# Introduction to Algebraic Topology

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**Abstract**

These notes are written to accompany the lecture course 'Introduction to Algebraic Topology' that was taught to advanced high school students during the Ross Mathematics Program in Columbus, Ohio from July 15th-19th, 2019. The course was taught over five lectures of 1-1.5 hours and the students were assumed to have some experience working with groups and rings, as well as some familiarity with the definition of a metric space and a continuous function.

These notes were adapted from the notes taken down by Alex Feiner and Ojaswi Acharya. Special thanks go to Alex for creating most of the diagrams. We would also like to thank Evan O'Dorney for comments and corrections. Further comments and corrections may be sent to j.k.nicholson@ucl.ac.uk.

## Contents

# 1 Spaces and Equivalences

In order to do topology, we will need two things. Firstly, we will need a notation of 'space' that will allow us to ask precise questions about objects like a sphere or a torus (the outside shell of a doughnut).

We will also need a notation of 'equivalence' of these spaces which, unlike in geometry, would say that a square and a hexagon are equivalent since we can stretch one shape into the other if they were made out of a flexible material.

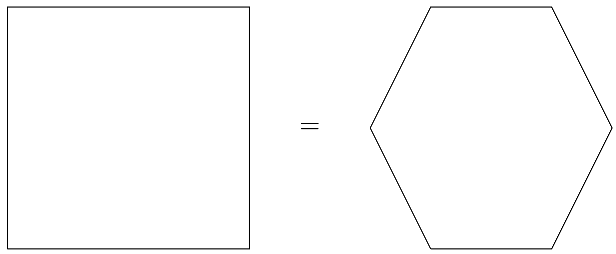


*Figure 1: We would like to view a square and a hexagon as equivalent spaces.*

## 1.1 Notions of Space

One way we can define a space is as a subset of $\mathbb{R}^n$. However, it does not suffice to consider these objects merely as sets since this loses all the information about the way the elements are arranged.

One way we can give sets $X$ and $Y$ extra structure is to decide which functions $f : X \to Y$ are going to be continuous (i.e. 'preserving the structure of $X$ and $Y$ as a space'). We will do this by asking that $X$ and $Y$ are metric spaces:

**Definition 1.1.** A metric space is defined as a set $X$ and a distance function $d : X \times X \to \mathbb{R}_{\geqslant 0}$ such that

i $d(x, y) = 0 \iff x = y$

ii $d(x, y) = d(y, x)$ for all $x, y \in X$

iii $d(x, z) \leqslant d(x, y) + d(y, z)$ for all $x, y, z \in X$.

**Definition 1.2.** If $X$ and $Y$ are metric spaces, we can then declare a function $f : X \to Y$ to be continuous if, for every $\varepsilon > 0$, there is a $\delta > 0$ such that $d(x, y) < \delta$ implies $d(f(x), f(y)) < \varepsilon$.

**Remark 1.3.** It turns out that there is actually a more generally notation known as a 'topological space'. This definition is the answer to the question: what is the least amount of information we need to add to a set in order to have a meaningful notion of continuous function? However, most spaces of interest are metric spaces and so we will not consider general topological spaces in this course.

### 1.1.1 Subsets of $\mathbb{R}^n$

In particular, any subset $X \subseteq \mathbb{R}^n, n \geq 1$ can be viewed as a metric space with the usual distance function

$$d((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}.$$

This will allow us define some well-known spaces:

**Example 1.4.**

i The simplest example is $n$-dimensional space $\mathbb{R}^n$

ii The $n$-sphere $S^n = \left\{ (x_0, \ldots, x_n) \ \subseteq \mathbb{R}^{n+1} \ \middle| \ \sum_{i=1}^{n} x_i^2 = 1 \right\} \subseteq \mathbb{R}^{n+1}$. $S^2$ is a sphere and $S^1$ is a circle (a one-dimensional sphere) iii The unit interval $I = [0, 1] \subseteq \mathbb{R}$

iv The point space $* = \{0\} \subseteq \mathbb{R}$

We can build new spaces from old ones in all the usual ways. For example, if $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$, then $X \times Y \subseteq \mathbb{R}^{n+m}$. Some spaces can be viewed as products in this way:

**Example 1.5.**

i The square $I^2$,

ii The cylinder $S^1 \times I$,

iii The torus $S^1 \times S^1$.

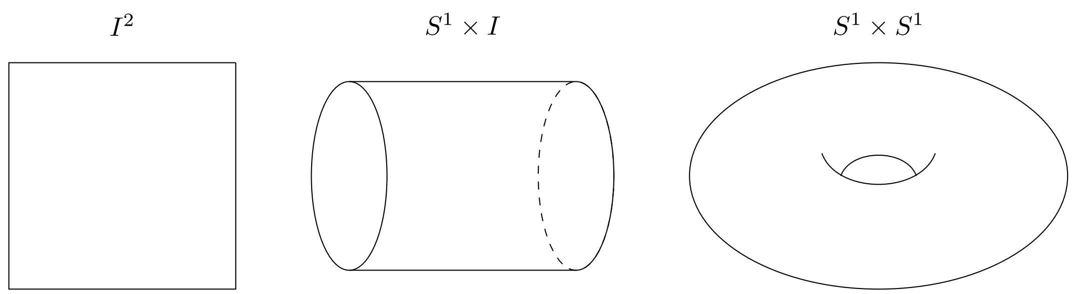



*Figure 2*

This last example may require some thought. Every point on the torus can be uniquely specified in terms of two points on the circle $S^1$. However $S^1 \times S^1 \subseteq \mathbb{R}^4$ we would usually think about the torus as a subset of $\mathbb{R}^3$. With some work, one can eventually come up with the following explicit formula:

$$T = \left\{ ((2 + \cos\theta)\cos\varphi, (2 + \sin\theta)\cos\varphi, \sin\varphi) \in \mathbb{R}^3 \;\middle|\; 0 \leqslant \theta, \varphi < 2\pi \right\}.$$

We will see later that $S^1 \times S^1$ and $T$ *are* equivalent spaces. Since the formula for $T$ was very ad-hoc, this is useful since we can talk about $S^1 \times S^1$ instead.

In fact, we can go one step further and define a new notion of space for which turning a drawing or intuitive picture of a space is much simpler.

### 1.1.2 Finite Cell Complexes

We will define $n$-complexes in the case $n = 1, 2$. This coincides with the definition of 'finite $n$-dimensional CW-complex' in the literature.

**Definition 1.6** (1-complex)**.** A 1-complex is defined as a graph $(V, E)$ where the vertex set $V = \{v_1, \ldots, v_n\}$ for some $n$ and the edge set $E = \{E_1, \ldots, E_m\} \subseteq V \times V$ for some $m$.

**Example 1.7.** $V = \{v_1, v_2, v_3, v_4\}, E = \{\{v_1, v_2\}, \{v_2, v_2\}, \{v_2, v_3\}, \{v_1, v_3\}, \{v_3, v_4\}\}$.

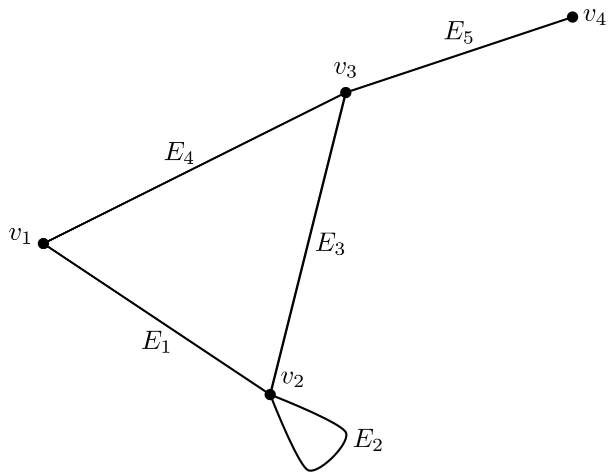



*Figure 3: A 1-complex.*

We can turn a 1-complex $(V, E)$ into a metric space $X$ using the diagram above. The set $X$ will the the union of intervals $[0, 1]$ corresponding to the edges, who overlap at the vertices. Here distances should be the shortest length of a path through the edges and vertices, i.e. if $x, y \in X$

$$d(x, y) = \min\{\text{length}(p) \mid p \text{ is a path from } x \text{ to } y\}$$

where a 'path' is defined as a sequence of segments of edges $E_i$ and each edge is given length one.

**Definition 1.8** (2-complex)**.** A 2-complex is defined as a triple $(V, E, F)$ consisting of a set of vertices $V = \{v_1, \ldots, v_n\}$, a set of edges $E = \{E_1, \ldots, E_m\}$ and a set of faces $F = \{F_1, \ldots, F_r\}$ such that $(V, E)$ is a graph and the faces are defined in either of the following ways:

- As sequences of edges $F_i = (E_{i_1}, \ldots, E_{i_k})$ where adjacent edges $(E_{i_t}, E_{i_{t+1}})$ share a common vertex for $t = 1, \ldots, k$ (taking $i_{k+1}$ to be $i_1$). We will think of this as 'the face attached along the path $(E_{i_1}, \ldots, E_{i_k})$.
- As a single vertex $v_i$. This will be the face attached at a single vertex $v_i$ by wrapping it up into the shape of a sphere.

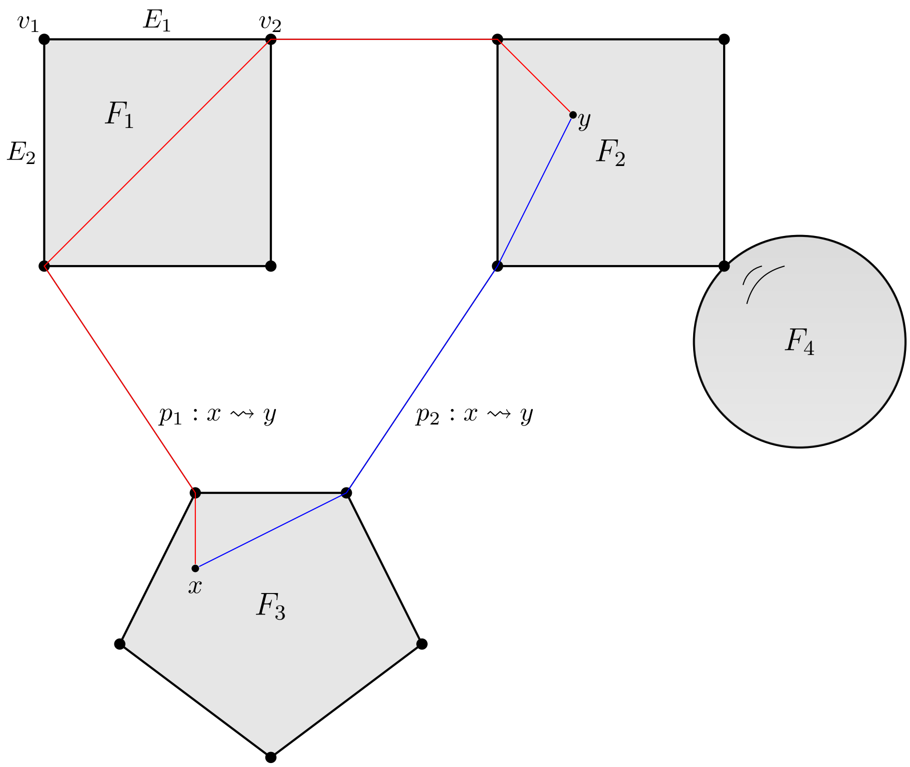



*Figure 4: A 2-complex.*

This can be turned into a metric space $X$ similarly to the case of 1-complexes. In order to define our notion of distance in $X$, we will build the faces $F_i$ attached along paths $(E_{i_1}, \ldots, E_{i_k})$ as $k$-gons with unit radius, and the faces $F_i$ attached along vertices as spheres of unit radius with distance the length of arcs along great circles.

## 1.2 Notions of Equivalence

We will now explore two different notions of equivalence. We will first consider homeomorphism: this can be thought of as 'topologically equivalent' and means that the spaces are the same in as we can bend and push the two shapes into each other. This is a relatively mild notion of equivalence since it 'preserves dimension' in the sense that $\mathbb{R}^n$ and $\mathbb{R}^m$ will not be homeomorphic unless $n = m$.

The second notion we will consider is homotopy equivalence which, in contrast, allows for much more violent deformations of spaces. For example, we will see that $\mathbb{R}^n$ is homotopy equivalent to a point for all $n$.

### 1.2.1 Homeomorphism

For the rest of the course, all functions between spaces will be assumed to be continuous unless otherwise stated.

**Definition 1.9.** We say that metric spaces $X$ and $Y$ are homeomorphic, denoted $X \cong Y$, if there are maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g = id_Y, g \circ f = id_X$. We say that such a map $f$ is a homeomorphism from $X$ to $Y$.

**Exercise 1.10.** Prove that $\cong$ is an equivalence relation.

The following result from analysis often makes life easier:

**Lemma 1.11.** Let $X$ and $Y$ be cell complexes. Then if $f : X \to Y$ is a continuous bijection there is a $g : Y \to X$ such that $f \circ g = id_Y$ and $g \circ f = id_X$.

**Remark 1.12.** In fact this holds for a larger class of metric spaces, namely those which are compact.

**Example 1.13.** $X = [0,1]$ and $Y = [0,2]$. Let $f(x) = 2x$ and $g(x) = \frac{1}{2}x$.

**Example 1.14.** $X = \mathbb{R}$ and $Y = (0,1)$. Let $f(x) = \frac{1}{1+e^{-x}}$, the sigmoid function. This has an explicit inverse $g(x) = -\log\left(\frac{1}{x} - 1\right)$. One can check both of these functions are continuous using the standard techniques from a first course in analysis.

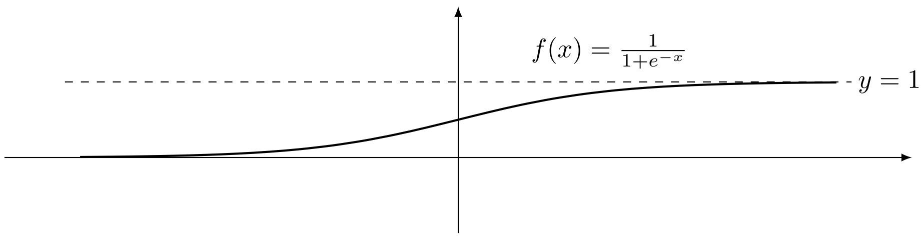



*Figure 5: The sigmoid function.*

**Example 1.15.** Let $X = \mathbb{R}^n$ for $n \geq 1$ and $Y = *$. Suppose there exists a homeomorphism $f : \mathbb{R}^n \to *$. Then there exists $g : * \to \mathbb{R}^n$ for which $f \circ g = id_*$ and $g \circ f = id_{\mathbb{R}^n}$. However, we must have that $f(x) = 0$ for all $x \in \mathbb{R}^n$ and $g(0) = x_0$ for some $x_0 \in \mathbb{R}^n$. In particular, $g(f(x)) = x_0$ for all $x \in \mathbb{R}^n$ and so $g \circ f \neq id_{\mathbb{R}^n}$. This is a contradiction and so $\mathbb{R}^n \ncong *$.

In fact, as mentioned earlier, a much stronger statement is true:

**Fact 1.16.** $\mathbb{R}^n \cong \mathbb{R}^m$ if and only if $n = m$.

This can be proven using techniques from algebraic topology, though the proof is difficult and well beyond the scope of this course.

### 1.2.2 Homotopy Equivalence

We would like to pick our definition of homotopy equivalent to allow for deformations which do not preserve dimension. It turns out that it will be easiest to first talk about homotopy equivalence of maps rather than spaces.

**Definition 1.17.** We say that two maps $f, g : [0,1] \to [0,1]$ are homotopy equivalent, denoted $f \simeq g$, if there is a one-parameter family of maps $H_t : [0,1] \to [0,1]$ for $0 \leqslant t \leqslant 1$ which vary continuously in $t$ and are such that $H_0 = f$ and $H_1 = g$. We say that $H$ is a homotopy from $f$ to $g$.

**Remark 1.18.** Here by 'vary continuously in $t$', we simply mean that the induced function $H : [0,1] \times [0,1] \to [0,1]$, $(x,t) \mapsto H_t(x)$ is continuous.

The picture we should have in our head is as follows. We views the function $f$ as fading into the shape of the function $g$ as time $t$ flows from 0 to 1.

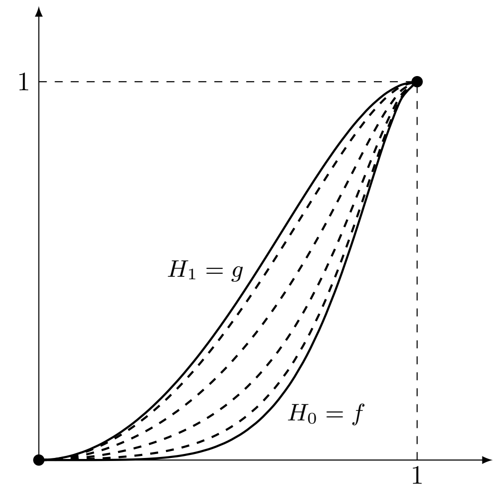



*Figure 6: Two functions $f$ and $g$ which are homotopy equivalent via $H$.*

Once we have this definition, it can be easily generalized as follows to other metric spaces.

**Definition 1.19.** We say that two maps $f, g : X \to Y$ are homotopy equivalent, denoted $f \simeq g$, if there is a one-parameter family of maps $H_t : X \to Y$, for $0 \leqslant t \leqslant 1$, which is continuous in $t$ and has the property that $H_0 = f$ and $H_1 = g$.

We can now define homotopy equivalence of spaces by taking our definition of homeomorphism but now only requiring that $f$ and $g$ are inverses up to homotopy equivalence.

**Definition 1.20.** Two metric spaces $X$ and $Y$ are homotopy equivalent, denoted $X \simeq Y$, if there exists functions $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \simeq id_Y$ and $g \circ f \simeq id_X$.

**Exercise 1.21.** Prove that $\simeq$ is an equivalence relation.

We will refer to the equivalence class of a space $X$ up to homotopy as the homotopy type of $X$. As expected, the definition implies that $X \cong Y \implies X \simeq Y$.

**Example 1.22.** We will show that $\mathbb{R}^n \simeq *$. Define functions $f : \mathbb{R}^n \to *, x \mapsto 1$ and $g : * \to \mathbb{R}^n, 1 \mapsto 0$. Then $f \circ g : * \to *, 1 \mapsto 1$, meaning that $f \circ g = id_*$. We also have that $g \circ f : \mathbb{R}^n \to \mathbb{R}^n, x \mapsto 0$.

The real work now amount to showing that $g \circ f \simeq id_{\mathbb{R}^n}$. If $H_t(x) = tx$, then $H_0 = 0 = g \circ f$ and $H_1 = id_{\mathbb{R}^n}$. Hence $g \circ f \simeq id_{\mathbb{R}^n}$, which implies that $\mathbb{R}^n \simeq *$.

**Example 1.23.** We will show that the circle $S^1$ is homotopy equivalent to the punctured complex plane $\mathbb{C} \setminus \{0\}$. The intuition is that you take the complex numbers and push everything inside $S^1$ outwards and everything outside $S^1$ inwards. As is often the case in these types of proofs, there are only two sensible choices for $f$ and $g$:

- $f : S^1 \to \mathbb{C} \setminus \{0\}, e^{i\theta} \mapsto e^{i\theta}$,
- $g : \mathbb{C} \setminus \{0\} \to S^1, re^{i\theta} \mapsto e^{i\theta}$.

It is easy to see that $g \circ f = id_{S^1}$. We now claim that $f \circ g \simeq id_{\mathbb{C} \setminus \{0\}}$. Since $f\left(g\left(re^{i\theta}\right)\right) = e^{i\theta}$, we need to find a one-parameter family of maps $H_t : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$ such that

$$H_0\left(re^{i\theta}\right) = re^{i\theta}, \quad H_1\left(re^{i\theta}\right) = e^{i\theta}.$$

This is achieved by the function $H_t\left(re^{i\theta}\right) = r^{1-t}e^{i\theta}$ satisfies this. Thus $S^1 \simeq \mathbb{C} \setminus \{0\}$.

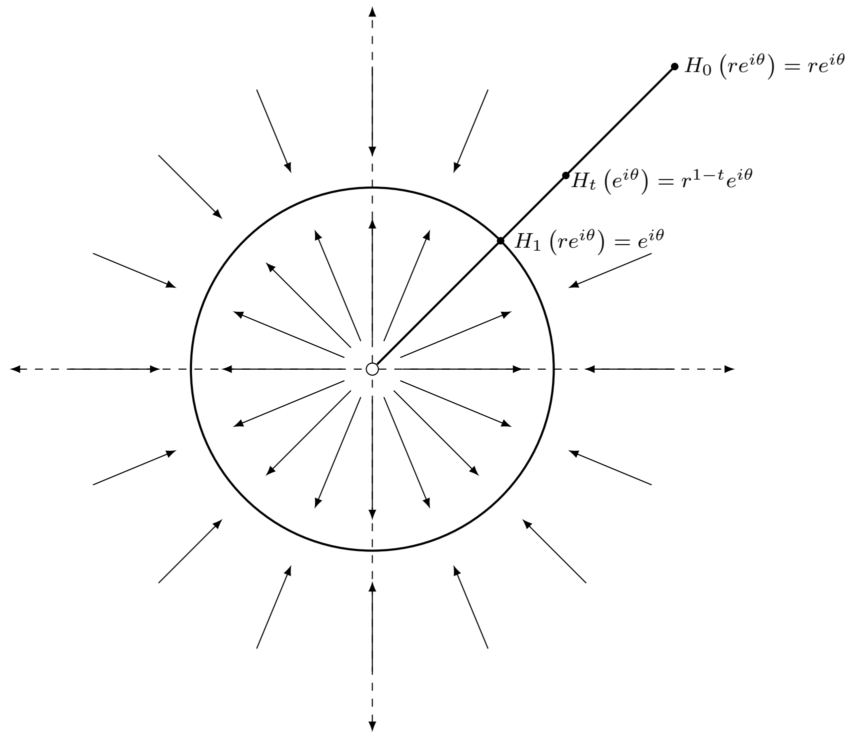



*Figure 7: The homotopy $H$ between $f \circ g$ and $id_{\mathbb{C}\setminus\{0\}}$.*

It is worth taking some time coming to terms with the diagram for the homotopy $H$. As time passes from 0 to 1, the point $re^{i\theta}$ moves towards the point $e^{i\theta}$ along a radial line.

# 2 Invariants

**Example 2.1.** Are the spaces $S^1$ and $*$ homotopy equivalent? The only possible maps between these spaces are as follows, where $x_0 \in S^1$:

- $f : S^1 \to *, e^{i\theta} \mapsto 0$,
- $g : * \to S^1, 0 \mapsto x_0$.

We have that $f \circ g = id_*$ and $c_{x_0} = g \circ f : S^1 \to S^1, e^{i\theta} \mapsto x_0$. Is $c_{x_0} \simeq id_*$?

We could try searching for a while for a homotopy between $c_{x_0}$ and $id_*$ but we would not find one. How can we prove that a homotopy equivalence does not exist?

The basic idea is to construct an invariant, namely a way to assign quantities $I(X)$ to spaces $X$ in such a way that $X \simeq Y$ implies that $I(X)$ and $I(Y)$ are equal. If we have such an invariant and we can show that $I(X)$ and $I(Y)$ are not equal, then we show that $X \not\simeq Y$. This applies to any equivalence relation on spaces.

## 2.1 The Euler Characteristic

Our first example of an invariant will be the Euler characteristic.

**Definition 2.2.** Let $X$ be a $n$-complex and let $f_i$ denote the number of cells in dimension $i$. Then the Euler characteristic of $X$ is $\chi(X) = \sum_{i=0}^{n} (-1)^i f_i$.

So $f_0$ denotes the number of vertices, $f_1$ denotes the number of edges and $f_2$ denotes the number of faces. For low-dimensional examples, we will write $V = f_0$, $E = f_1$ and $F = f_2$.

**Example 2.3.** We can compute the Euler characteristic for a few familiar figures:

i $\chi(*) = 1$

ii $\chi(\text{loop with one point}) = V - E = 1 - 1 = 0$,

iii $\chi(\underbrace{\text{loop with two points}}_{\simeq S^1}) = V - E = 2 - 2 = 0$

iv $\chi(\underbrace{\text{hollow cube with solid faces}}_{\simeq S^2}) = V - E + F = 8 - 12 + 6 = 2$

v $\chi(\text{dodecahedron}) = 2$

From these examples, we might be tempted to conjecture that, if two spaces are homeomorphic, then they have the same Euler characteristic. In fact, much more is true:

**Theorem 2.4.** Let $X$ and $Y$ be cell complexes. Then $X \simeq Y$ implies that $\chi(X) = \chi(Y)$, i.e. $\chi$ is a homotopy invariant.

This gives us a way to define Euler characteristic for metric spaces which do not come with the structure of a cell complex but which are simply homotopy equivalent to a cell complex. For example, we can define $\chi(S^1)$ to be $\chi(X)$ for either of the two examples considered above.

**Example 2.5.** If $S^1 \simeq *$, then $\underbrace{\chi\left(S^1\right)}_{=0} = \underbrace{\chi(*)}_{=1}$. Thus $S^1 \not\simeq *$.

We will not prove this theorem since it is tricky. We instead set the following as a challenge:

**Challenge 2.6.** Prove the $\chi$ is a homotopy invariant, but you may assume it is true in the case where $X \simeq *$, i.e. you may assume that $X \simeq * \implies \chi(X) = \chi(*) = 1$.

So the Euler characteristic is an invariant which is easy to compute but for which it is difficult to prove that it is homotopy invariant. Another drawback is that the invariant is integer valued and unlikely to contain much information.

## 2.2 The Fundamental Group

We would now like to construct a new invariant which is different to the Euler characteristic in these three respects. This time, homotopy will appear in the definition of our invariant. This will allow us to throw away enough information in our definition to ensure that homotopy equivalent of spaces have the same invariant. However, since homotopy appears in the definition itself, it will be hard to compute (at least from first principles).

Let $X$ be a metric space (whenever topologists think about a space they think about a torus). Fix a single point $x_0 \in X$ and consider all the paths in the space where you traverse the space and return to the point, formally we want to consider the set:

$$\{\text{loops at } x_0\} = \{\gamma : I \to X \mid \gamma(0) = \gamma(1) = x_0,\ \gamma \text{ is a map}\}.$$

The point $x_0$ is known as the basepoint, and we will often simply talk about 'loops' with the understanding that there is some basepoint which is fixed. We would like to use our homotopy to compare loops.

**Definition 2.7.** We say that loops $\gamma, \gamma' : I \to X$ are homotopy equivalent, written $\gamma \simeq \gamma'$, if there exists a continuously varying one-parameter family of loops $H_t : I \to X$ such that $H_0 = \gamma,\ H_1 = \gamma'$.

**Remark 2.8.** Note that this is not quite the same as a homotopy between $\gamma, \gamma' : I \to X$ considered merely as functions. Here we require that the homotopy $H_t$ is a loop for all $t$, i.e. that $H_t(0) = H_t(1) = x_0$ for all $0 \leqslant t \leqslant 1$. This is often referred to as a based homotopy.

To picture a homotopy, we should view two loops as stretchy pieces of string. We say they are equivalent if you can stretch that piece of stretchy string into the other while keeping its contact point fixed. In the diagram below $\gamma_0 \simeq \gamma_0'$, $\gamma_1 \simeq c_{x_0}$.

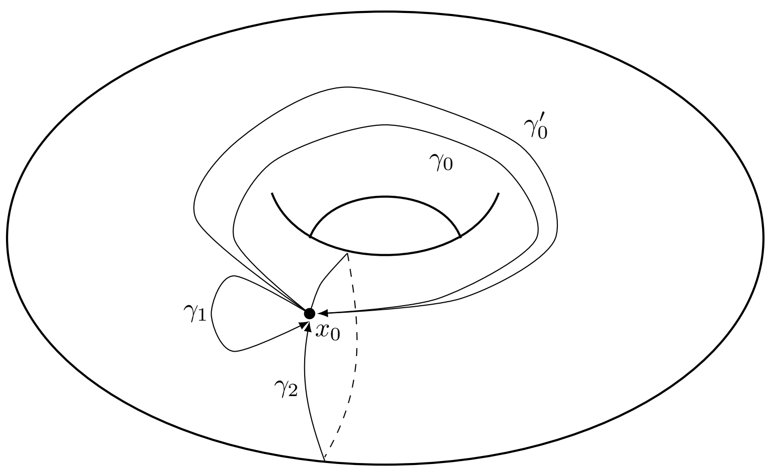



*Figure 8: A torus with multiple different paths drawn around its surface.*

It is also true that $\gamma_0$, $\gamma_1$ and $\gamma_2$ are not homotopy equivalent, though this will require some work to prove.

Our new invariant will be defined as the homotopy classes of these loops:

**Definition 2.9.** $\pi_1(X, x_0) := \{\text{loops at } x_0\}/\simeq$.

**Remark 2.10.** If $S$ is a set and $\simeq$ is an equivalence relation of $S$, then we write $S/\simeq$ for the equivalence classes of $\simeq$, i.e. the set $S$ modulo $\simeq$.

**Example 2.11.** Let $X = I = [0, 1], x_0 = 0$. Consider an arbitrary loop $\gamma : I \to I$ with $\gamma(0) = \gamma(1) = 0$. We claim that $\gamma \simeq c_0$. Recall that $c_0 : I \to I, x \mapsto 0$. Consider the function $H_t(x) = (1 - t)\gamma(x)$. We have that $H_0 = \gamma$ and $H_1 = 0$. Thus $\pi_1(I, 0) = \{c_0\}$.

This shows how violent the deformations induced by a homotopy can be: we are allowed to destroy things, which we can't do with a homeomorphism. We can picture this using the following diagram. At time $t = 0$, every point on $\gamma$ in pushed vertically downwards and reached the horizontal axis at the $t - 1$.

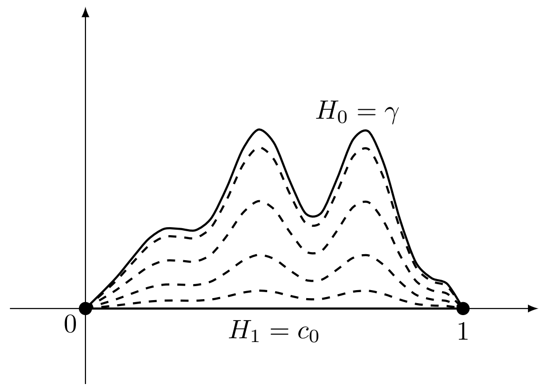



*Figure 9: The homotopy $H$ between $\gamma$ and $c_0$*

**Example 2.12.** Consider one loop in $S^1$ given by $\gamma_1 : I \to S^1$,

$$\gamma_1(\theta) = e^{2\pi i\theta}.$$

We could also have

$$\gamma_n(\theta) = (e^{2\pi i\theta})^n$$

which wraps around the circle $n$ times and $\gamma_0(\theta) = c_1$ is the constant path.

We can ask a similar question to the one we asked earlier about path on $S^1$, namely:

**Question 2.13.** Is $\gamma_1 \simeq \gamma_0$?

In the other direction, we could also ask whether or not these loops are all there is:

**Question 2.14.** Let $\gamma$ be a loop. Then is $\gamma \simeq \gamma_n$ for some $n \in \mathbb{Z}$?

How should we picture an arbitrary path from $\gamma : I \to S^1$? We can take the horizontal axis to be $I$ and can also picture the vertical axis as $I$ as long as we allow maps to pass freely between 0 and 1 (corresponding to the fact that $S^1$ is the path as $I = [0, 1]$ with the endpoints identified). The picture is as follows:

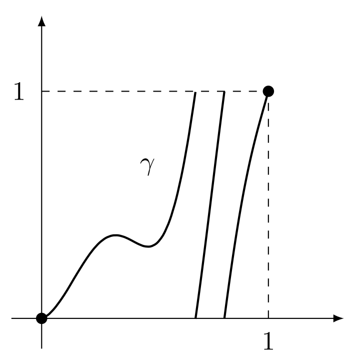



*Figure 10*

However, much more useful is to stack different copies of $I$ up the vertical axis and allow the curve $\gamma$ to pass into each region whenever it passes through 0 or 1. The picture is on the left below, with the dashed lines corresponding to other possible starting points for $\gamma$:

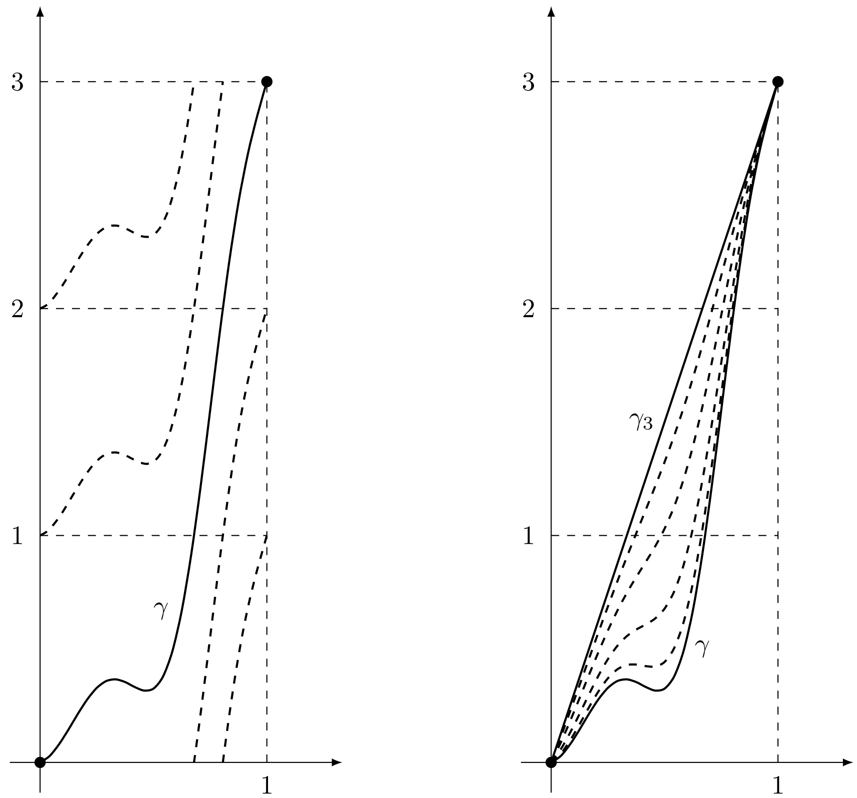



*Figure 11: The path $\gamma : I \to S^1$ and homotopy from $\gamma$ to $\gamma_3$*

If we understand this picture well enough, it it easy to see that $\gamma \simeq \gamma_3$. We show the homotopy in the diagram on the right above. We can picture $\gamma$ being made a highly elastic piece of string and snapping to the vertical line connecting $(0, 0)$ to $(1, 3)$. This argument suffices to prove that, for any loop $\gamma$, $\gamma \simeq \gamma_n$ for some $n$. We will return to this later.

### 2.2.1 Homotopy Invariance

For sets $S, S'$, we will write $S \simeq_{\text{bij}} S'$ if there exists a bijection $f : S \to S'$. This is an equivalence relation and we would like to show that this set is a homotopy invariant up to bijection.

From now on, we will assume all spaces $X$ are path-connected, i.e. for all $x, y \in X$, there exists a map $p : I \to X$ such that $p(0) = x$ and $p(1) = y$. We will say that $p$ is a path from $x$ to $y$ and write $p : x \rightsquigarrow y$.

We will now show that, assuming all spaces are path-connected, the set $\pi_1(X, x_0)$ is independent of the choice of basepoint up to bijection.

**Lemma 2.15.** Let $X$ be a metric space and $x_0, x_1 \in X$. Then $\pi_1(X, x_0) \simeq_{\text{bij}} \pi_1(X, x_1)$.

*Proof.* Let $x_0, x_1 \in X$. The key idea is that a path $p : x_0 \rightsquigarrow x_1$ induces a map $p_* : \pi_1(X, x_0) \to \pi_1(X, x_1)$, $ga \mapsto p^{-1} \cdot \gamma \cdot p$ where $p^{-1} \cdot \gamma \cdot p$ denotes the path formed by first moving along $p^{-1}$ ($p$ in reverse), then moving along $\gamma$ and finally along $p$. Explicitly, we can write this as:

$$(p^{-1} \cdot \gamma \cdot p)(t) = \begin{cases} p^{-1}(3t) & t \leqslant 1/3 \\ \gamma(3t-1) & 2/3 \leqslant t \leqslant 2/3 \\ p(3t-2) & 2/3 < t \leqslant 1. \end{cases}$$

We can picture this re-parametrisation as asking that time flows at three times the usual speed to allow for moving along three curves.

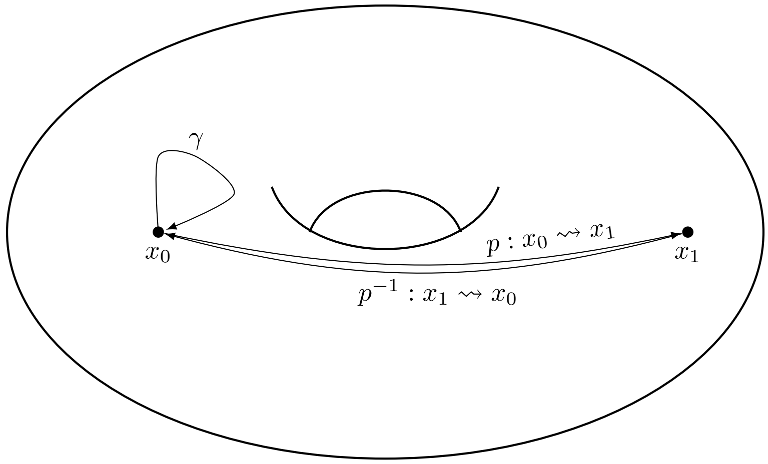



*Figure 12*

In order to show that $p_*$ is a bijection, it will suffice to check that $(p^{-1})_* \circ p_* : \pi_1(X, x_0) \to \pi_1(X, x_0)$ is equal to $\mathrm{id}_{\pi_1(X,x_0)}$. Note that $(p^{-1})_* \circ p_* : \gamma \mapsto p \cdot p_*(\gamma) \cdot p^{-1} = (p \cdot p^{-1}) \cdot \gamma \cdot (p \cdot p^{-1})$.

In order to show that $(p \cdot p^{-1}) \cdot \gamma \cdot (p \cdot p^{-1}) \simeq \gamma$, it will suffice to prove that $p \cdot p^{-1} \simeq c_{x_0}$. We can picture this as follows:

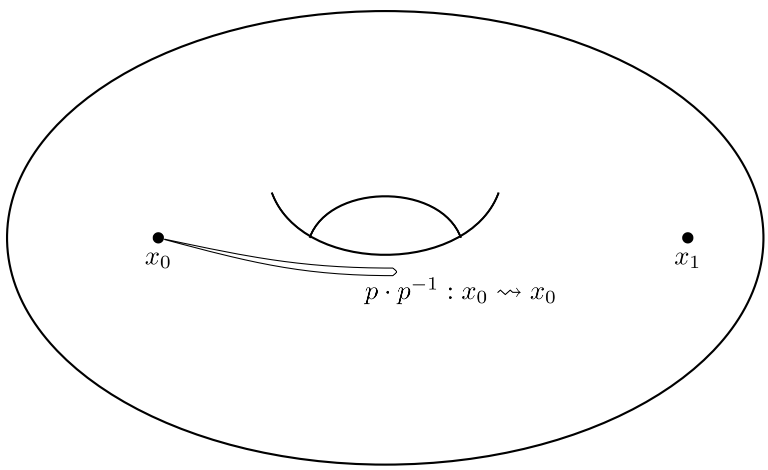



*Figure 13*

Recall that $p \cdot p^{-1}$ is defined as:

$$(p \cdot p^{-1})(x) = \begin{cases} p(2x) & 0 \leqslant x \leqslant 1/2 \\ p^{-1}(2x-1) & 1/2 \leqslant x \leqslant 1. \end{cases}$$

Explicitly, the diagram above corresponds to taking the homotopy:

$$H_t(x) = \begin{cases} (1-t)p(2x) & 0 \leqslant x \leqslant 1/2 \\ (1-t)p^{-1}(2x-1) & 1/2 \leqslant x \leqslant 1. \end{cases}$$

□

Since we have this lemma, we can now write $\pi_1(X)$ without referring to the basepoint (although one basepoint does still have to be chosen).

**Theorem 2.16.** Let $X$ and $Y$ be metric spaces. Then $X \simeq Y$ implies that $\pi_1(X) \simeq_{\text{bij}} \pi_1(Y)$, i.e. $\pi_1$ is a homotopy invariant.

*Proof.* Let $f : X \to Y, x_0 \mapsto y_0$ and assume for simplicity that there exists $g : Y \to X, y_0 \mapsto x_0$ such that $f \circ g \simeq id_Y$ and $g \circ f \simeq id_X$ (in general, $g$ would map $y_0 \mapsto x_1$ where $x_0 \neq x_1$).

The idea of the proof is similar to the previous lemma in that we define a map

$$f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0),$$
$$\underbrace{\gamma}_{\gamma:I\to X} \mapsto \underbrace{f \circ \gamma}_{f\circ\gamma:I\to Y}$$

and we would like to show it is a bijection. It will suffice to show that the composition

$$g_* \circ f_* : \pi_1(X, x_0) \to \pi_1(X, x_0)$$

is equal to the identity $\mathrm{id}_{\pi_1(X,x_0)}$ and also that $f_* \circ g_* = \mathrm{id}_{\pi_1(Y,y_0)}$ which will imply that $f_*$ and $g_*$ are invertible and hence bijections.

Note the following:

**Lemma 2.17.** Suppose we have a function $f : X \to X$ s.t. $f \simeq id_X$. Then $f_* = id_{\pi_1(X,x_0)}$.

This follow from that fact that, if $H_t$ is a homotopy from $f$ to $\mathrm{id}_X$, then $\widetilde{H}_t = H_t \circ \gamma$ is a homotopy from $f \circ \gamma$ to $\gamma$ and hence $f_*(\gamma) = \gamma$.

In particular, since $g_* \circ f_* = (g \circ f)*$ (they both send $\gamma \mapsto g \circ f \circ \gamma$) and $g \circ f \simeq \mathrm{id}_X$, we have that $(g \circ f)_* = \mathrm{id}_{\pi_1(X,x_0)}$. It follows similarly that $(f \circ g)_* = \mathrm{id}_{\pi_1(Y,y_0)}$. □

In the above proof, we assumed that $g(y_0) = x_0$. To prove the general case, we would need to prove a slightly more general lemma which we leave as an exercise.

**Exercise 2.18.** Show that, if $f : X \to X, x_0 \mapsto x_1$ has $f \simeq id_X$, then $f_*$ is bijective. Hence finish the proof of Theorem 2.16.

We will return once again to our motivating example of the circle.

**Example 2.19.** Consider $\pi_1(S^1)$ with the loops

$$\gamma_n : I \to S^1, \theta \mapsto \left(e^{2\pi i\theta}\right)^n.$$

We sketched a proof that $\pi_1(S^1) = \{\gamma_n \mid n \in \mathbb{Z}\}/\simeq$ and we conjectured that $\gamma_n \simeq \gamma_m$ if and only if $n = m$ which would imply that $\pi_1(S^1) = \{\gamma_n \mid n \in \mathbb{Z}\} = \mathbb{Z}$.

This identification with $\mathbb{Z}$ suggests that $\pi_1(S^1)$ may contain more structure than that of a set. Namely $\mathbb{Z}$ has an operation $(n, m) \mapsto n + m$ which should correspond to an operation $(\gamma_n, \gamma_m) \mapsto \gamma_{n+m}$. Using our notation earlier, this corresponds to concatenation $\gamma_{n+m} \simeq \gamma_n \cdot \gamma_m$.

It turns out that this definition works for general metric spaces $X$.

**Definition 2.20.** If $\gamma, \gamma' \in \pi_1(X)$, then we define the product $\gamma \cdot \gamma' \in \pi_1(X)$ to be the loop $\gamma \cdot \gamma' : I \to X$ with

$$(\gamma \cdot \gamma')(t) = \begin{cases} \gamma(2t) & 0 \le t \le 1/2 \\ \gamma'(2t-1) & 1/2 \le t \le 1. \end{cases}$$

This is the curve formed by 'doing $\gamma$ then $\gamma'$.

This operation has some special properties. Recall the definition of a group.

**Definition 2.21.** A group is a set $G$ and a map $\cdot : G \times G \to G$ such that

i Identity: There is an element $1 \in G$ such that $g \cdot 1 = g = 1 \cdot g$,

ii Inverse: For all $g \in G$, there exists $g^{-1}$ such that $g^{-1} \cdot g = 1 = g \cdot g^{-1}$.

iii Associativity: For all $g, h, k \in G$, $(g \cdot h) \cdot k = g \cdot (h \cdot k)$.

**Remark 2.22.** Note that commutativity is not a requirement to be a group. Groups that have the property are called abelian groups.

**Definition 2.23.** If $G$ and $H$ are groups and $f : G \to H$ is a function then we say that $f$ a homomorphism if $f(gh) = f(g)f(h)$ for all $g, h \in G$ and $f(1) = 1$. We say that $G$ are $H$ are isomorphic, written $G \cong H$, if there are homomorphisms $f : G \to H$ and $g : H \to G$ such that $f \circ g = id_H$ and $g \circ f = id_G$.

This should remind us of the definition of continuous function and homeomorphism of spaces.

It turns out that we need not ever find an inverse to show that two groups are isomorphic. We leave the following as an exercise:

**Exercise 2.24.** Let $G$ and $H$ be groups. If there exists a group homomorphism $f : G \to H$ that is bijective, then $G \cong H$.

**Theorem 2.25.** $\pi_1(X)$ is a group.

*Proof.* Consider the constant map $c_0$. Then we can check that

$$c_0 \cdot \gamma \simeq \gamma \simeq \gamma \cdot c_0.$$

If $\gamma^{-1}(t) = \gamma(1 - t)$, we showed earlier that

$$\gamma \cdot \gamma^{-1} \simeq c_0 \simeq \gamma^{-1} \cdot \gamma.$$

We also have associativity, i.e. that

$$(\gamma \cdot \gamma') \cdot \gamma'' \simeq \gamma \cdot (\gamma' \cdot \gamma'').$$

□

The proofs in the first and third case just amount to re-parametrisation (i.e. 'slowing down and speeding up time') and a careful proof simply requires finding the right formula for the homotopy.

It turns out that the group structure is homotopy equivalent too:

**Theorem 2.26.** Let $X$ and $Y$ be metric spaces. Then $X \simeq Y$ implies that $\pi_1(X) \cong \pi_1(Y)$, i.e. $\pi_1$ is a homotopy invariant as a group.

*Proof.* By the exercise above, all we need to show is that, if $p : x_0 \rightsquigarrow x_1$ is a path and $f : X \to Y$ is a map, then the maps $p_* : \pi_1(X, x_0) \to \pi_1(X, x_1)$, $\gamma \mapsto p^{-1} \cdot \gamma \cdot p$ and $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$, $\gamma \mapsto f \circ \gamma$ are group homomorphisms. □

We will finally give a name to our new invariant.

**Definition 2.27.** $\pi_1(X)$ is called the fundamental group.

### 2.2.2 The Fundamental Group of $S^1$

We will finally finish off our computation of $\pi_1(S^1)$. Recall the idea that we used in Figure 11 to draw loops $\gamma : I \to S^1$. Formally, these drawing were functions $\widetilde{\gamma} : I \to \mathbb{R}$ which reduce back to $\gamma$ when maps are composed with a canonical map $\mathbb{R} \to S^1$. We can clarify this idea with the following definition:

**Definition 2.28.** Define the map

$$p : \mathbb{R} \to S^1, x \mapsto e^{2\pi i x}.$$

If $\gamma : I \to S^1$ is a loop, then we say a map $\widetilde{\gamma} : I \to \mathbb{R}$ is a lift of $\gamma$ if $\widetilde{\gamma}(0) = 0$ and $p \circ \widetilde{\gamma} : I \mapsto S^1$.

**Definition 2.29.** We will say that lifts $\widetilde{\gamma}$ and $\widetilde{\gamma'}$ will be homotopy equivalent if there exists a continuously varying one-parameter family of homotopies $H_t : I \to \mathbb{R}$ such that $H_0 = \widetilde{\gamma}$, $H_1 = \widetilde{\gamma'}$ and $H_t(0)$, $H_t(1)$ are fixed for all $0 \leq t \leq 1$.

Note that we require both endpoints to be fixed throughout the homotopy.

**Theorem 2.30.** $\pi_1(S^1) \cong (\mathbb{Z}, +)$.

*Proof.* We want to take the $\gamma_n : I \to S^1, \theta \mapsto e^{2\pi i n\theta}$ and show that these are the only loops up to homotopy. We then have to show that none of these loops are homotopy equivalent to eachother, i.e. we must show:

i For all loops $\gamma$, we need $\gamma \simeq \gamma_n$ for some $n$.

ii $\gamma_n \simeq \gamma_m$ iff $n = m$.

We sketched a proof of the first claim earlier and we will now make this argument more precise. Let $\gamma : I \to S^1$ be a loop. We need the following:

**Lemma 2.31.** Every loop $\gamma : I \to S^1$ has a unique lift $\widetilde{\gamma} : I \to \mathbb{R}$.

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ I \arrow[r, "\gamma"] \arrow[ur, dashed, "\exists!\widetilde{\gamma}"] & S^1 \end{tikzcd}$$

*Proof.* The entire proof is contained in the figure on the left below (which we reproduce from Figure 11).

In particular, we can prove existence by piece-wise defining the function based on all the points at that it meets the basepoint in the sphere (we can show this happens finitely many times using some basic results from analysis). Uniqueness follows similarly. $\square$

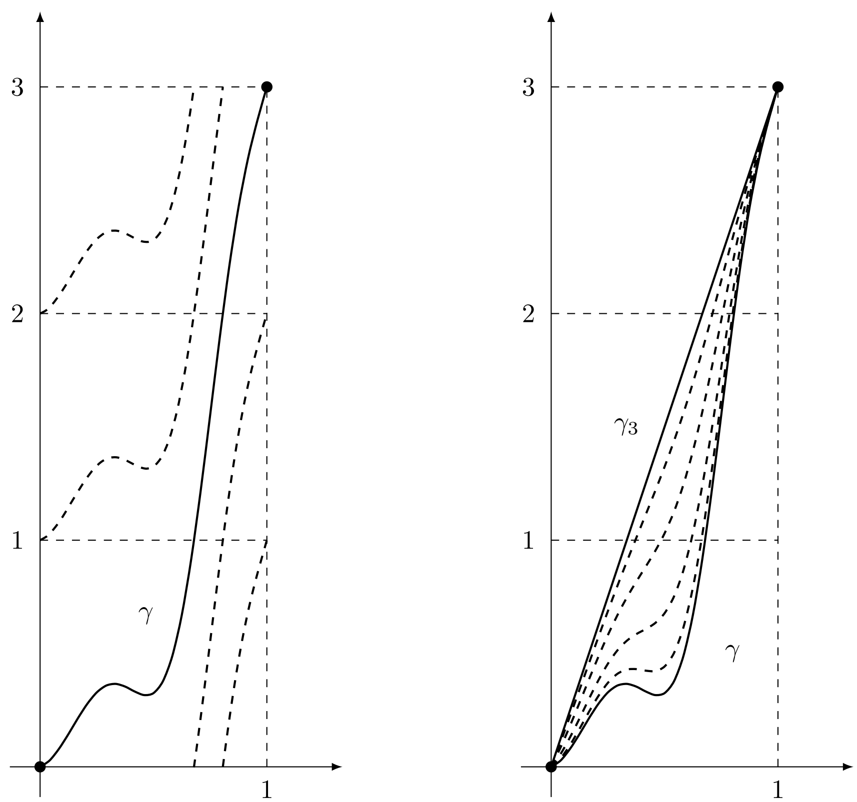



*Figure 14: The path $\gamma : I \to S^1$ and homotopy from $\gamma$ to $\gamma_3$*

We can now pick a lift $\widetilde{\gamma} : I \to \mathbb{R}$, i,e. a map with $\widetilde{\gamma}(0) = 0$ and $p \circ \widetilde{\gamma} = \gamma$. Let $n = \widetilde{\gamma}(1)$. This is an integer since $\widetilde{\gamma}$ is a lift, i.e. since $e^{2\pi i n} = p(n) = p(\widetilde{\gamma}) = \gamma(1) = 1$.

Note that $\widetilde{\gamma}_n(\theta) = n\theta$ is the lift of $\gamma_n$. It is now easy to see that $\widetilde{\gamma} \simeq \widetilde{\gamma}_n$ from the figure on the right above (we can define a homotopy analogously to in Figure **??**).

Does this imply that $\gamma \simeq \gamma_n$? We want to prove something along the lines of if two lifts are homotopic then the original things are homotopic. The fact that $\widetilde{\gamma} \simeq \widetilde{\gamma}_n$ means that there is a homotopy $\widetilde{H}_t : I \to \mathbb{R}$ such that $\widetilde{H}_0 = \widetilde{\gamma}$ and $\widetilde{H}_1 = \widetilde{\gamma}_n$. We want a a map $H_t : I \to S^1$ such $H_0 = \gamma$ and $H_1 = \gamma_n$. It suffices to take $H_t = p \circ \widetilde{H}_t$ since $H_0 = p_0 \circ \widetilde{\gamma} = \gamma$ and $H_1 = p \circ \widetilde{\gamma}_n = \gamma_n$. This finishes the proof of the first claim.

The proof of the second claim will be similar. Suppose that $\gamma_n \simeq \gamma_m$.

Note that $\widetilde{\gamma}_n \simeq \widetilde{\gamma}_m$ implies that $n = \widetilde{\gamma}(1) = \widetilde{\gamma'}(1) = m$ since homotopies of lifts must fix endpoints. Hence $n = m$ and so it will suffice to prove that $\widetilde{\gamma}_n \simeq \widetilde{\gamma}_m$. This can be done using an explicit homotopy. This finishes the proof of the second claim.

This implies that $\pi_1(S^1) = \{\gamma_n : n \in \mathbb{Z}\} \cong \mathbb{Z}$ since we know that $\gamma_n \cdot \gamma_m = \gamma_{n+m}$. $\square$

Some of the above discussion can also be phrased in terms of the following results which says that homotopies themselves can be lifted uniqely:

**Lemma 2.32.** If $H_t : I \to S^1$ is a homotopy of loops, then there is a unique homotopy of lifts $\widetilde{H}_t : I \to \mathbb{R}$ such that $p \circ \widetilde{H} = H$.

Generalising this proof to other spaces is the beginning of a long and interesting story.

**Example 2.33.** What if we want the fundamental group of the torus $T \cong S^1 \times S^1$? We can use methods similar to what we did for $S^1$ to find $\pi_1(T)$. A torus is somehow a bit like $\mathbb{R}^2$ but just folded down. We can play a similar gsme with $p : \mathbb{R}^2 \to T$ to show that $\pi_1(T) \cong \mathbb{Z} \times \mathbb{Z}$.

In the general setting, the map $p : \mathbb{R} \to S^1$ is called a covering map and the space $\mathbb{R}$ lying over $S^1$ is known as a covering space.

## 2.3 Other Invariants

Are $\chi$ and $\pi_1$ the only invariants of interest? We know that

$$X \simeq Y \implies \chi(X) = \chi(Y), \pi_1(X) \cong \pi_1(Y)$$

but is it true that

$$\chi(X) = \chi(Y), \pi_1(X) \cong \pi_1(Y) \implies X \simeq Y?$$

This turns out to be false:

**Example 2.34.** We claim that $\pi_1(S^2) = \{1\}$. If $\gamma : I \to S^2$ is a loop, we would like to show that it is homotopic to the constant loop $c_{x_0}$ by picking a point $x \in S^2$ which is not in the image of $\gamma$ and projecting $\gamma$ towards $x_0$ along the arc connecting $x_0$ to $x$. We can picture this as follows:

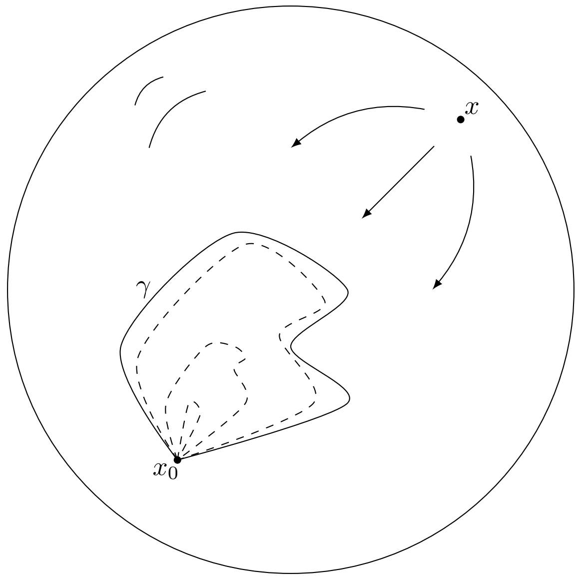



*Figure 15*

However this proof does not work since, surprisingly, there are continuous surjective function $f : I \to S^2$. These things are called space filling curves. An an example, we can get a continuous surjection $f : I \to I^2$ as the limit of the following sequence of continuous function $f_1, f_2, f_3, f_4, f_5 \dots$.

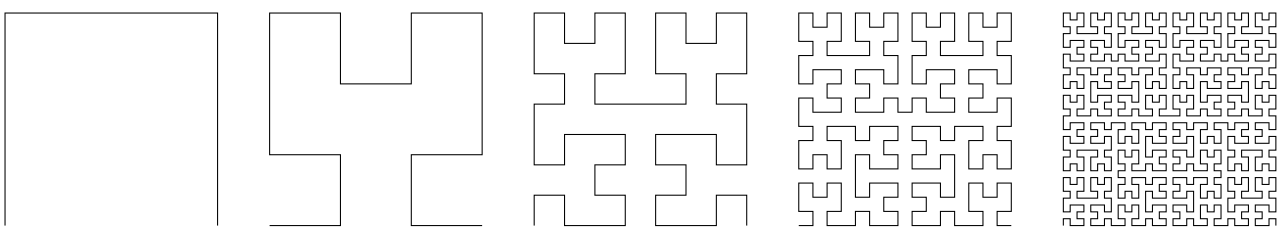


*Figure 16: Hilbert's space filling curve.*

This proof can be made to work: one needs to prove that you can 'wiggle' $\gamma$ in some way to make it not surjective, though it requires many additional details.

We can show similarly that $\pi_1\left(S^4\right) = \{1\}$. Since $\chi(S^2) = \chi(S^4) = 0$, both spaces have the same $\chi$ and $\pi_1$. But are these spaces homotopy invariant? The answer is no but to prove this we will, of course, need even more invariants.
There are two main types of homotopy invariant. Firstly we have the homotopy groups:

$$\pi_1(X), \pi_2(X), \pi_3(X), \dots .$$

These 'higher' homotopy groups are defined analogously to $\pi_1(X)$ in that $\pi_n(X)$ is the homotopy classes of maps $S^n \to X$ which send a point on $S^n$ to a fixed basepoint in $X$. For values of $i \geqslant 2$ we have that $\pi_i(X)$ is an abelian group. These are important though $\pi_1(X)$ contains more information in general since abelian groups can be classified and only take one of a few forms.

We also have the homology groups which are all abelian:

$$H_0(X), H_1(X), H_2(X), H_3(X), \dots$$

These can be defined in a clever way using linear algebra. The definition is easy in the case of cell complexes (though it is hard to prove it is homotopy invariant with this definition) and the analogous definition for general spaces is more difficult (though it is, perhaps, a little easier to prove it is homotopy invariant). It turns out that the Euler characteristic can be computed from the homology groups.

**Fact 2.35.** Every (finitely-generated) abelian group $A$ is of the form

$$A \cong \mathbb{Z}^n \times F,$$

where $F$ is finite. We define $\mathrm{rank}(A) = n$.

**Theorem 2.36.** The Euler characteristic can be expressed as

$$\chi(X) = \sum_i (-1)^i \mathrm{rank}(H_i(X)).$$

Therefore the fact that $\chi(X)$ is a homotopy invariant is implied by the fact that $H_i(X)$ is a homotopy invariant, and this is the usual way to prove this fact.

# 3 Applications

We will now give two applications of the fact that $\pi_1(S^1) \cong \mathbb{Z}$ to algebra and analysis.

## 3.1 The Fundamental Theorem of Algebra

**Theorem 3.1** (The Fundamental Theorem of Algebra)**.** For every non-constant polynomial

$$f(x) = \sum_{k=0}^{n} a_k x^k,$$

with $n \geqslant 1$ and $a_k \in \mathbb{C}$, there exists $z \in \mathbb{C}$ such that $f(z) = 0$.

*Proof.* Suppose $f$ has no roots over $\mathbb{C}$ and assume, without any loss of generality, that $f$ is monic. Then $f : \mathbb{C} \to \mathbb{C} \setminus \{0\}$ is continuous. We now have this continuous function and need to study it a bit more. The fact that it exists should be enough to get a contradiction. Recall that we have a map

$$\begin{aligned} &f_* : \pi_1(\mathbb{C}) \to \pi_1(\mathbb{C} \setminus 0), \\ &\gamma \mapsto f \circ \gamma. \end{aligned}$$

Before we move on we should identify what $\pi_1(\mathbb{C})$ and $\pi_1(\mathbb{C} \setminus \{0\})$ are.

Note that $\mathbb{R}^n \simeq *$. Letting $n = 2$, we get that $\mathbb{R}^2 \cong \mathbb{C}$ as spaces, which implies $\mathbb{C} \simeq *$. We also know that $\mathbb{C} \setminus \{0\} \simeq S^1$. Thus, by homotopy of $\pi_1$, we get that

$$\pi_1(\mathbb{C}) \cong \pi_1(*) = \{1\}, \quad \pi_1(\mathbb{C} \setminus \{0\}) \cong \mathbb{Z}.$$

We thus have a group homomorphism

$$f_* : \{1\} \to \mathbb{Z}, \quad 1 \mapsto 0$$

and so $f \circ \gamma \simeq c_0$ for all $\gamma : I \to \mathbb{C}$. We would now like to get a contradiction by exhibiting a loop $\gamma : I \to \mathbb{C}$ for which $f \circ \gamma \not\simeq c_0$.

Note that, for $r$ sufficiently large, we have that

$$f\left(re^{i\theta}\right) = \sum_{k=0}^{n} a_k r^k e^{ik\theta} \approx r^n e^{in\theta}.$$

In particular, if $\gamma = r\gamma_1$ is the path which goes once around the circle of radius $r$ then, provided $r$ is sufficiently large, there is a homotopy $f \circ \gamma \simeq r^n \gamma_n$. The 'approximately equal to' can be turned into a homotopy of loops since we can move between the two paths without passing through the origin (which is not in the space). The picture is as follows:

*Figure 17*

Since $r^n\gamma_n \simeq \gamma_n$, by shrinking the radius, we get that $f \circ \gamma \mapsto n \in \mathbb{Z} \cong \pi_1(S^1)$ under the group isomorphism $\pi_1(\mathbb{C} \setminus \{0\}) \to \pi_1(S^1)$ established previously. Since $n \geq 1$, $f \circ \gamma \not\simeq c_1$ which completes the proof. □

Using the Euclidean algorithm for polynomials, we can state this theorem in another form.

**Corollary 3.2.** Polynomials factorize over $\mathbb{C}$.

## 3.2 Brouwer Fixed Point Theorem

Consider the disc

$$D^2 = \left\{(x,y) \in \mathbb{R}^2 \;\middle|\; x^2 + y^2 \leqslant 1\right\}.$$

**Theorem 3.3** (Brouwer Fixed Point Theorem)**.** If $f : D^2 \to D^2$ is continuous, then $f$ has a fixed point, i.e. there exists $x \in D^2$ for which $f(x) = x$.

*Proof.* Suppose $f$ does not have a fixed point. How are we going to apply algebraic topology to this? We want to find a way to encode the non-existence of a fixed point into the existence of a continuous function with certain properties and then use the fundamental group in a similar way to how we did previously.

One thing we can do, which we could not be if $f$ has a fixed point, is to consider the unique line that connects $x$ and $f(x)$. This defines a map $g : D^2 \to S^1$ where $x$ maps to the point on the same side of $x$ on the line connecting $x$ and $f(x)$. One can check that $g$ is continuous by, for example, finding a formula for it by solving for the intersection points.

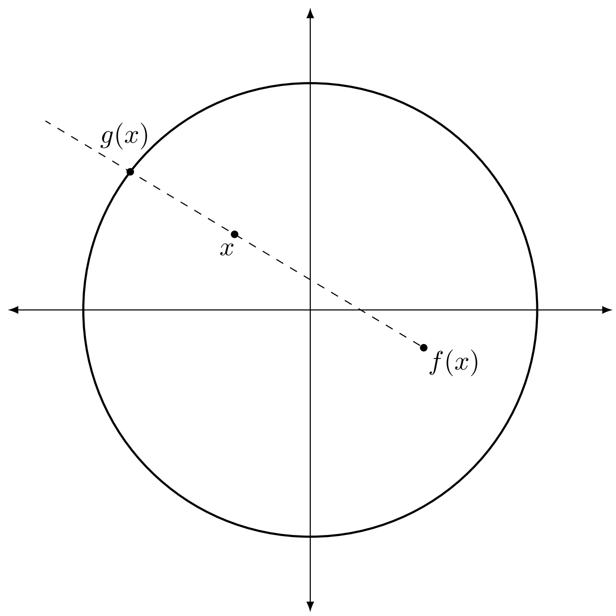



*Figure 18: An illustration of the function g.*

Now consider the function

$$g_* : \pi_1(D^2) \to \pi_1(S^1), \gamma \mapsto g \circ \gamma.$$

Recall that $\pi_1(S^1) \cong \mathbb{Z}$. We also have that $D^2 \simeq *$, meaning that $\pi_1(D^2) \cong \pi_1(*) = \{1\}$. This implies that $g \circ \gamma \simeq c_1$ for all $\gamma \in \pi_1(D^2)$.

Note that, if $x \in D^2$ is already on the boundary circle, then $g(x) = x$. In particular $g|_{S^1} = \mathrm{id}_{S^1}$. Consider the loop $\gamma_n : I \to D^2$, $\theta \mapsto (e^{2\pi i\theta})^n$. Then

$$g(\gamma_n(\theta)) = g\left(e^{2\pi in\theta}\right) = e^{2\pi in\theta},$$

because $e^{2\pi in\theta}$ is on the circle and $g|_{S^1} : S^1 \to S^1$. This is a contradiction since $g \circ \gamma_n$ corresponds to the element $n \in \mathbb{Z} \cong \pi_1(S^1)$ and so is not homotopic to the constant path for any choice of $n \neq 1$. $\square$

A more slick way to do the last part of this argument is to consider

$$S^1 \overset{i}{\hookrightarrow} D^2 \xrightarrow{g} S^1$$

where $i : S^1 \hookrightarrow D^2$ is the inclusion map. We can then apply $\pi_1$ to get

$$\pi_1(S^1) \xrightarrow{i_*} \pi_1(D^2) \xrightarrow{g_*} \pi_1(S^1).$$

The definition of $g$ implies that $g(x) = x$ if $x$ is already on the boundary circle, i.e. $g \circ i = \mathrm{id}_{S^1}$.

Since $g_* \circ i_* = (g \circ i)_*$, this implies that the composition

$$g_* \circ i_* : \mathbb{Z} \to \{1\} \to \mathbb{Z}$$

equals the function $id_{\pi_1(S^1)} : \mathbb{Z} \to \mathbb{Z}$ which is bijective. However $g_* \circ i_*$ is not bijective since it maps everything to $0 \in \mathbb{Z}$, which is a contradiction.

**Remark 3.4.** This approach can be viewed as relying on the 'functoriality' of $\pi_1$. We say that $\pi_1$ is a functor since it does two things. Firstly, it assigns groups to spaces

$$\pi_1 : \{\text{Spaces}\} \to \{\text{Groups}\}, \quad X \mapsto \pi_1(X)$$

and, secondly, it assigns maps between groups (group homomorphisms) to maps between spaces (continuous functions)

$$\pi_1 : \{X \to Y \text{ continuous}\} \to \{\pi_1(X) \to \pi_1(Y) \text{ homomorphism}\}, \quad f \mapsto f_*$$

in such a way that we have nice properties such that $(f \circ g)_* = f_* \circ g_*$ and $(\mathrm{id}_X)_* = \mathrm{id}_{\pi_1(X)}$.

A good reference for the further applications of this sort is Daniel Shapiro's book on sums of squares.

# 4 Classification Problems

## 4.1 Classification of 1-Complexes

We want to focus on classifying complexes. Classifying 1-complexes is doable, but classifying 2-complexes is a good deal harder. We can model classification of other more complicated things off of classifying 1-complexes. As usual, we will assume that all complexes are path-connected.

We will break the classification of 1-complexes down into two distinct stages which are reminiscent of our proof that $\pi_1(S^1) \cong \mathbb{Z}$.

(i) Show that every graph can be placed uniquely in some kind of 'standard form'

(ii) Show that two graphs in standard form are homotopy equivalent if and only if they are the same graph.

We begin by focusing on this first stage. Consider the graph below on the left. To put this in a standard form, we consider how to alter the graph to a homotopy equivalent one which is simpler. We do this by successively shrinking edges with distinct vertices to a point.

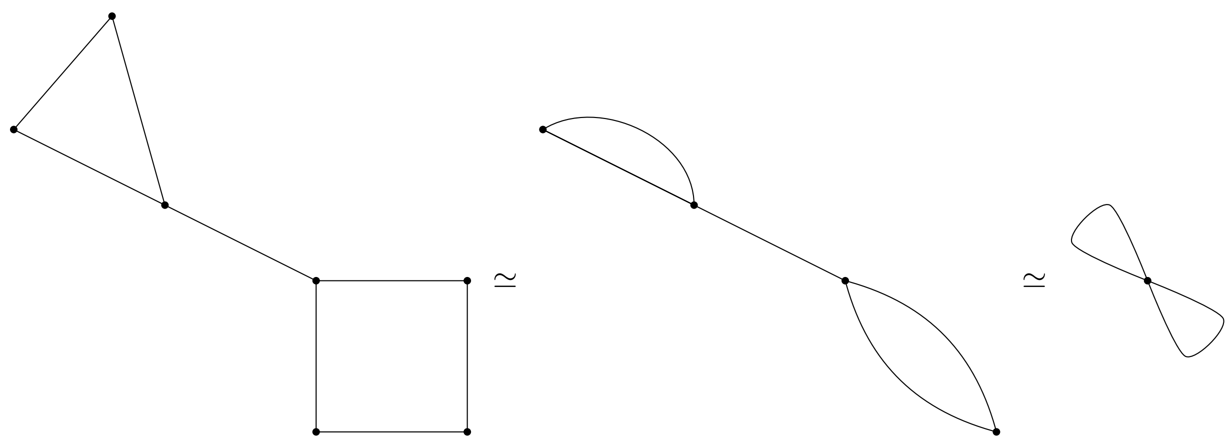



*Figure 19: An example of contracting the edges on a graph to reduce it down to standard form.*

This suggests that the standard form we want might be the graphs with a single vertex. We make the following definition.

**Definition 4.1.** The flower with $n$ petals $X_n$ will be the unique graph with a single vertex and $n \geq 0$ edges.

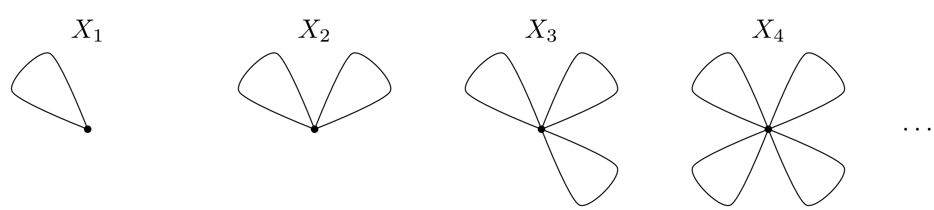



*Figure 20: $X_1, X_2, X_3$, and $X_4$.*

**Definition 4.2.** Let $X$ be a 1-complex and $e = (v_1, v_2) \in E(X)$. Then $X/e$ is the graph formed by deleting the edge $e$ and combining the vertices $v_1$ and $v_2$. We refer to this process as edge contraction.

**Exercise 4.3.** Given a 1-complex $X$ with vertices $v_1 \neq v_2$ and $e = (v_1, v_2) \in E(X)$, then $X \simeq X/e$.

**Theorem 4.4.** Let $X$ be a 1-complex. Then $X \simeq X_n$ for some $n \geqslant 0$.

*Proof.* We will use Induction. We can start with a 1 complex with more than 1 vertex, and then reduce it using Exercise 4.3 to work it down to $X_n$. □

We can ado this using the fact from graph theory that every graph has a spanning tree $T$ with $T \subseteq X$ and $T$ containing no loops. It is then easy to see that $X \simeq X/T = X_n$ for some $n \geqslant 0$ and that $X/T$ can be attained as a sequence of edge contractions of the above form.

We now prove the second part, namely that these standard forms are distinct.

**Lemma 4.5.** *$X_n \simeq X_m$ if and only if $n = m$.*

*Proof.* If $X_n \simeq X_m$, then $\chi(X_n) = \chi(X_m)$. However $\chi(X_n) = 1 - n$ and so $1 - n = 1 - m$ which implies that $n = m$. □

In particular, we have shown the following:

**Theorem 4.6.** (Classification of 1-complexes) *If $X$ and $Y$ are 1-complexes, then $X \simeq Y$ if and only if $\chi(X) = \chi(Y)$.*

**Remark 4.7.** In contract to the examples of $S^2$ and $S^4$ which showed that $\chi$ and $\pi_1$ are not enough to classify spaces, this result shows that even just $\chi$ is enough to classify 1-complexes.

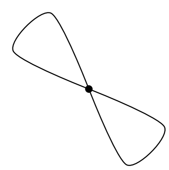
## 4.2 Classification of 2-Complexes

We would like to do a similar thing for 2-complexes.

**Definition 4.8.** Let $X^{(i)} = \{\text{cells of dimension } i\}$. This is called the $i$-skeleton

**Example 4.9.** How can we describe the torus $T = S^1 \times S^1$?

$T^{(0)} = \bullet \qquad T^{(1)} =$



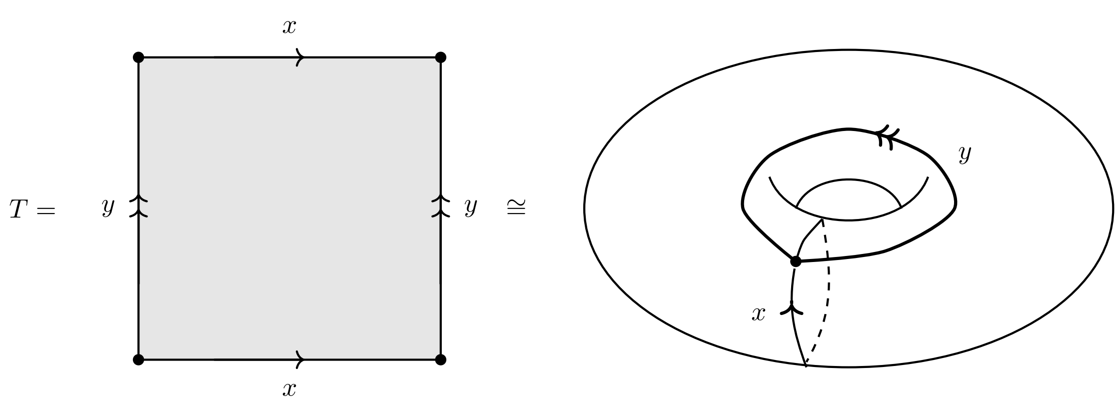



*Figure 21*

This is quite simple since $T^{(1)}$ is in standard form as a 1-complex. It turns out this is always possible.

**Lemma 4.10.** *If $X$ is a 2-complex then $X \simeq X'$, where $X'$ is a 2-complex with a single vertex.*

*Proof.* The proof amounts to checking that $X \simeq X/e$ is still true for edges $e = (v_1, v_2)$ with $v_1 \neq v_2$. □

This lemma allows us to put 2-complexes in some sort of standard form, though it will not be quite as restrictive as the case of 1-complexes.

Up to homotopy equivalence, 2-complexes can be described by the following data:

(i) Flower with $n$ petals. We can label each petal with the symbols $x_1, \ldots, x_n$.

(ii) The attaching paths for the edges then correspond to words in the symbols

$$\left\{x_1, x_1^{-1}, x_2, x_2^{-1}, \ldots, x_n, x_n^{-1}\right\}.$$

Note that the trivial word 1 is also allowed which corresponds to attachment of a sphere at a vertex (one can show that the choice of vertex does not affect the resulting homotopy type).

**Example 4.11.** The torus can be described by loops labelled by $x, y$ and an attaching path $xyx^{-1}y^{-1}$. This should be clear from the picture on the left though a little harder to see from the picture on the right.

This data may remind you of something from group theory.

### 4.2.1 Group Presentations

A group presentation is a way of writing a group $G$ in a form like

$$G = \langle x, y \mid xy = yx \rangle$$

where the elements of the group are generated by $x$ and $y$ (i.e. are words in $\{x, y, x^{-1}, y^{-1}\}$) subject to the equation $xy = yx$.

**Example 4.12.**

(i) Suppose we have $\mathbb{Z}/p\mathbb{Z}$, which is a group under addition. We can find a generator under addition as $\mathbb{Z}/p\mathbb{Z} = \left\{1, x, x^2, \ldots, x^{p-1}\right\}$, where $x^n = \sum_{i=1}^{n} x$. We write this as $\mathbb{Z}/p\mathbb{Z} = \langle x \mid x^p = 1\rangle$.

(ii) We have $\mathbb{Z} = \{\ldots, -1, 0, 1, \ldots\}$ as a group under addition. This is the same as $\mathbb{Z} = \{\ldots, x^{-1}, 1, x^1, \ldots\}$ for $x = 1$ and $x^n = \sum_{i=1}^{n} x$ as above, and we can write this as $\mathbb{Z} = \langle x \mid \cdot \rangle$ where '$\cdot$' denotes the fact that there are no relations (other than requiring that $x \cdot x^{-1} = 1$ which is assumed).

(iii) Consider $G = \left\langle x, y \;\middle|\; x^3 = y^2 = 1,\ xy = yx \right\rangle$. Then we can show that $G = \left\{1, x, x^2, y, xy, x^2y\right\}$. It is straightfoward to show that arbitary words can be put into one of these six forms, though showing these words are not the same requires a slightly more delicate argument. We have that $G \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$.

We claim that there is a one-to-one correspondence between group presentations and 2-complexes.

**Example 4.13.** The circle $S^1$ can be described by the flower with one petal. It therefore has a single edge $x$ and no attaching paths. We have that

$$\begin{aligned} \pi_1(S^1) &\cong \mathbb{Z} = \left\{\ldots, x^{-1}, 1, x^1, \ldots\right\} \\ &= \langle x \mid \cdot \rangle. \end{aligned}$$

**Example 4.14.** The torus $T$ can be described by edges $x, y$ and a face attached along the path $xyx^{-1}y^{-1}$. We would like to convince ourselves that:

$$\pi_1(T) \cong \left\langle x, y \;\middle|\; xyx^{-1}y^{-1} = 1 \right\rangle.$$

First consider $\pi_1\left(T^{(1)}, \cdot\right)$ where $T^{(1)} = X_2$ is the flower with two petals. Similarly to the case of the circle, it can be shown that the edge paths $x$ and $y$ are not homotopic, i.e.

$$\begin{aligned} \pi_1(T^{(1)}, \cdot) &= \{\text{loops at } \cdot\} / \simeq \\ &= \left\{\text{words in } \left\{x, x^{-1}, y, y^{-1}\right\}\right\} / \simeq. \end{aligned}$$

Attaching the face along the path $xyx^{-1}y^{-1}$ forces this path to be homotopic to the constant path: we can simply move through the square down to the basepoint. It turns out that this is the only new relation this introduces to the fundamental group, hence:

$$\pi_1(T) \cong \left\langle x, y \;\middle|\; xyx^{-1}y^{-1} = 1 \right\rangle.$$

**Definition 4.15.** If $\mathscr{P} = \langle x_1, \cdots, x_n \mid r_1, \cdots, r_m \rangle$ is a presentation for a group $G$, where the $r_i$ are words in the $x_i$, $x_i^{-1}$. Then the presentation complex for $\mathscr{P}$, denoted $X_{\mathscr{P}}$, is the 2-complex defined by taking a flower with $n$-petals labelled by $x_1, \cdots, x_n$ and attaching faces along the paths corresponding the the relation $r_1, \cdots, r_m$.

Note that $\chi(X_{\mathscr{P}}) = 1 - n + m$.

**Theorem 4.16.** If $\mathscr{P}$ is a presentation for the group $G$, then $\pi_1(X_{\mathscr{P}}) \cong G$.

We will not prove this here as a proof would require knowing far more about $\pi_1$ than we already do. The above discussion now implies that there is a one-one correspondence

$$\{2 \text{ complexes } X \text{ such that } \pi_1(X) \cong G\} \leftrightarrow \{\text{Presentations } \mathscr{P} \text{ for } G\}.$$

This has two nice consequences.

**Definition 4.17.** We say a group $G$ is finitely-presented if it can be described by a presentation $\mathscr{P}$ with finitely many generators and relations.

This includes all finite groups and probably all infinite groups we have come across before. The above theorem implies:

**Corollary 4.18.** If $G$ is a finitely-presented group, then there exists a 2-complex $X$ with $\pi_1(X) \cong G$.

In particular, classifying 2-complexes in general is at least as hard as classifying finitely-presented groups. One can prove theorems which imply that finitely-presented groups can never be classified. A more concrete issue is that even classifying finite groups is well beyond the realm of what most mathematicians think is possible.

Perhaps surprisingly, this gives us a formula for the fundamental group of any cell complex. We need the following lemma.

**Lemma 4.19.** If $X$ is a cell complex, then $\pi_1(X) \cong \pi_1(X^{(2)})$.

This can be shown by proving that attaching cells of dimension $\geq 3$ does not alter the fundamental group. This is related to this fact that $\pi_1(S^n) = \{1\}$ for all $n \geq 2$ (we saw the case $n = 2$ earlier).

In particular, if $X$ is a cell complex, then we can compute $\pi_1(X)$ by finding a 2-complex $X'$ with a single vertex which is homotopy equivalent to $X^{(2)}$ (for example, by contracting a spanning tree) and then writing down the presentation $\mathscr{P}$ corresponding to $X^{(2)}$.

It may now seem as though fundamental groups are actually very easy to compute. However, determining the isomorphism class (or even just the size) of a group from a presentation is very difficult. It can be shown, for example, that there is no single algorithm that can determine whether two presentations $\mathscr{P}_1, \mathscr{P}_2$ represent the same group.

### 4.2.2 Stable Equivalence of 2-Complexes

We now consider the question of whether or not $\chi(X)$ and $\pi_1(X)$ are enough to determine the 2-complex $X$. In order to answer this question, we consider a certain principle for classification problems: to first classify with respect to a stronger equivalence relation.

**Definition 4.20** (Wedge)**.** If $X$ and $Y$ are 2-complexes, then define their wedge, $X \vee Y$, to be the 2-complex set with vertex set $V = V(X) \cup V(Y)$ with two vertices $v_X \in V(X)$ and $v_Y \in V(Y)$ identified, and edge set $E = E(X) \sqcup E(Y)$.

We can picture this as the spaces $X$ and $Y$ glued together at a single point. One can check the the homotopy type of $X \vee Y$ is independent of the choice of vertices.

**Example 4.21.** If $X = T$ is the torus and $Y = S^2$ is the sphere, then $T \vee S^2$ is a 'torus with a pimple' as drawn below.

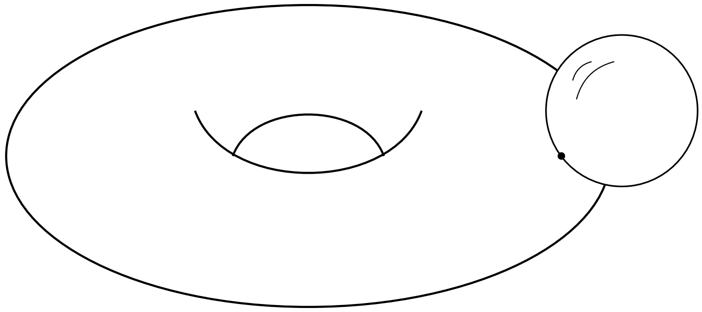


*Figure 22*

We will use this to define an equivalence relation which is stronger than homotopy.

**Definition 4.22** (Stable homotopy)**.** Two 2-complexes $X$ and $Y$ are stably homotopic, denoted $X \sim Y$, if there exists $r \geqslant 0$ for which

$$X \vee \underbrace{S^2 \vee \cdots \vee S^2}_{r} \simeq Y \vee \underbrace{S^2 \vee \cdots \vee S^2}_{r}.$$

**Remark 4.23.** Note that 'stable homotopy' is often used to refer to a different concept in the literature.

From now on, we will abbreviate $X \vee \underbrace{S^2 \vee \cdots \vee S^2}_{r}$ to $X \vee rS^2$.

We would now ask the weaker question of whether or not $\chi(X)$ and $\pi_1(X)$ determine $X$ up to stable homotopy. However, we need to first check that $\chi$ and $\pi_1$ stable homotopy invariants.

**Lemma 4.24.** If $X$ and $Y$ are 2-complexes, then $X \sim Y$ implies that $\chi(X) = \chi(Y)$ and $\pi_1(X) \cong \pi_1(Y)$, i.e. $\chi$ and $\pi_1$ are stable homotopy invariants.

*Proof.* Suppose that $X \sim Y$, i.e. $X \vee rS^2 \simeq Y \vee rS^2$ for some $r \geq 0$. Since $\chi$ and $\pi_1$ are homotopy invariants, we have that

$$\chi\left(X \vee rS^2\right) = \chi\left(Y \vee rS^2\right), \quad \pi_1\left(X \vee rS^2\right) \cong \pi_1\left(Y \vee rS^2\right).$$

But $\chi\left(X \vee rS^2\right) = \chi(X) + r$ and $\chi\left(Y \vee rS^2\right) = \chi(Y) + r$, and so $\chi(X) = \chi(Y)$.

We can also show that $\pi_1\left(X \vee rS^2\right) \cong \pi_1(X)$ and $\pi_1\left(Y \vee rS^2\right) \cong \pi_1(Y)$. To see this note that, if $X = X_{\mathscr{P}}$ for some presentation

$$\mathscr{P} = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_m \rangle,$$

then $X_{\mathscr{P}} \vee S^2 \cong X_{\mathscr{P}'}$ for

$$\mathscr{P}' = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_m, 1 \rangle.$$

This represents the same group as $\mathscr{P}$ and so $\pi_1(X_{\mathscr{P}} \vee S^2) \cong \pi_1(X_{\mathscr{P}'})$. $\square$

The above proof illustrates an important method for proving things about 2-complexes. Since 2-complexes correspond to group presentations, we can interpret operations on 2-complexes in terms of operations on group presentations (and vice versa). This often makes proofs significantly easier, though depends on Theorem 4.16 which we did not prove.

**Remark 4.25.** This also means that we can not use $\chi$ and $\pi_1$ to find some $X \not\simeq Y$ with $X \sim Y$.

**Challenge 4.26.** Prove that, if $\chi(X) = \chi(Y)$ and $\pi_1(X) \cong \pi_1(Y)$, then $X \sim Y$.

The proof should involve turning two presentations for the same group $G$ into each other by a sequence of operations of presentations, and then carefully considering how each of these operations changes the homotopy type of the underlying complex.

### 4.2.3 The Cancellation Problem for 2-Complexes

We will now deal with the problem of classifying 2-complexes in general.

We saw above that $\chi\left(X \vee S^2\right) = \chi(X) + 1$. Pick two 2-complexes over $G$, i.e. $\pi_1(X) \cong \pi_1(Y) \cong G$. Suppose that $\chi(X) = a$ and $\chi(Y) = b$. Then $\chi\left(X \vee bS^2\right) = a + b = \chi\left(Y \vee aS^2\right)$, which implies that $X \vee bS^2 \sim Y \vee aS^2$. Thus if $c = r + a$ and $d = r + a$, then $X \vee cS^2 \simeq Y \vee dS^2$.

In particular, any two 2-complexes over $G$ are connected by a series of wedges with $S^2$. This allows us to represent the set of homotopy types of 2-complexes over $G$ in the following way:

**Definition 4.27.** Let $G$ be a finitely-presented group and define $H(G)$ to be the graph whose vertices are the homotopy types of 2-complexes over $G$, and whose edge connect $X$ to $X \vee S^2$.

The discussion above implies that $H(G)$ is path-connected. We can also show:

**Exercise 4.28.** $H(G)$ is a tree.

We will refer to $H(G)$ as the *tree of homotopy types of 2-complexes over $G$*.

This follows easily from the fact that $H(G)$ can be drawn as follows, where two vertices are at the same height in the graph if and only if they have the same Euler characteristic:

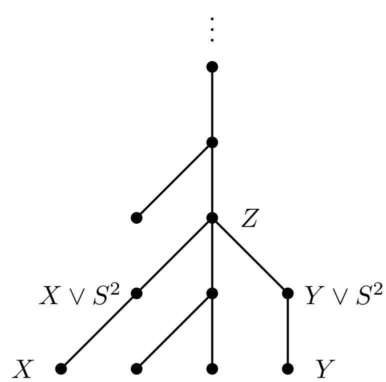



*Figure 23*

In this example, the vertices $X \vee S^2$ and $Y \vee S^2$ both connect to $Z$ since $Z \simeq X \vee S^2$ and $Z \simeq Y \vee S^2$, i.e. $X \vee S^2 \simeq Y \vee S^2$.

Fix a group $G$ and let $X$ and $Y$ be 2-complexes over $G$.

**Question 4.29.** Does $X \vee S^2 \simeq Y \vee S^2$ imply that $X \simeq Y$?

This question was known to J.H.C. Whitehead when he developed the homotopy theory of cell complexes in the early 20th century. However, it was not until 1976 that the first example of non-cancellation was discovered.

**Example 4.30.** Let $n \geqslant 3$ and let $p$ be a prime. Then, for each $i \geq 1$ with $(i, p) = 1$, define the presentation

$$\mathscr{P}(i) := \left\langle x_1, \dots, x_n \;\middle|\; x_1^p = \cdots = x_n^p = 1,\ x_1^i x_2 = x_2 x_1^i,\ x_a x_b = x_b x_a, 1 \leq a < b \leq n, (a,b) \neq (1,2) \right\rangle.$$

It is easy to see that $\mathscr{P}(1)$ is a presentation for $(\mathbb{Z}/p\mathbb{Z})^n$. Also, by changing generator from $x_1$ to $x_1^i$, we see that $\mathscr{P}(i)$ are also presentations for $(\mathbb{Z}/p\mathbb{Z})^n$.

Let $X(i) = X_{\mathscr{P}(i)}$. Then $\pi_1(X(i)) \cong (\mathbb{Z}/p\mathbb{Z})^n$ and, by the definition of $X_{\mathscr{P}(i)}$, we have that

$$\chi(X(i)) = 1 - \underbrace{n}_{\text{number of 1-cells}} + \underbrace{n + \binom{n}{2}}_{\text{number of two cells}}$$

which is independent of $i$.

From the example, we get the following theorem.

**Theorem 4.31.** $X(a) = X(b)$ if and only if $ab^{-1} \equiv \pm k^{n-1} \pmod p$, for some $k \in (\mathbb{Z}/p\mathbb{Z})^\times$.

Proving this is quite difficult and involves finding an invariant of the homotopy type which is not an invariant of the stable homotopy type.

In the other direction, it is possible to prove the following cancellation theorem in the case of finite fundamental group:

**Theorem 4.32.** Suppose $G$ is finite and $X$ and $Y$ are 2-complexes over $G$ with $\chi(X) = \chi(Y)$. Then $X \vee S^2 \simeq Y \vee S^2$.

This show that the trees of homotopy types of 2-complexes over finite groups $G$ take the following form:

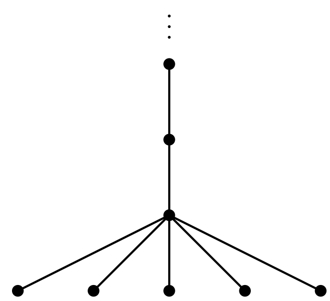


*Figure 24*

## 4.3 Other Classification Problems

We will end the course by briefly mention other classification problems in topology.

Consider the case of $n$-manifolds. These are spaces which look like $D^n$ at every point. It turns out that a nice thing to do is instead of just classifying manifolds up to homeomorphism or homotopy, we can instead try and classify them up to stable homeomorphism or stable homotopy.

A good notion of stable classification has, so far, only been found for even-dimensional spaces.

**Definition 4.33.** If $M$ and $N$ are $2n$-manifolds, then we define the connected sum $M\#N$ by cutting out discs $D_1^{2n} \subseteq M$ and $D_2^n \subseteq N$ and attaching $M \setminus D_1^{2n}$ and $N \setminus D_2^n$ along a higher dimensional cylinder $S^{2n-1} \times I$ (which works since the discs have boundary $S^{2n-1}$).

We can now define our notions of stable equivalence on $2n$-manifolds.

**Definition 4.34.** We say that $2n$-manifolds $M$ and $N$ are stably homeomorphic if there exists $r \geq 0$ for which $M\#r(S^n \times S^n) \cong N\#r(S^n \times S^n)$. Similarly we say that $M$ and $N$ are stably homotopy equivalent if there exists $r \geq 0$ for which $M\#r(S^n \times S^n) \simeq N\#r(S^n \times S^n)$.

Here we write $M\#r(S^n \times S^n)$ to mean $M\#\underbrace{(S^n \times S^n)\#\cdots\#(S^n \times S^n)}_{r}$.

In these cases, it is often possible to classify $2n$-manifolds up to stable homotopy or homeomorphism. This then reduces the classification of these manifolds to a cancellation problem similar to the one considered above.

It turns out the cancellation problem for $n$-complexes is closely related to the cancellation problem for $2n$-manifolds. Every $n$-complex $X$ is homotopy equivalent to an $n$-complex $X'$ which can be viewed as a subset of $\mathbb{R}^{2n+1}$ (this is a typical 'embedding theorem'). This complex can be 'thickened' inside of $\mathbb{R}^{2n+1}$ to give a $(2n+1)$-dimensional manifold-with-boundary $N(X)$. The boundary of $N(X)$ is a $2n$-manifold $M(X)$, and this is well-defined up to homotopy. It is possible to show that $M(X \vee S^n) \simeq M(X)\#(S^n \times S^n)$, which gives the connection between these classification problems.