Introduction to Algebraic Topology

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Lecture 3. The fundamental group of the circle

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Recap

Let X be a metric space and let $x_0 \in X$

A loop at x_0 is a map $\gamma : [0,1] \to X$ with $\gamma(0) = \gamma(1) = x_0$

We defined $\pi_1(X, x_0) := \{\text{loops at } x_0\}/\simeq$

- Two loops $\gamma, \gamma' : [0,1] \to X$ have $\gamma \simeq \gamma'$ if there exists a homotopy $H_t : [0,1] \to X$ such that $H_0 = \gamma,\ \ H_1 = \gamma'$ and $H_t(0) = H_t(1) = x_0$ (a *based* homotopy)

Theorem
If X is path-connected, then $\pi_1(X, x_0) \cong_{bij} \pi_1(X, x_1)$

- We let $\pi_1(X)$ denote the set up to equivalence of sets

Theorem
If $X \simeq Y$, then $\pi_1(X) \cong \pi_1(Y)$ are equivalent as sets

- So $\pi_1(X)$ is a homotopy invariant

Definition

We say X is *simply connected* if $\pi_1(X) = 1$, i.e. if it is equivalent to the set of size one

The property of being simply connected is a homotopy invariant

Example: $\pi_1([0,1]) = \{c_0\}$ and so $[0,1]$ is simply connected
An alternate proof is that $[0,1] \simeq *$ and $\pi_1(*) = 1$

If X is a metric space with $\pi_1(X) \neq 1$, then $X \not\simeq *$

However, it is difficult to show that a space has $\pi_1(X) \neq 1$

Group theory

Definition

A *group* is a set G and a map $\cdot : G \times G \to G$ such that

- There is an element $1 \in G$ such that $g \cdot 1 = g = 1 \cdot g$
- For all $g \in G$, there exists $h \in G$ such that $h \cdot g = 1 = g \cdot h$
- If $g, h, k \in G$, then $(g \cdot h) \cdot k = g \cdot (h \cdot k)$ (associativity)

Example: $(\mathbb{R} \setminus \{0\}, \cdot)$ where $\cdot$ is multiplication

- Let $x \in \mathbb{R} \setminus \{0\}$. Then $x \cdot 1 = x = 1 \cdot x$
- Let $x \in \mathbb{R} \setminus \{0\}$. Then $\frac{1}{x} \in \mathbb{R} \setminus \{0\}$ and $\frac{1}{x} \cdot x = 1 = x \cdot \frac{1}{x}$
- Associativity is clear

Example: $(\mathbb{Z}, +)$ is a group, where $+$ is addition

- Let $n \in \mathbb{Z}$. Then $n + 0 = 0 = 0 + n$ (so the '1' is 0)
- Let $n \in \mathbb{Z}$. Then $-n \in \mathbb{Z}$ and $n + (-n) = 0 = (-n) + n$

Example: $(\mathbb{Z}/n\mathbb{Z}, +)$ is a group

Example: Let $(\mathbb{Z}/n\mathbb{Z})^\times = \{m : (n, m) = 1\} \subseteq \mathbb{Z}/n\mathbb{Z}$

If $m_1, m_2 \in (\mathbb{Z}/n\mathbb{Z})^\times$, then $m_1 \cdot m_2 \in (\mathbb{Z}/n\mathbb{Z})^\times$

Exercise: Show that $((\mathbb{Z}/n\mathbb{Z})^\times, \cdot)$ is a group

Definition

Let G and H are groups. Then a function $f : G \to H$ is a *homomorphism* if $f(gh) = f(g)f(h)$ for all $g, h \in G$ and $f(1) = 1$.

Example: $f : \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$, $m \mapsto m \pmod{n}$ is a homomorphism

We need to decide what it means for two groups to be equivalent

Definition

Two groups G, H are *isomorphic* ($G \cong H$) if there are homomorphisms $f : G \to H$ and $g : H \to G$ such that $f \circ g = \mathrm{id}_H$ and $g \circ f = \mathrm{id}_G$

We say $f : G \to H$ is an *isomorphism*

This should remind us of the definition of homeomorphism

Exercise: Prove that, if $f : G \to H$ is a bijective homomorphism, then f is an isomorphism

Example: Consider $(\mathbb{Z}/5\mathbb{Z})^\times = \{1, 2, 3, 4\}$ (with multiplication)

Note that $2^2 = 4, 2^3 = 3, 2^4 = 1$ so $(\mathbb{Z}/5\mathbb{Z})^\times = \{1, 2, 2^2, 2^3\}$

$f : \mathbb{Z}/4\mathbb{Z} \to (\mathbb{Z}/5\mathbb{Z})^\times, n \mapsto 2^n$ is a bijective homomorphism

Hence f is an isomorphism and $\mathbb{Z}/4\mathbb{Z} \cong (\mathbb{Z}/5\mathbb{Z})^\times$

Group structure on $\pi_1(X)$

If p and p' are two paths, we previously defined $p \cdot p'$

This works for loops since loops are paths

So, if $\gamma, \gamma' : [0,1] \to X$ are loops, then $\gamma \cdot \gamma' : [0,1] \to X$ is given by

$$(\gamma \cdot \gamma')(t) = \begin{cases} \gamma(2t) & 0 \leq t \leq 1/2 \\ \gamma'(2t-1) & 1/2 \leq t \leq 1. \end{cases}$$

Exercise: If $\gamma_1 \simeq \gamma_1'$, then show $\gamma_1 \cdot \gamma_2 \simeq \gamma_1' \cdot \gamma_2$ and $\gamma_2 \cdot \gamma_1 \simeq \gamma_2 \cdot \gamma_1'$

Hence this gives a well-defined operation on $\pi_1(X)$:

Definition

If $[\gamma], [\gamma'] \in \pi_1(X)$, then $[\gamma] \cdot [\gamma'] := [\gamma \cdot \gamma']$

Theorem

$\pi_1(X)$ *a group with the operation* $[\gamma] \cdot [\gamma'] := [\gamma \cdot \gamma']$

Proof.

It is an exercise to check the following facts:

- We can take $1 := c_{x_0}$ since $[c_{x_0}] \cdot [\gamma] = [\gamma] = [\gamma] \cdot [c_{x_0}]$
- If $[\gamma] \in \pi_1(X)$, then $[\gamma] \cdot [\gamma^{-1}] = [c_{x_0}]$
- $([\gamma_1] \cdot [\gamma_2]) \cdot [\gamma_3] = [\gamma_1] \cdot ([\gamma_2] \cdot [\gamma_3])$ □

The proofs amount to re-parametrisation: 'slowing down and speeding up time'

Since $\pi_1(X)$ is a group, we would hope it contains more information about X than the equivalence class of its underlying set

It turns out that the group structure is a homotopy invariant too:

Theorem

If $X \simeq Y$, then $\pi_1(X) \cong \pi_1(Y)$ are isomorphic as groups

Proof.

Our proof that there is bijection $\pi_1(X) \cong \pi_1(Y)$ involved showing the following two maps are bijective:

- If p is a path from x_0 to x_1, then

$$p_* : \pi_1(X, x_0) \to \pi_1(X, x_1), \quad \gamma \mapsto p^{-1} \cdot \gamma \cdot p$$

- If $f : X \to Y$ is a map and $y_0 = f(x_0)$, then the map

$$f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0), \quad \gamma \mapsto f \circ \gamma$$

It is easy to see that p_* and f_* are group homomorphisms

The result follows in a similar way □

Computing $\pi_1(S^1)$

We want to compute $\pi_1(S^1)$ as a group

Take $S^1 := \{e^{i\theta} : \theta \in [0, 2\pi]\} \subseteq \mathbb{C}$ and basepoint $1 \in S^1$

Consider the loop $\gamma_1 : [0,1] \to S^1$, $\gamma_1(\theta) = e^{2\pi i\theta}$ which wraps around S^1 once anticlockwise

For each $n \in \mathbb{Z}$, we have $\gamma_n : [0,1] \to S^1$, $\gamma_n(\theta) = (e^{2\pi i\theta})^n$ which wraps around n times anticlockwise (clockwise) for $n \geq 0$ ($n \leq 0$)

We have $\gamma_n \simeq \gamma_1^n$ and, more generally, $\gamma_n \cdot \gamma_m \simeq \gamma_{n+m}$

The main goal of this lecture is to prove:

Theorem

There is an isomorphism:

$$f : \mathbb{Z} \to \pi_1(S^1), \quad n \mapsto \gamma_n$$

In particular, $\pi_1(S^1) \cong \mathbb{Z}$

In order to prove this, we need to show:

- If $\gamma \in \pi_1(S^1)$, then $\gamma \simeq \gamma_n$ for some $n \in \mathbb{Z}$ (f is surjective)
- If $n \neq m$, then $\gamma_n \not\simeq \gamma_m$ (f is injective)

f is a group homomorphism since

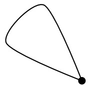
$$f(n+m) = \gamma_{n+m} \simeq \gamma_n \cdot \gamma_m = f(n) \cdot f(m)$$

Hence, if these two things hold, then f is an isomorphism

Task 1: If $\gamma \in \pi_1(S^1)$, then $\gamma \simeq \gamma_n$ for some $n \in \mathbb{Z}$

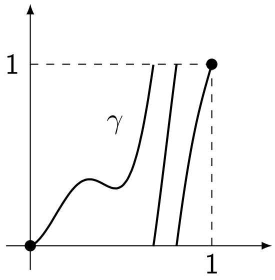
How should we picture an arbitrary loop $\gamma : [0,1] \to S^1$?

- S^1 is homeomorphic to the graph $V = \{v\}$, $E = \{(v,v)\}$:



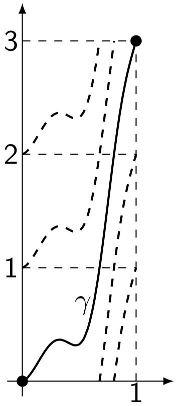
- Hence $S^1 \cong [0,1]/(0 \sim 1)$, i.e. the interval $[0,1]$ but with the points 0 and 1 identified
- In this model for S^1, we take the basepoint to be 0 $(= 1)$

So we can view $\gamma : [0,1] \to S^1$ as a map $\gamma : [0,1] \to [0,1]/(0 \sim 1)$:



A more useful way to draw this:

- Stack different copies of $[0, 1]$ up the vertical axis
- Allow γ to pass into each region when it passes through 0 or 1

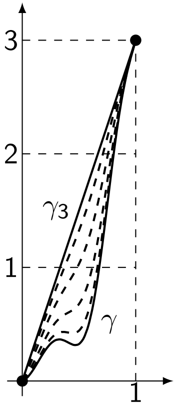


Here the dashed lines correspond to possible starting points for γ

Recall that $\gamma_3 : [0,1] \to S^1$, $\theta \mapsto e^{6\pi i\theta}$

This is the same as $\gamma_3 : [0,1] \to [0,1]/(0 \sim 1)$, $\theta \mapsto 3\theta$

The following picture shows that $\gamma \simeq \gamma_3$:



We now want to turn this into a rigorous argument

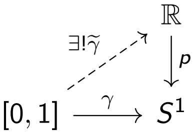
Formally, these drawing are maps $\widetilde{\gamma} : [0,1] \to \mathbb{R}$ such that $p \circ \widetilde{\gamma} = \gamma$ where $p : \mathbb{R} \to [0,1]$, $x \mapsto x \pmod 1$

Definition

If $\gamma : [0,1] \to S^1$ is a loop, then a map $\widetilde{\gamma} : I \to \mathbb{R}$ is a *lift* of γ if $p \circ \widetilde{\gamma} = \gamma$

Lemma (Lifting)

Every loop $\gamma : [0,1] \to S^1$ *has a unique lift* $\widetilde{\gamma} : [0,1] \to \mathbb{R}$ *with* $\widetilde{\gamma}(0) = 0$



Proof: By analysis, γ crosses the basepoint only finitely many times

We can define $\widetilde{\gamma}$ piecewise between each crossing point

Uniqueness follows similarly □

So let $\gamma \in \pi_1(S^1)$ and let $\widetilde{\gamma} : [0, 1] \to \mathbb{R}$ be its lift

Since $p(\widetilde{\gamma}(1)) = \gamma(1) = 0$, we have that $\widetilde{\gamma}(1) = n \in \mathbb{Z}$

We now want to formalise the second picture to show that $\widetilde{\gamma} \simeq \widetilde{\gamma}_n$ where $\widetilde{\gamma}_n(x) = nx$

We defined '$\simeq$' for loops. A similar definition works for all paths:

Definition

Two paths $p, p' : [0, 1] \to X$ are *based homotopy equivalent* ($p \simeq p'$) if there exists a continuously varying one-parameter family of maps $H_t : [0, 1] \to X$ such that $H_0 = p$, $H_1 = p'$, **and $H_t(0)$, $H_t(1)$ are fixed for all $t \in [0, 1]$**

We want to find a based homotopy $\widetilde{\gamma} \simeq \widetilde{\gamma}_n$

Define $\widetilde{H}_t : [0, 1] \to \mathbb{R}$ by:

$$\widetilde{H}_t(x) = (1 - t)\widetilde{\gamma}(x) + t\widetilde{\gamma}_n(x)$$

so that $\widetilde{H}_0 = \widetilde{\gamma}$, $\widetilde{H}_1 = \widetilde{\gamma}_n$ and $\widetilde{H}_t(0) = 0$, $\widetilde{H}_t(1) = n$ for all $t \in [0, 1]$

Hence $\widetilde{\gamma} \simeq \widetilde{\gamma}_n$

Then $H_t = p \circ \widetilde{H}_t : [0, 1] \to S^1$ has $H_0 = p_0 \circ \widetilde{\gamma} = \gamma$,
$H_1 = p \circ \widetilde{\gamma}_n = \gamma_n$ and $H_t(0) = 0$ for all $t \in [0, 1]$

Hence $\gamma \simeq \gamma_n$, i.e. they are equivalent as loops $\square$

Task 2: If $n \neq m$, then $\gamma_n \not\simeq \gamma_m$

This can be proven using the same idea

$\gamma_n \simeq \gamma_m$ and let $\widetilde{\gamma}_n$ and $\widetilde{\gamma}_m$ be lifts. Is $\widetilde{\gamma}_n \simeq \widetilde{\gamma}_m$?

Recall that a homotopy $H_t : [0,1] \to X$ for $t \in [0,1]$ is really a map $H : [0,1] \times [0,1] \to X$ with $H(t,x) = H_t(x)$

Lemma (Homotopy lifting)

Every based homotopy $H : [0,1] \times [0,1] \to S^1$ has a unique lift $\widetilde{H} : [0,1] \times [0,1] \to \mathbb{R}$ which is a based homotopy

$$\begin{array}{ccc} & \overset{\exists! \widetilde{H}}{\nearrow} & \mathbb{R} \\ & & \downarrow p \\ [0,1] \times [0,1] & \xrightarrow{H} & S^1 \end{array}$$

This implies $\widetilde{\gamma}_n \simeq \widetilde{\gamma}_m$

Since $\widetilde{H}(t,0) = 0$, $\widetilde{H}(t,1)$ are fixed: $n = \widetilde{\gamma}_n(1) = \widetilde{\gamma}_m(1) = m$ $\square$

So $\pi_1(S^1) \cong \mathbb{Z}$ and S^1 is not simply connected

We saw previously that $\chi(S^1) = \chi(T) = 0$ where T is the torus

Is $S^1 \simeq T$?

By an earlier exercise, $T \cong S^1 \times S^1$

Exercise: Show that the map

$$\psi : \pi_1(X) \times \pi_1(Y) \to \pi_1(X \times Y), \quad (\gamma, \gamma') \mapsto \gamma \times \gamma'$$

is a group isomorphism, where $(\gamma \times \gamma')(x) = (\gamma(x), \gamma'(x)) \in X \times Y$

Hence $\pi_1(T) \cong \pi_1(S^1 \times S^1) \cong \pi_1(S^1) \times \pi_1(S^1) \cong \mathbb{Z} \times \mathbb{Z}$
(this is a the group with $(a, b) + (c, d) := (a + c, b + d)$)

Lemma

$\mathbb{Z} \ncong \mathbb{Z} \times \mathbb{Z}$ as groups

Proof.

Let $f : \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ is a group isomorphism and let $(a, b) = f(1)$

Since f is a homomorphism, we have $f(n) = n \cdot f(1) = (na, nb)$

Since f is bijective $(1, 0), (0, 1) \in \mathrm{Im}(f)$

- $(1, 0) \in \mathrm{Im}(f) \Rightarrow (1, 0) = (na, nb)$ for some $n \in \mathbb{Z} \Rightarrow a = 0$
- $(0, 1) \in \mathrm{Im}(f) \Rightarrow (0, 1) = (na, nb)$ for some $n \in \mathbb{Z} \Rightarrow b = 0$

Hence $f(1) = (0, 0)$ which is a contradiction □

If $S^1 \simeq T$, then $\mathbb{Z} \cong \pi_1(S^1) \cong \pi_1(T) \cong \mathbb{Z} \times \mathbb{Z}$

Hence $S^1 \not\simeq T$ by the lemma above