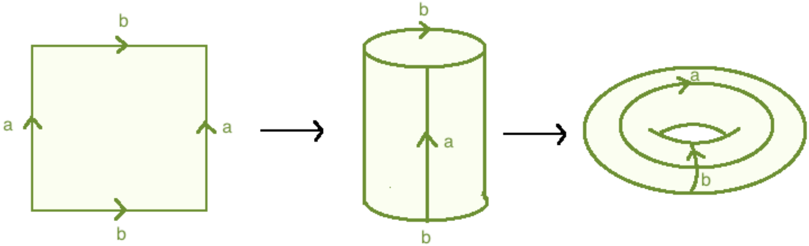
$=$



Proposition

$X \cong T$, i.e. X is homeomorphic to a torus

- The proof is based on the following picture



- This can be used to write down a bijection $f : X \to T$
- It should be clear from the formula for f and f^{-1} that they are continuous functions □

Exercise: Prove $T \cong S^1 \times S^1$

Proposition

If $n \geq 1$, then $\mathbb{R}^n \ncong *$

Proof.

Suppose there exists a homeomorphism $f : \mathbb{R}^n \to *$, i.e. there exists $g : * \to \mathbb{R}^n$ for which $f \circ g = \mathrm{id}_*$ and $g \circ f = \mathrm{id}_{\mathbb{R}^n}$

Then f is a bijection

This is a contradiction since $* = \{0\}$ and $\mathbb{R}^n$ is infinite □

Theorem

$\mathbb{R}^n \cong \mathbb{R}^m$ *if and only if* $n = m$

- Generalises the result above since $\mathbb{R}^0 \cong *$
- So homeomorphisms cannot change the 'dimension' of a space
- The proof is difficult and beyond the scope of this course

Equivalences: homotopy equivalence

We will now define another notion of equivalence for metric spaces which will allow for deformations which do not preserve dimension

Definition

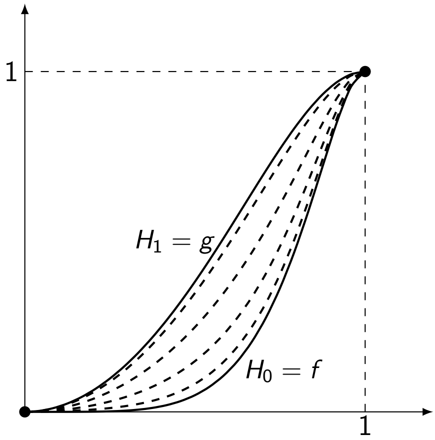
Two maps $f, g : [0,1] \to [0,1]$ are *homotopy equivalent* ($f \simeq g$) if there is a one-parameter family of maps $H_t : [0,1] \to [0,1]$ such that such that $H_0 = f$ and $H_1 = g$ and which varies continuously for $t \in [0,1]$

We say that H is a homotopy from f to g

Here by 'vary continuously', we simply mean that the induced function $H : [0,1] \times [0,1] \to [0,1]$, $(x,t) \mapsto H_t(x)$ is continuous

We should think of t as **time** and f fades into g as time flows from time $t = 0$ to time $t = 1$

The picture we should have in our head is:



We can generalise this to arbitrary metric spaces:

Definition

Let X and Y be metric spaces. Two maps $f, g : X \to Y$ are *homotopy equivalent* ($f \simeq g$) if there is a one-parameter family of maps $H_t : X \to Y$ such that $H_0 = f$ and $H_1 = g$ and which varies continuously for $t \in [0, 1]$

This is homotopy equivalence of *maps*. How can we use this to define homotopy equivalence of *spaces*?

The idea is to replicate the definition of homeomorphism but replace "$= \mathrm{id}_X$" with "$\simeq \mathrm{id}_X$"

Definition

Two metric spaces X and Y are *homotopy equivalent* ($X \simeq Y$) if there exists maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \simeq \mathrm{id}_Y$ and $g \circ f \simeq \mathrm{id}_X$

We say that $f : X \to Y$ is a *homotopy equivalence*

Exercise: Prove that $\simeq$ is an equivalence relation

Proposition

If $X \cong Y$, *then* $X \simeq Y$

Proof.

If $f \circ g = \mathrm{id}_Y$, then $f \circ g \simeq \mathrm{id}_Y$
If $g \circ f = \mathrm{id}_X$, then $g \circ f \simeq \mathrm{id}_X$ □

Proposition

If $n \geq 0$, then $\mathbb{R}^n \simeq *$

Proof.

Define functions $f : \mathbb{R}^n \to *, x \mapsto 0$ and $g : * \to \mathbb{R}^n, 0 \mapsto 0$

Then $f \circ g : * \to *, 0 \mapsto 0$, i.e. $f \circ g = \mathrm{id}_*$

We also have $g \circ f : \mathbb{R}^n \to \mathbb{R}^n, x \mapsto 0$ and must show $g \circ f \simeq \mathrm{id}_{\mathbb{R}^n}$

Let $H_t(x) = tx$ for $t \in [0,1]$. Then $H_0 = 0 = g \circ f$, $H_1 = \mathrm{id}_{\mathbb{R}^n}$ and H_t varies continuously

Hence $g \circ f \simeq \mathrm{id}_{\mathbb{R}^n}$ and so $\mathbb{R}^n \simeq *$ □

Corollary

For $n, m \geq 0$, $\mathbb{R}^n \simeq * \simeq \mathbb{R}^m$

So homotopy equivalences can change the 'dimension' of a space