Introduction to Algebraic Topology

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Lecture 2. Invariants

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Recap

We considered two types of metric spaces:

- Subsets of $\mathbb{R}^n$
- 1 and 2-dimensional cell complexes

Definition

Two metric spaces X and Y are *homotopy equivalent* ($X \simeq Y$) if there exists maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \simeq \mathrm{id}_Y$ and $g \circ f \simeq \mathrm{id}_X$

We say that $f : X \to Y$ is a *homotopy equivalence*

- $g \circ f \simeq \mathrm{id}_X$ means there exists a a one-parameter family of maps $H_t : X \to X$ such that $H_0 = g \circ f$ and $H_1 = \mathrm{id}_X$ and which varies continuously for $t \in [0, 1]$

We will show that the circle S^1 is homotopy equivalent to the punctured complex plane $\mathbb{C} \setminus \{0\}$

We will take $S^1 = \{e^{i\theta} : \theta \in [0, 2\pi]\} \subseteq \mathbb{C}$

Exercise: Prove that this is homeomorphic to $\{(x, y) : x^2 + y^2 = 1\} \subseteq \mathbb{R}^2$ (our previous definition)

We need to find $f : S^1 \to \mathbb{C} \setminus \{0\}$ and $g : \mathbb{C} \setminus \{0\} \to S^1$ such that $f \circ g \simeq \mathrm{id}_{\mathbb{C}\setminus\{0\}}$ and $g \circ f \simeq \mathrm{id}_{S^1}$

Fortunately, there are only two sensible choices for f and g:

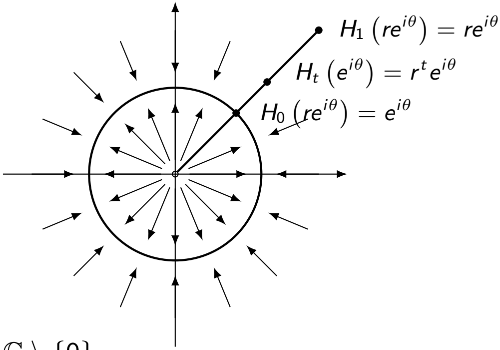
- $f : S^1 \to \mathbb{C} \setminus \{0\}, e^{i\theta} \mapsto e^{i\theta}$
- $g : \mathbb{C} \setminus \{0\} \to S^1, re^{i\theta} \mapsto e^{i\theta}$ (where $r \neq 0$)

We have $g \circ f = id_{S^1}$ and so need to show that $f \circ g \simeq id_{\mathbb{C}\setminus\{0\}}$

Since $f\left(g\left(re^{i\theta}\right)\right) = e^{i\theta}$, we need to find a continuous one-parameter family of maps $H_t : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$ such that

$$H_0\left(re^{i\theta}\right) = e^{i\theta}, \quad H_1\left(re^{i\theta}\right) = re^{i\theta}$$

One example is the function $H_t\left(re^{i\theta}\right) = r^t e^{i\theta}$:



Hence $S^1 \simeq \mathbb{C} \setminus \{0\}$

How do we prove that two spaces are not homotopy equivalent? For example, is $S^1 \simeq \mathbb{R}$?

One approach is:

- Suppose there exists continuous functions $f : S^1 \to \mathbb{R}$ and $g : \mathbb{R} \to S^1$ such that $f \circ g \simeq \mathrm{id}_{\mathbb{R}}$ and $g \circ f \simeq \mathrm{id}_{S^1}$
- Find general forms for the functions f and g
- Try to arrive at a contradiction

However, this approach fails in general since there are many choices for f and g

The best approach is to instead use **algebraic topology**

What is Algebraic Topology?

Algebraic topology gives a method to prove that $X \not\simeq Y$ using *invariants*:

- An invariant is a quantity $I(X)$ which we can attach to each space X such that, if $X \simeq Y$, then $I(X) = I(Y)$
- Hence, if $I(S^1) \neq I(\mathbb{R})$, then $S^1 \not\simeq \mathbb{R}$

One example is $I(X) = X$ (the space itself). However this is a bad example since:

- We want $I(X)$ to be such that determining if $I(X) = I(Y)$ is easier than determining if $X \simeq Y$

Typically this means that $I(X)$ will be a quantity from algebra

The goal of this lecture will be to define two invariants:

- The Euler characteristic $\chi(X)$
- The fundamental group $\pi_1(X)$

Euler characteristic

Definition

Let X be an n-dimensional cell complex and let f_i denote the number of cells in dimension i

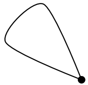
The *Euler characteristic* of X is $\chi(X) = \sum_{i=0}^{n} (-1)^i f_i$

For low-dimensional examples, we will write:

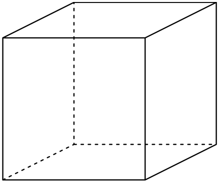
- $V = f_0 =$ number of vertices
- $E = f_1 =$ number of edges
- $F = f_2 =$ number of faces

Examples:

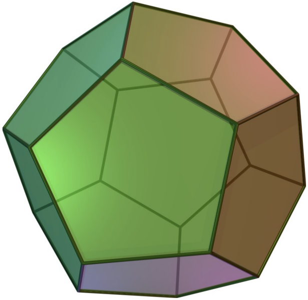
- $\chi(*) = 1$
- Let X be a loop with one vertex. Then:
  $\chi(X) = V - E = 1 - 1 = 0$

$X =$ 

- Let C be a cube (hollow but with solid faces). Then:
  $\chi(C) = V - E + F = 8 - 12 + 6 = 2$



- Let D be a regular dodecahedron. Then:
  $\chi(D) = V - E + F = 20 - 30 + 12 = 2$



Note that $C \cong D \cong S^2$ are both homeomorphic to the sphere S^2

We also have $\chi(C) = \chi(D) = 2$

Does $X \cong Y$ imply $\chi(X) = \chi(Y)$?

In fact, even more is true:

Theorem

χ is a homotopy invariant, i.e. if X and Y are cell complexes and $X \simeq Y$, then $\chi(X) = \chi(Y)$

The proof is beyond the scope of this course

Challenge problem: Prove that χ is a homotopy invariant on the class of 2-dimensional cell complexes. You may assume that, if $X \simeq *$, then $\chi(X) = 1$

Since χ is a homotopy invariant, we can extend the definition of χ to metric spaces which are homotopy equivalent to cell complexes:

Definition

If X is a metric space and Y is a cell complex such that $X \simeq Y$, then define $\chi(X) := \chi(Y)$

Examples:

- $\mathbb{R}^n \cong *$. Hence $\chi(\mathbb{R}^n) = 1$
- $S^1 \cong X$ where X is a loop with one vertex. Hence $\chi(S^1) = 0$

This implies that $S^1 \not\cong \mathbb{R}$ which would be difficult to prove by other means

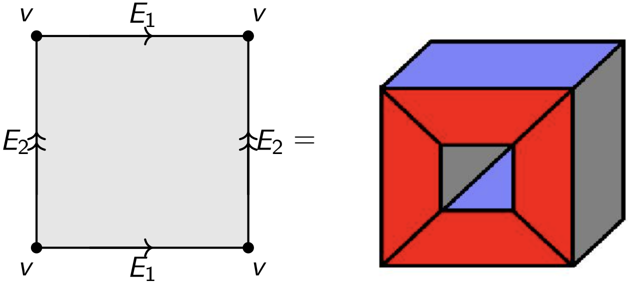
If $S^1 \cong \mathbb{R}$, then $S^1 \simeq \mathbb{R}$. Hence we also have $S^1 \not\cong \mathbb{R}$

- $S^2 \cong C$ where C is the cube. Hence $\chi(S^2) = 2$

If X is the surface of a polyhedra, then $X \simeq S^2$ and so $\chi(X) = 2$

If X has V vertices, E edges and F faces, then $V - E + F = 2$
This was first discovered by Leonhard Euler in 1758

- If T is a torus, then T is homeomorphic to either of the following cell complexes



The first example shows $\chi(T) = V - E + F = 1 - 2 + 1 = 0$

Since $\chi(S^1) = 0$, we cannot distinguish S^1 and T using χ

So how can we prove that $T \not\cong S^1$?

Summary of the Euler characteristic as an invariant:

- Hard to prove that it is a homotopy invariant
- Easy to compute
- Only consists of an integer value, so can only distinguish a limited number of spaces

The fundamental group

We will now define a new invariant $\pi_1(X)$

This time, homotopy will appear in the definition of our invariant and so it will be:

- Easy to prove that it is a homotopy invariant
- Hard to compute

Furthermore:

- It will have the structure of a group, and so has the power to distinguish between a larger number of spaces

Let X be a metric space and let $x_0 \in X$ (known as the basepoint)

Consider all paths in X which start and end at x_0:

$$\{\text{loops at } x_0\} = \{\gamma : [0,1] \to X \mid \gamma(0) = \gamma(1) = x_0,\ \gamma \text{ continuous}\}$$

Definition

We say that loops $\gamma, \gamma' : [0,1] \to X$ are *homotopy equivalent* ($\gamma \simeq \gamma'$) if there exists a continuously varying one-parameter family of loops $H_t : [0,1] \to X$ such that $H_0 = \gamma,\ H_1 = \gamma'$.

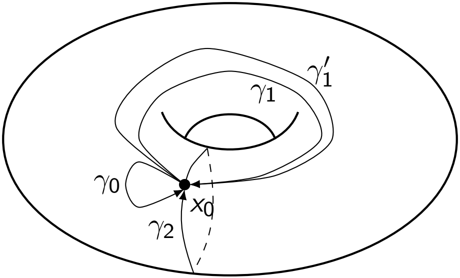
This is **not the same** as a homotopy between $\gamma, \gamma' : [0,1] \to X$ considered as functions between metric spaces

Here H_t is a loop for all t, i.e. $H_t(0) = H_t(1) = x_0$ for all $t \in [0,1]$. This is also known as a *based homotopy*

To picture a homotopy:

- View two loops as stretchy pieces of string attached at $x_0 \in X$
- Two loops are equivalent if you can stretch one piece of string into the other while keeping x_0 fixed

Example: Consider loops γ_0, γ_0', γ_1 and γ_2 on the torus T



Let $c_{x_0} : [0,1] \to X$, $t \mapsto x_0$ denote the constant loop

Then $\gamma_1 \simeq \gamma_1'$, $\gamma_0 \simeq c_{x_0}$. Are γ_0, γ_1 and γ_2 homotopy equivalent?

If S is a set and $\equiv$ is an equivalence relation of S, then we write $S/\equiv$ for the equivalence classes of $\equiv$, i.e. "the set S modulo $\equiv$"

Example: if $S = \mathbb{Z}$ and $a \simeq b$ if $a \equiv b \mod n$, then $\mathbb{Z}/\simeq\, \cong \mathbb{Z}/n\mathbb{Z}$ are isomorphic as rings

Definition

$\pi_1(X, x_0) := \{\text{loops at } x_0\}/\simeq$

If $\gamma : [0,1] \to X$ is a loop at x_0, then we often write $[\gamma] \in \pi_1(X, x_0)$

Example: $\pi_1(T, x_0) = \{[c_{x_0}], [\gamma_1], [\gamma_2], \cdots\}/\simeq$

If we could show $\gamma_1 \not\simeq c_{x_0}$, then $\pi_1(T, x_0)$ would contain more than one element

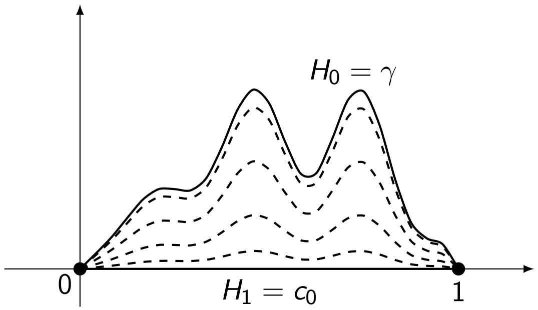
Example: $X = [0,1]$, $x_0 = 0$. We want to compute $\pi_1(X, x_0)$

Let $\gamma : [0,1] \to X$ be a loop with $\gamma(0) = \gamma(1) = 0$

We claim that $\gamma \simeq c_0$

This is achieved by the based homotopy $H_t(x) = (1-t)\gamma(x)$ which has $H_0 = \gamma$, $H_1 = 0$, and is a loop for all $t \in [0,1]$ since $H_t(0) = 0$

At time t increases, every point on γ is pushed towards 0:



Hence $\pi_1(X, x_0) = \{c_0\}$.

Homotopy invariance of $\pi_1(X, x_0)$

What would it mean to say that $\pi_1(X, x_0)$ is a homotopy invariant?

- Two sets are 'equal' if there is a bijection between them
- If $X \simeq Y$, then we need a bijection $\pi_1(X, x_0) \to \pi_1(Y, y_0)$
- Is this true for *all* choices of x_0, y_0 or just some choices?

We say that a metric space X is *path-connected* if, for all $x_0, x_1 \in X$, there exists a path from x_0 to x_1, i.e. there exists a continuous function $p : [0, 1] \to X$ with $p(0) = x_0$, $p(1) = x_1$

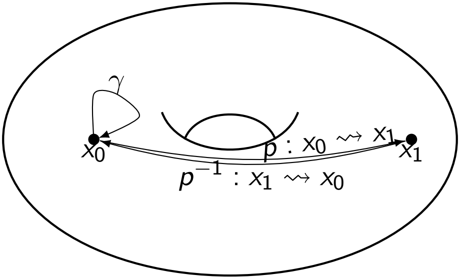
Theorem

Let X be a path-connected metric space and let $x_0, x_1 \in X$
If p is a path from x_0 to x_1, then there exists a bijection

$$p_* : \pi_1(X, x_0) \to \pi_1(X, x_1)$$

Proof: If $\gamma \in \pi_1(X, x_0)$, then define $p_*(\gamma) := p^{-1} \cdot \gamma \cdot p$

This is the loop at x_1 which travels $x_1 \xrightarrow{p^{-1}} x_0 \xrightarrow{\gamma} x_0 \xrightarrow{p} x_1$:



Explicitly, this has the form:

$$(p^{-1} \cdot \gamma \cdot p)(t) = \begin{cases} p^{-1}(3t) & t \leqslant 1/3 \\ \gamma(3t-1) & 2/3 \leqslant t \leqslant 2/3 \\ p(3t-2) & 2/3 \leqslant t \leqslant 1. \end{cases}$$

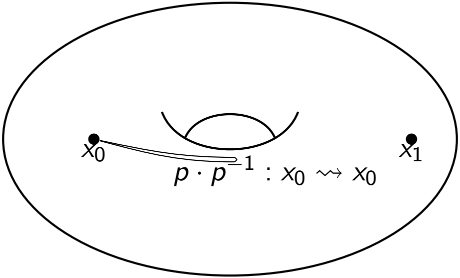
In order to show that p_* is a bijection, it will suffice to check that $(p^{-1})_* \circ p_* : \pi_1(X, x_0) \to \pi_1(X, x_0)$ is equal to $\mathrm{id}_{\pi_1(X,x_0)}$

Note that $(p^{-1})_* \circ p_* : \gamma \mapsto p \cdot p_*(\gamma) \cdot p^{-1} = (p \cdot p^{-1}) \cdot \gamma \cdot (p \cdot p^{-1})$

Hence we need to show that $(p \cdot p^{-1}) \cdot \gamma \cdot (p \cdot p^{-1}) \simeq \gamma$

It suffices to prove that $p \cdot p^{-1} \simeq c_{x_0}$

The proof that $p \cdot p^{-1} \simeq c_{x_0}$ is the following picture:



Recall that $p \cdot p^{-1}$ is defined as:

$$(p \cdot p^{-1})(x) = \begin{cases} p(2x) & 0 \leqslant x \leqslant 1/2 \\ p^{-1}(2x-1) & 1/2 \leqslant x \leqslant 1. \end{cases}$$

Explicitly, the diagram above corresponds to taking the homotopy:

$$H_t(x) = \begin{cases} p(t \cdot 2x) & 0 \leqslant x \leqslant 1/2 \\ p^{-1}(1 + 2t \cdot (x-1)) & 1/2 \leqslant x \leqslant 1 \end{cases}$$

□

We say that two sets A, B are equivalent if there is a bijection $f : A \to B$

This is an equivalence relation on the class of sets

If X is a metric space and $x_0, x_1 \in X$, then $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ are equivalent as sets

Definition

Let $\pi_1(X)$ denote the set equivalence class containing $\pi_1(X, x_0)$ (for any choice of $x_0 \in X$)

From now on, we will assume that all spaces are path-connected metric spaces

Theorem (π_1 is a homotopy invariant)

If $X \simeq Y$, then $\pi_1(X) \cong \pi_1(Y)$ are equivalent as sets

This will follow from:

Theorem
If $f : X \to Y$ is a homotopy equivalence, $x_0 \in X$ and $y_0 = f(x_0)$, then there is a bijection

$$f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0), \quad \gamma \mapsto f \circ \gamma$$

Proof: Suppose there is a map $g : Y \to X$ such that $f \circ g \simeq \mathrm{id}_Y$, $g \circ f \simeq \mathrm{id}_X$ and $g(y_0) = x_0$ (in general, $g(y_0) \neq x_0$)

It will suffice to show that the composition

$$(g \circ f)_* = g_* \circ f_* : \pi_1(X, x_0) \to \pi_1(X, x_0), \gamma \mapsto (g \circ f) \circ \gamma$$

is equal to $\mathrm{id}_{\pi_1(X,x_0)}$ (and similarly $f_* \circ g_* = \mathrm{id}_{\pi_1(Y,y_0)}$)

This would imply that f_* and g_* are invertible and hence bijections

It suffices to prove:

Lemma

Let $f : X \to X$ be a map such that $f \simeq \mathrm{id}_X$ and $f(x_0) = x_0$
Then $f_* = \mathrm{id}_{\pi_1(X,x_0)}$, i.e. $f \circ \gamma \simeq \gamma$ for all $\gamma \in \pi_1(X, x_0)$

Proof.

Let $H_t : X \to X$ be a homotopy from f to id_X

Then $\widetilde{H}_t = H_t \circ \gamma : [0,1] \to X$ is a based homotopy $f \circ \gamma \simeq \gamma$ □

This completes the proof since:

- $f \circ g \simeq \mathrm{id}_Y$ implies $f_* \circ g_* = (f \circ g)_* = \mathrm{id}_{\pi_1(Y,y_0)}$
- $g \circ f \simeq \mathrm{id}_X$ implies $g_* \circ f_* = (g \circ f)_* = \mathrm{id}_{\pi_1(X,x_0)}$

□

However, in the proof we assumed that $g(y_0) = x_0$

Exercise: Show that, if $f : X \to X$ is a map such that $f \simeq \mathrm{id}_X$, then f_* is bijective. That is, finish the proof of the Theorem.