Introduction to Algebraic Topology

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Lecture 4. Classification problems

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Recap

Let X be a metric space

We showed that $\pi_1(X) := \{\text{loops at } x_0\}/\simeq$ is a group under concatenation of loops $\gamma \cdot \gamma'$

Theorem

$\pi_1(S^1) \cong \mathbb{Z}$

We showed that $\pi_1(T) \cong \mathbb{Z} \times \mathbb{Z}$

$\chi(T) = \chi(S^1) = 0$ and $\mathbb{Z} \cong \mathbb{Z} \times \mathbb{Z}$ as sets (both countable)

However $\mathbb{Z} \ncong \mathbb{Z} \times \mathbb{Z}$ as groups and so $S^1 \not\simeq T$

Hence the fundamental group π_1 contains new information

Applications of $\pi_1(S^1) \cong \mathbb{Z}$

Theorem (The Fundamental Theorem of Algebra)

For every non-constant polynomial has a root, i.e. for every

$$f(x) = \sum_{k=0}^{n} a_k x^k,$$

with $n \geqslant 1$, $a_k \in \mathbb{C}$ and $a_n \neq 0$, there exists $z \in \mathbb{C}$ with $f(z) = 0$

Corollary

If $n \geq 1$, every degree n polynomial has at most n complex roots

Proof.

Use the Euclidean algorithm for polynomials and induction $\square$

Proof: Suppose f has no roots over $\mathbb{C}$ and assume f is monic

Then f is a continuous function $f : \mathbb{C} \to \mathbb{C} \setminus \{0\}$

Recall that there is a group homomorphism

$$f_* : \pi_1(\mathbb{C}) \to \pi_1(\mathbb{C} \setminus \{0\}), \quad \gamma \mapsto f \circ \gamma.$$

- $\mathbb{C} \cong \mathbb{R}^2 \simeq * \Rightarrow \pi_1(\mathbb{C}) \cong \pi_1(*) = 1$
- $\mathbb{C} \setminus \{0\} \simeq S^1 \Rightarrow \pi_1(\mathbb{C} \setminus \{0\}) \cong \mathbb{Z}$

Hence f_* is the group homomorphism

$$f_* : \{0\} \to \mathbb{Z}, \quad 0 \mapsto 0$$

and so $f \circ \gamma \simeq c_{x_0}$ for all loops $\gamma : [0,1] \to \mathbb{C}$

We now want to obtain a contradiction by exhibiting a loop $\gamma : [0,1] \to \mathbb{C}$ for which $f \circ \gamma \not\simeq c_{x_0}$.

Recall from earlier that $\gamma_n : [0,1] \to S^1 \hookrightarrow \mathbb{C}$, $\theta \mapsto e^{2\pi i n\theta}$ is the loop which wraps around the circle n times

For r large, we have that

$$f\left(re^{i\theta}\right) = \sum_{k=0}^{n} a_k r^k e^{ik\theta} \approx r^n e^{in\theta}$$

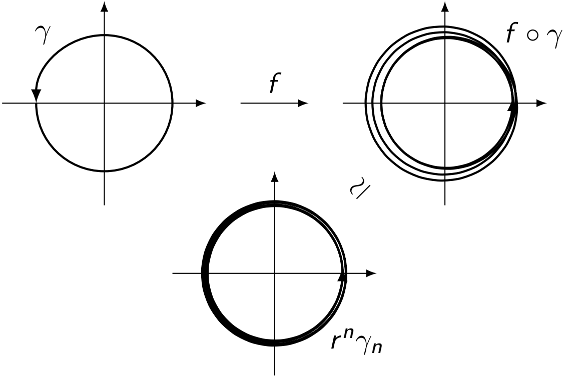
Let $\gamma = r\gamma_1$ be the loop which goes once around the circle of radius r anticlockwise

Then $f(\gamma(\theta)) = f(re^{2\pi i\theta}) \approx r^n e^{2\pi i n\theta} = r^n\gamma_n(\theta)$ for r large

The '$\approx$' can be turned into a homotopy since we can move between the loops without passing through $0 \notin \mathbb{C} \setminus \{0\}$

Hence $f \circ \gamma \simeq r^n\gamma_n$.

The picture is as follows:



By shrinking the radius, we have $r^n \gamma_n \simeq \gamma_n$

So $f \circ \gamma \simeq \gamma_n \in \pi_1(S^1)$ which, under $\pi_1(S^1) \cong \mathbb{Z}$, is $n \in \mathbb{Z}$

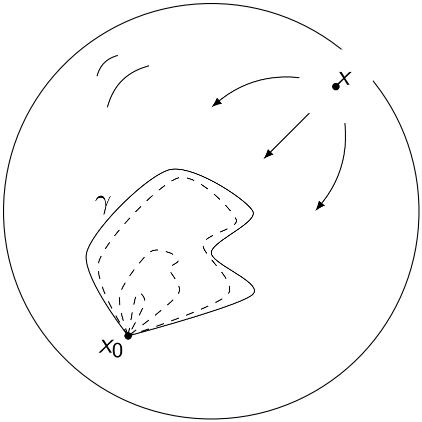
Since $n \geq 1$, $f \circ \gamma \simeq \gamma_n \not\simeq c_{x_0}$

This is a contradiction $\square$

Example: $\pi_1(S^2) = 1$

If $\gamma : [0.1] \to S^2$ is a loop, pick $x \in S^2$ not in the image of γ

Then $\gamma \simeq c_{x_0}$ by moving each point on γ towards x_0 along the arc from x_0 to x:

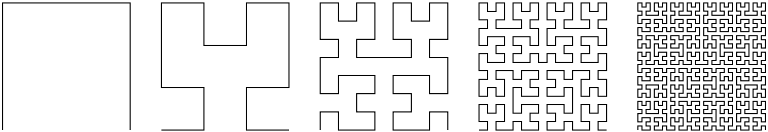


However, this proof does not work. Why?

There exists continuous surjective functions $f : [0,1] \to S^2$ (!)

Such a map is called a *space filling curve*

Example: we obtain a continuous surjection $f : [0,1] \to [0,1]^2$ as the limit of the following sequence of continuous functions



To fix the proof that $\pi_1(S^2) = 1$: we need to 'wiggle' γ so that it cannot be surjective

This requires some work

Does there exists spaces X and Y such that $\chi(X) = \chi(Y)$, $\pi_1(X) \cong \pi_1(Y)$ but $X \not\simeq Y$?

- We have $\pi_1(S^2) = 1$, $\chi(S^2) = 2$
- More generally $\pi_1(S^n) = 1$, $\chi(S^n) = 1 + (-1)^n$ for $n \geq 2$
- It can be shown that $S^n \simeq S^m$ if and only if $n = m$

Hence $\chi(S^2) = \chi(S^4)$ and $\pi_1(S^2) \cong \pi_1(S^4)$ but $S^2 \not\simeq S^4$

What about if we restrict to 2-complexes?

Question

Let X and Y be 2-complexes with $\chi(X) = \chi(Y)$ and $\pi_1(X) \cong \pi_1(Y)$. Is $X \simeq Y$?

If so, then these two invariants would be all we need to classify 2-dimensional cell complexes up to homotopy equivalence

Classification of 1-dimensional cell complexes

We will break the classification of 1-complexes (graphs) into two distinct stages:

- Show that every graph is homotopy equivalent to a graph in 'standard form'
- Show that two graphs in standard form are homotopy equivalent if and only if they are the same graph

As usual, we will assume that the graphs are path-connected

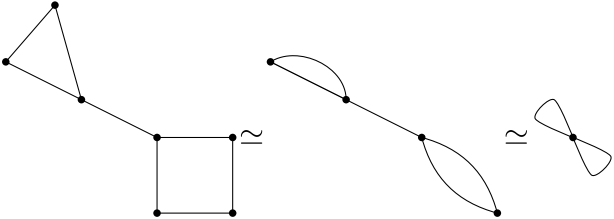
Definition

Let $X = (V, E)$ be a graph and $e = (v_1, v_2) \in E$

Define X/e to be the graph formed by deleting the edge e and combining the vertices v_1 and v_2 (this is *edge contraction*)

Exercise: Given a graph X with vertices $v_1 \neq v_2$ and $e = (v_1, v_2) \in E(X)$, then $X \simeq X/e$

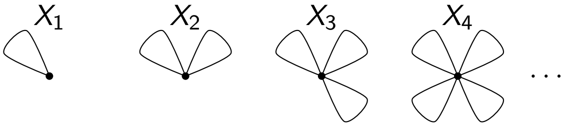
Example:



This suggests the following as our standard form for graphs:

Definition

The flower with n petals X_n will be the unique graph with a single vertex and $n \geq 0$ edges



Theorem

If X is a graph, then $X \simeq X_n$ for some $n \geqslant 0$

Proof.

If X has more than one vertex then, since X is path-connected, there exists an edge $e = (v_1, v_2)$ with $v_1 \neq v_2$

$X \simeq X/e$ and X/e has one less vertex

By induction, $X \simeq \{\text{graph with a single vertex}\}$ $\square$

Lemma

$X_n \simeq X_m$ if and only if $n = m$.

Proof.

If $X_n \simeq X_m$, then $\chi(X_n) = \chi(X_m)$

For all $n \geq 1$, we have $\chi(X_n) = 1 - n$

Hence $1 - n = 1 - m$ which implies that $n = m$ $\square$

In particular, we have shown the following:

Theorem (Classification of 1-complexes)

Let X and Y be 1-dimensional cell complexes

Then $X \simeq Y$ if and only if $\chi(X) = \chi(Y)$

Classification of 2-dimensional cell complexes

We want to follow the same approach again

What does it mean to put a 2-complex into standard form?

Lemma

If X is a 2-complex, then $X \simeq X'$ where X' is a 2-complex with a single vertex

Proof.

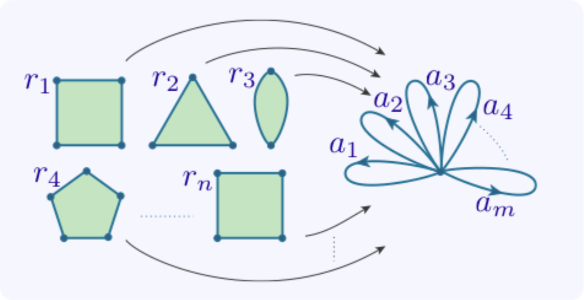
The proof amounts to checking that $X \simeq X/e$ is still true for edges $e = (v_1, v_2)$ with $v_1 \neq v_2$ $\square$

This allows us to restrict out attention to 2-complexes with a single vertex (!)

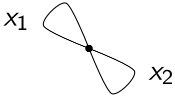
A 2-complex X with a single vertex has the following description:

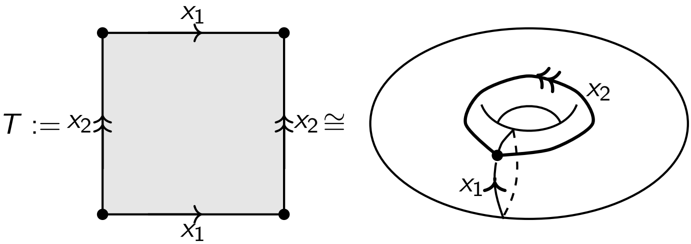
- Take the flower with n petals X_n and label each petal with the symbols $x_1, \ldots, x_n$
- The attaching paths for the faces then correspond to words $w_1, \cdots, w_m$ in the symbols

$$\{x_1, x_1^{-1}, x_2, x_2^{-1}, \ldots, x_n, x_n^{-1}\}$$



Example: For the torus T be have





So T can be specified by the data:

- x_1, x_2 (labels for the edges)
- $w_1 := x_2 x_1 x_2^{-1} x_1^{-1}$ (attaching maps for the faces)

This should remind us of group presentations
(if we have seen them before)

Group presentations

Let $x_1, \cdots, x_n$ be formal labels and let $w_1, \cdots, w_n$ be words in $\left\{x_1, x_1^{-1}, x_2, x_2^{-1}, \cdots, x_n, x_n^{-1}\right\}$, e.g. $w_1 = x_{i_1}^{\pm 1} \cdots x_{i_k}^{\pm 1}$

We can construct a group out of this data as follows:

- Let S be the set of words in $\left\{x_1, x_1^{-1}, x_2, x_2^{-1}, \ldots, x_n, x_n^{-1}\right\}$
- Let $\sim$ be the unique equivalence relation on S such that $w_i \sim 1$, $x_i x_i^{-1} \sim 1$ and $a \sim b$, $c \sim d$ implies $a \cdot c \sim b \cdot d$
- Then the equivalence classes $S/\sim$ is a group under $\cdot$

We let $S/\sim$ be denoted by $\mathcal{P} = \langle x_1, \cdots, x_n \mid w_1, \cdots, w_m \rangle$

We call $\mathcal{P}$ a *group presentation*

Let $X_{\mathcal{P}}$ denote the 2-complex with $x_1, \cdots, x_n$ (labels for the edges) and $w_1, \cdots, w_m$ (attaching maps for the faces)

Example: Let S be the set of words in $\left\{x_1, x_1^{-1}, x_2, x_2^{-1}\right\}$

We want to determine $G = \langle x_1, x_2 \mid x_2 x_1 x_2^{-1} x_1^{-1} \rangle$ as a group

Firstly, $x_2 x_1 x_2^{-1} x_1^{-1} \sim 1$ is equivalent to $x_2 x_1 \sim x_1 x_2$
(multiply both sides on the right by $x_1 x_2$)

$\Rightarrow$ The order of multiplication of x_1 and x_2 doesn't matter

$\Rightarrow$ Every $w \in S$ has $w \simeq x_1^n x_2^m$ for some $n, m \in \mathbb{Z}$

Exercise: Check that there is a group isomorphism

$$f : \mathbb{Z} \times \mathbb{Z} \to \langle x_1, x_2 \mid x_2 x_1 x_2^{-1} x_1^{-1} \rangle, \quad (n, m) \mapsto x_1^n x_2^m$$

Hence $\langle x_1, x_2 \mid x_2 x_1 x_2^{-1} x_1^{-1} \rangle \cong \mathbb{Z} \times \mathbb{Z} \cong \pi_1(T)$

This phenomena is completely general:

Theorem

Let X be a 2-complex with a single vertex which is specified by

- $x_1, \cdots, x_n$ *(labels for the edges)*
- $w_1, \cdots, w_m$ *(attaching maps for the faces)*

Then $\pi_1(X) \cong \langle x_1, \cdots, x_n \mid w_1, \cdots, w_m \rangle$ *and* $\chi(X) = 1 - n + m$

Does $\chi(X) = \chi(Y)$ and $\pi_1(X) \cong \pi_1(Y)$ imply $X \simeq Y$?

Definition

We say that a pair of group presentations

$$\mathcal{P} = \langle x_1, \cdots, x_n \mid w_1, \cdots, w_m \rangle, \quad \mathcal{P}' = \langle x_1', \cdots, x_{n'}' \mid w_1', \cdots, w_{m'}' \rangle$$

are *exotic* if $\mathcal{P} \cong \mathcal{P}'$ as groups, $n - m = n' - m'$ and $X_{\mathcal{P}} \not\simeq X_{\mathcal{P}'}$

Question

Do exotic presentations exist?

- If $\mathcal{P}$ and $\mathcal{P}'$ are exotic presentations, then $\chi(X_{\mathcal{P}}) = \chi(X_{\mathcal{P}'})$ and $\pi_1(X_{\mathcal{P}}) \cong \pi_1(X_{\mathcal{P}'})$ but $X_{\mathcal{P}} \not\simeq X_{\mathcal{P}'}$
- Hence exotic presentations exist if and only if if is *not true* that "$\chi(X) = \chi(Y)$ and $\pi_1(X) \cong \pi_1(Y)$ implies $X \simeq Y$"

Despite being considered by J. H. C. Whitehead in the 1940s, it wasn't until 1976 that the first examples were found

We will write z^w to mean $w^{-1}zw$

Theorem (Martin Dunwoody, 1976)

There are exotic presentations

$$\mathcal{P} = \langle x_1, x_2 \mid x_1^2 x_2^{-3}, 1\rangle, \quad \mathcal{P}' = \langle x_1, x_2 \mid w_1, w_2\rangle$$

where $w_1 = x_1^2x_2^{-3}(x_1^2x_2^{-3})^{x_1}(x_1^2x_2^{-3})^{x_1^2}$ *and*
$w_2 = x_1^2x_2^{-3}(x_1^2x_2^{-3})^{x_2}(x_1^2x_2^{-3})^{x_2^2}(x_1^2x_2^{-3})^{x_2^3}$

Theorem (Wolfgang Metzler, 1976)

$$\mathcal{P} = \langle x_1, x_2, x_3 \mid x_1^3, x_2^3, x_3^3, x_1x_2x_1^{-1}x_2^{-1}, x_2x_3x_2^{-1}x_3^{-1}, x_3x_1x_3^{-1}x_1^{-1}\rangle$$
$$\mathcal{P}' = \langle x_1, x_2, x_3 \mid x_1^3, x_2^3, x_3^3, x_1^2x_2x_1^{-2}x_2^{-1}, x_2x_3x_2^{-1}x_3^{-1}, x_3x_1x_3^{-1}x_1^{-1}\rangle$$

are exotic presentations for the group $\mathbb{Z}/5 \times \mathbb{Z}/5 \times \mathbb{Z}/5$

Hence the answer to our question is no

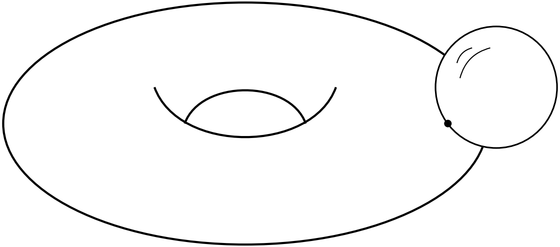
What information about 2-complexes does $\chi(X)$, $\pi_1(X)$ contain?

Definition

If X and Y are cell complexes, then $X \vee Y$ is the cell complex formed by attaching X and Y at vertices $v_x \in X$ and $v_Y \in Y$

- This is independent of the choice of v_X, v_Y (up to homotopy)

Example: If $X = T$ is the torus and $Y = S^2$ is the sphere, then $T \vee S^2$ is a 'torus with a pimple'



Definition

Two 2-complexes X and Y are *stable homotopy equivalent* $(X \sim Y)$ if there exists $r \geqslant 0$ for which

$$X \vee \underbrace{S^2 \vee \cdots \vee S^2}_{r} \simeq Y \vee \underbrace{S^2 \vee \cdots \vee S^2}_{r}$$

(Note: this is not the same as 'stable homotopy theory')

Theorem

Let X and Y be 2-complexes. Then $\chi(X) = \chi(Y)$ and $\pi_1(X) \cong \pi_1(Y)$ if and only if $X \sim Y$

The existence of exotic presentations shows that $X \sim Y$ does not imply $X \simeq Y$

Conclusion

In this course, we introduced invariants $\chi(X)$, $\pi_1(X)$ and showed:

- $\chi(X)$ determines 1-complexes up to homotopy equivalence
- $\chi(X)$, $\pi_1(X)$ determines 2-complexes up to stable homotopy equivalence (but not up to homotopy equivalence)

What other invariants do we need to determine 2-complexes up to homotopy equivalence?

What about for cell complexes in higher dimensions?

Example of other invariants:

- $\pi_n(X)$ for $n \geq 1$ (the **higher homotopy groups**)
- $H_i(X)$ for $i \geq 0$ (the **homology groups**)
- $H^*(X) = \bigoplus_{i \geq 0} H^i(X)$ (the **cohomology ring**)

This is just the start of a long and interesting story...

Thank you for listening!