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# Quantum cluster algebras from geometry

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joint work with Marta Mazzocco, Loughborough University

- Combinatorial description of $\mathfrak{T}^H_{g,s}$
- Confluences of holes and reductions of algebras of geodesic functions: from skein relations to Ptolemy relations
- Shear coordinates for bordered Riemann surfaces and new laminations: closed curves and arcs.
- From quantum shear coordinates to quantum cluster algebras and back.

Monodromy manifolds of Painlevé equations

Confluence of singularities = confluence of holes in monodromy manifolds

- How to catch the Stokes phenomenon in terms of fundamental group?
- What is the *character variety* of a Riemann surface with Stokes lines on its boundary?

Cusped character variety

- The character variety of a Riemann surface with holes is well understood.
- Use chewing-gum and cusp pull out moves to produce Stokes lines.

The cusped character variety is a cluster algebra! **(LCh–M. Mazzocco– V. Rubtsov)**

## Fat graph description for Riemann surfaces with holes

A **fat graph** (a graph with the prescribed cyclic ordering of edges entering each vertex) $\Gamma_{g,s}$ is a *spine of the Riemann surface* $\Sigma_{g,s}$ with $g$ handles and $s > 0$ holes if

(a) this graph can be embedded without self-intersections in $\Sigma_{g,s}$;

(b) all vertices of $\Gamma_{g,s}$ are three-valent;

(c) upon cutting along all edges of $\Gamma_{g,s}$ the Riemann surface $\Sigma_{g,s}$ splits into $s$ polygons each containing exactly one hole.

We set a real number $Z_\alpha$ [the Penner–Thurston $h$-lengths (logarithms of cross-ratios)] into correspondence to the $\alpha$th edge of $\Gamma_{g,s}$ if it is not a loop. To each edge that is a loop we set into correspondence the number

$$\omega_i = \begin{cases} 2\cosh(P_i/2), \\ 2\cos(\pi/p_i) \end{cases}$$

where $P_i \geq 0$ is the perimeter of the hole or $p_i \in \{2, 3, \dots\}$ the order of the orbifold point inside the monogon

## Combinatorial description of $\mathfrak{T}^H_{g,s}$—general construction

**The Fuchsian group $\Delta_{g,s} \subset PSL(2, \mathbb{R})$ and geodesic functions**, $\Sigma_{g,s} = H_2^+/\Delta_{g,s}$

| | | |
|---|---|---|
| closed geodesics on $\Sigma_{g,s}$ | $\longleftrightarrow$ | conj. classes of $\pi_1(\Sigma_{g,s})$ |
| $\updownarrow$ | | $\updownarrow$ |
| closed paths in $\Gamma_{g,s}$ | $\longleftrightarrow$ | conj. classes of $\Delta_{g,s}$ |

Every time the path homeomorphic to a (closed) geodesic $\gamma$ passes along the edge with the label $\alpha$ we insert [Fock] the *edge matrix*

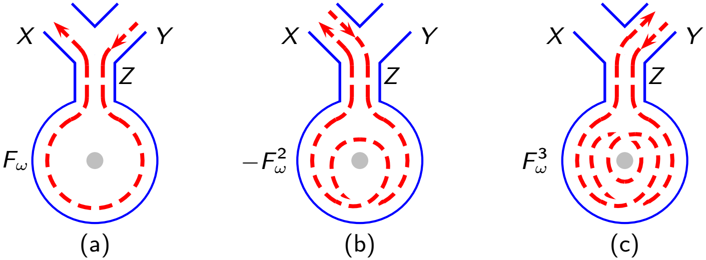
$$X_{Z_\alpha} = \begin{pmatrix} 0 & -e^{Z_\alpha/2} \\ e^{-Z_\alpha/2} & 0 \end{pmatrix};$$

every time we turn "right" or "left" at vertices, we introduce the turn matrices

$$R = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \qquad L = R^2 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}.$$

**Passing along a loop**

$$\begin{array}{ll} \text{(a)} & \ldots X_X L X_Z F_\omega X_Z L X_Y \ldots, \\ \text{(b)} & \ldots X_X L X_Z (-F_\omega^2) X_Z R X_X \ldots, \\ \text{(c)} & \ldots X_Y R X_Z (F_\omega^3) X_Z L X_Y \ldots. \end{array}, \qquad F_i = \begin{pmatrix} 0 & 1 \\ -1 & -w_i \end{pmatrix}.$$



A typical element of a Fuchsian group reads

$$P_\gamma = LX_{Y_n} RX_{Y_{n-1}} \cdots RX_{Y_2} LX_{Z_1} (-1)^{k+1} F^k_{\omega_i} X_{Z_1} RX_{Y_1},$$

The *main algebraic objects* are **geodesic functions**:

$$G_\gamma \equiv tr\, P_\gamma = 2\cosh(\ell_\gamma/2),$$

where $\ell_\gamma$ are actual lengths of the closed geodesics on $\Sigma_{g,s}$.

**Positivity** The combinations

$$RX_Y = \begin{pmatrix} e^{-Y/2} & -e^{Y/2} \\ 0 & e^{Y/2} \end{pmatrix}, \quad LX_Y = \begin{pmatrix} e^{-Y/2} & 0 \\ -e^{-Y/2} & e^{Y/2} \end{pmatrix},$$

$$RX_Z F_\omega X_Z = \begin{pmatrix} e^{-Z} + \omega & -e^{Z} \\ -\omega & e^{Z} \end{pmatrix}, \quad LX_Z F_\omega X_Z = \begin{pmatrix} e^{-Z} & 0 \\ -e^{-Z} - \omega & e^{Z} \end{pmatrix},$$

$$RX_Z(-F_\omega^{-1})X_Z, \quad LX_Z(-F_\omega^{-1})X_Z$$

as well as products of any number of these matrices have the sign structure $\begin{pmatrix} + & - \\ - & + \end{pmatrix}$, so the trace of any of $P_\gamma$ with first powers of $F_\omega$ and/or $-F_\omega^{-1}$ is a sum of exponentials with positive integer coefficients.

### Poisson structure

**Theorem**

*In the coordinates $Z_\alpha$ on any fixed spine corresponding to a surface with or without orbifold points, the Weil–Petersson bracket $B_{WP}$ reads*

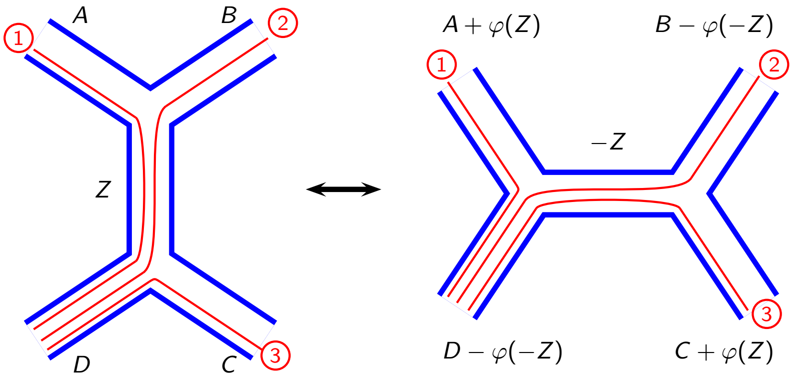
$$\{f(\mathbf{Z}), g(\mathbf{Z})\} = \sum_{\substack{\text{3-valent}\\ \text{vertices } \alpha=1}}^{4g+2s+|\delta|-4} \sum_{i=1}^{3 \mod 3} \left( \frac{\partial f}{\partial Z_{\alpha_i}} \frac{\partial g}{\partial Z_{\alpha_{i+1}}} - \frac{\partial g}{\partial Z_{\alpha_i}} \frac{\partial f}{\partial Z_{\alpha_{i+1}}} \right),$$

*where the sum ranges all three-valent vertices of a graph and $\alpha_i$ are the labels of the cyclically (clockwise) ordered ($\alpha_4 \equiv \alpha_1$) edges incident to the vertex with the label $\alpha$. This bracket gives rise to the* Goldman bracket *on the space of geodesic length functions.*

The central elements are **perimeters of holes**.
This structure is invariant w.r.t. the **flip morphisms**.

## Flip morphisms of fat graphs

- Flipping inner edges.



**Lemma**

*Flip morphisms preserve the traces of products over paths (the geodesic functions) simultaneously defining morphisms of the Poisson structures on the shear coordinates.*

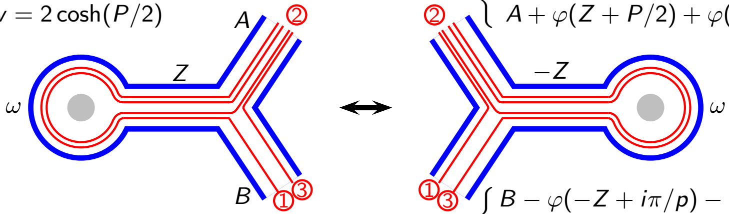
The proof of this lemma is based on the following useful *matrix* equalities (they correspond to three geodesic cases in the figure

$$\begin{aligned} X_D R X_Z R X_A &= X_{\tilde{A}} R X_{\tilde{D}}, \\ X_D R X_Z L X_B &= X_{\tilde{D}} L X_{\tilde{Z}} R X_{\tilde{B}}, \\ X_C L X_D &= X_{\tilde{C}} L X_{\tilde{Z}} L X_{\tilde{D}}. \end{aligned}$$

## Flipping the edge incident to a loop

$$\begin{cases} w = 2\cos(\pi/p) \\ w = 2\cosh(P/2) \end{cases}$$

$$\begin{cases} A + \varphi(Z + i\pi/p) + \varphi(Z - i\pi/p) \\ A + \varphi(Z + P/2) + \varphi(Z - P/2) \end{cases}$$




$$\begin{cases} B - \varphi(-Z + i\pi/p) - \varphi(-Z - i\pi/p) \\ B - \varphi(-Z + P/2) - \varphi(-Z - P/2) \end{cases}$$

This flip again preserves the geodesic functions and is a morphism of the Poisson structure.

The classical and Poisson algebras of geodesic functions are therefore invariant w.r.t. the choice of the spine $\Gamma_{g,s}$.

**Quantum MCG transformations**: $Z_\alpha \to Z^\hbar_\alpha$—elements of a $C^*$ algebra

$$[Z^\hbar_\alpha, Z^\hbar_\beta] = 2\pi i\hbar\{Z_\alpha, Z_\beta\}, \qquad *\text{-structure}: \ (Z^\hbar_\alpha)^* = Z^\hbar_\alpha.$$

[Ch, Fock] the *quantum flip morphisms*

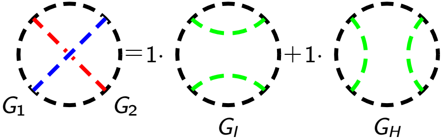
$$\begin{aligned}\{A,B,C,D,Z\} &\to \{A+\varphi^\hbar(Z), B-\varphi^\hbar(-Z), C+\varphi^\hbar(Z), D-\varphi^\hbar(-Z), -Z\}\\ &:= \{\tilde A, \tilde B, \tilde C, \tilde D, \tilde Z\},\end{aligned}$$

$$\varphi^\hbar(z) = -\frac{\pi\hbar}{2}\int_\Omega \frac{e^{-ipz}}{\sinh(\pi p)\sinh(\pi\hbar p)}dp,$$

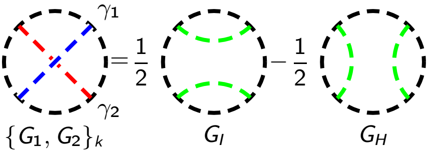
are morphisms of $C^*$ algebra.

# Combinatorial description of $\mathfrak{T}^H_{g,s}$—general construction

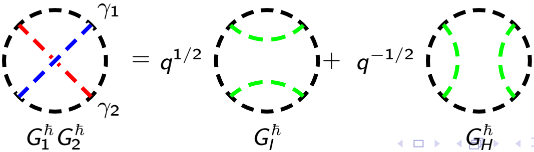
Classical skein relation




Goldman brackets for geodesic functions

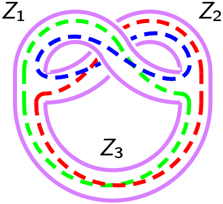



Quantum skein relation




$q = e^{-i\pi\hbar}$

Example: $\Sigma_{1,1}$:




$$G_{12} = \mathrm{Tr}(RX_{z_1}LX_{z_2}) = e^{z_1/2+z_2/2} + e^{-z_1/2-z_2/2} + e^{-z_1/2+z_2/2}$$
$$G_{23} = \mathrm{Tr}(RX_{z_2}LX_{z_3}) = e^{z_2/2+z_3/2} + e^{-z_2/2-z_3/2} + e^{-z_2/2+z_3/2}$$
$$G_{13} = \mathrm{Tr}(RX_{z_3}LX_{z_1}) = e^{z_3/2+z_1/2} + e^{-z_3/2-z_1/2} + e^{-z_3/2+z_1/2}$$

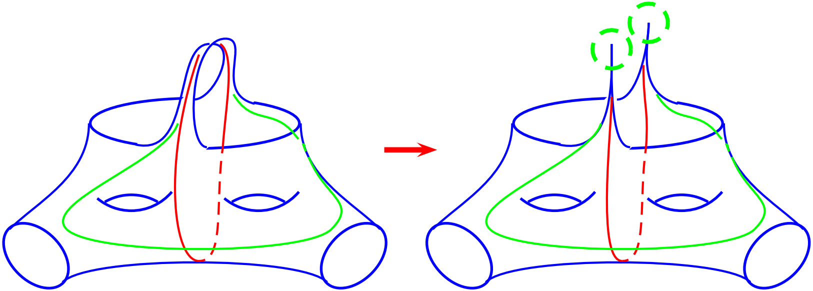
Central element: $2 - e^{z_1+z_2+z_3} - e^{-z_1-z_2-z_3} = G_{12}^2 + G_{13}^2 + G_{23}^2 - G_{12}G_{13}G_{23}$,

$$\{G_{12}, G_{13}\} = G_{23} - \tfrac{1}{2}G_{12}G_{13}, \quad [G^\hbar_{12}, G^\hbar_{13}]_q = (q - q^{-1})G^\hbar_{23},$$
$$\{G_{23}, G_{12}\} = G_{13} - \tfrac{1}{2}G_{12}G_{23}, \quad [G^\hbar_{23}, G^\hbar_{12}]_q = (q - q^{-1})G^\hbar_{13},$$
$$\{G_{13}, G_{23}\} = G_{12} - \tfrac{1}{2}G_{13}G_{23}, \quad [G^\hbar_{13}, G^\hbar_{23}]_q = (q - q^{-1})G^\hbar_{12}.$$

$[A, B]_q := q^{1/2}AB - q^{-1/2}BA.$

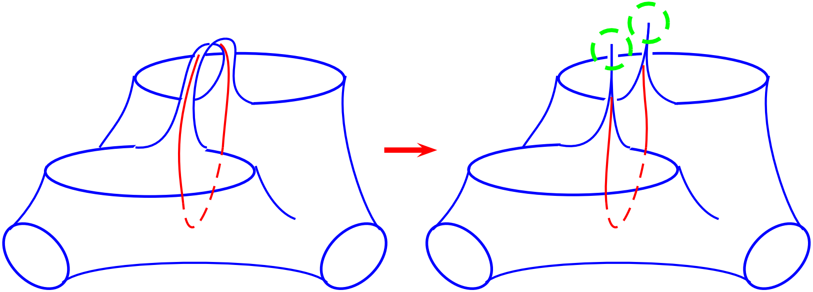
"Chewing gums" and degenerations of geodesic algebras

**1.** The result of a confluence of two holes of a Riemann surface $\Sigma_{g,s,n}$ of genus $g$ with $s$ holes/orbifold points, and $n$ bordered cusps is a Riemann surface of the same genus $g$, $s-1$ holes/orbifold points, and $n+2$ bordered cusps, and the hole obtained by the confluence of two original holes now contains two new bordered cusps:



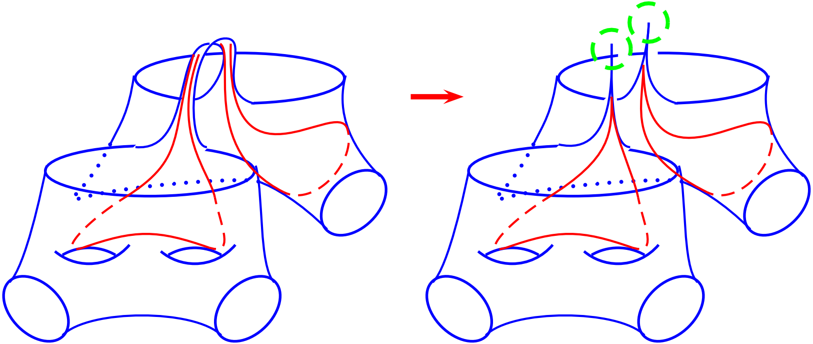
The result of confluence of sides of the same hole depends on whether breaking the chewing gum will result in one or two disjoint components.

**2a**. When breaking the chewing gum constituted by sides of the same hole in $\Sigma_{g,s,n}$ results in a one-component surface, the new surface $\Sigma_{g-1,s+1,n+2}$ will have genus lesser by one (so, originally, $g > 0$), the hole will split into two holes each containing one new bordered cusp.

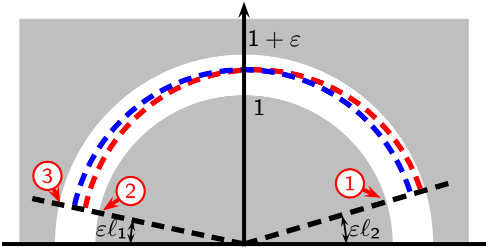


# Confluences of holes: new geodesic laminations

**2b.** When breaking the chewing gum constituted by sides of the same hole in $\Sigma_{g,s,n}$ results in a two-component surface, these new surfaces, $\Sigma_{g_1,s_1,n_1}$ and $\Sigma_{g_2,s_2,n_2}$, must be stable and such that $g_1+g_2=g$, $s_1+s_2=s+1$, and $n_1+n_2=n+2$ with $n_1>0$ and $n_2>0$



"Chewing gum" and limiting geodesics




We introduce the limit:

$$G_\gamma \to G_\gamma e^{-\widehat{D}_\gamma/2},$$

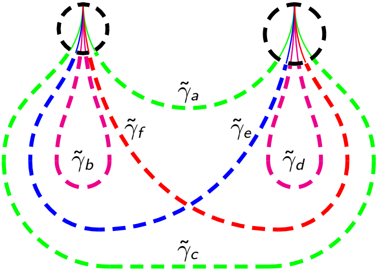
where $\widehat{D}_\gamma$ is the (total) geodesic length of the part(s) of the geodesic function passing through the chewing gum. In the limit as $\varepsilon \to 0$, *all* $\widehat{D}_\gamma$ are the same, that is parts of all geodesics confined between two collars have the same (infinite) length! In the limit as $\varepsilon \to 0$, collars become **horocycles** of a new R.s. New shear coordinate $e^{\pi_i} := \ell_i$, then

$$e^{D_{12}/2} = (\varepsilon)^{-1} e^{-\pi_1/2-\pi_2/2},$$

# Confluences of holes: new geodesic laminations

New geodesic functions: **arcs** $\mathfrak{a} = e^{\ell_\gamma/2}$, where $\ell_\gamma$ is the length of the finite part confined between two horocycles of a geodesic starting and terminating at bordered cusp(s).

In the limit, we obtain the **Ptolemy relation**:




$$\tilde{\gamma}_f \tilde{\gamma}_e = \tilde{\gamma}_a \tilde{\gamma}_c + \tilde{\gamma}_b \tilde{\gamma}_d$$

**Definition**

$\mathfrak{T}^H_{g,s,n} = \mathbb{R}^{6g-6+3s+2n}$ is the Teichmüller space of Riemann surfaces of genus $g$ with $s > 0$ holes, and $n > 0$ decorated bordered cusps. Every bordered cusp is associated to one of the holes, and we have a natural cyclic ordering of bordered cusps associated to the same hole due to the orientation of the Riemann surface. The space $\mathfrak{T}^H_{g,s,n}$ is the space of shear coordinates of a spine $\Gamma_{g,s,n}$ representing the corresponding Riemann surface.

A fat graph $\Gamma_{g,s,n}$ is a *spine of the Riemann surface* $\Sigma_{g,s,n}$ if

(a) this graph can be embedded without self-intersections in $\Sigma_{g,s,n}$;

(b) all vertices of $\Gamma_{g,s,n}$ are three-valent except exactly $n$ one-valent vertices placed at the corresponding bordered cusps;

(c) upon cutting along all edges of $\Gamma_{g,s,n}$ the Riemann surface $\Sigma_{g,s,n}$ splits into $s$ polygons each containing exactly one hole (or orbifold point);

(d) the above polygons are monogons for *all* orbifold points and *all* holes to which no bordered cusps are associated.

**Definition**

We call a geometric *geodesic lamination* (GL) on a bordered Riemann surface a set of nondirected curves up to a homotopical equivalence such that

(a) these curves are either closed curves ($\gamma$) or *arcs* ($\mathfrak{a}$) that start and terminate at bordered cusps (which can be the same cusp);

(b) these curves have no (self)intersections inside the Riemann surface (but can be incident to the same bordered cusp); they are neither empty loops nor empty loops starting and terminating at the same cusp.

The algebraic GL is

$$\prod_{\gamma\in GL}(2\cosh(l_\gamma/2))\prod_{\mathfrak{a}\in GL}e^{l_\mathfrak{a}/2}:=\prod_{\gamma\in GL}G_\gamma\prod_{\mathfrak{a}\in GL}G_\mathfrak{a}$$

where $l_\gamma$ are geodesic lengths of closed curves and $l_\mathfrak{a}$ are signed geodesic lengths of parts of arcs $\mathfrak{a}$ contained between two horocycles; $G_\gamma=2\cosh(l_\gamma/2)$ and $G_\mathfrak{a}=e^{l_\mathfrak{a}/2}$ are the corresponding geodesic functions.

Note that in thus defined GL sets of ends of arcs entering the same bordered cusp are *linearly ordered* w.r.t. the orientation of the Riemann surface.

# Combinatorial description of $\mathfrak{T}^H_{g,s,n}$

We endow *all* edges that are not loops with real numbers $Z_\alpha$ (for internal edges) and $\pi_j$ (for "pending" edges).
We then construct geodesic functions for closed geodesics as before, whereas those for $G_\mathfrak{a}$ are given by traces

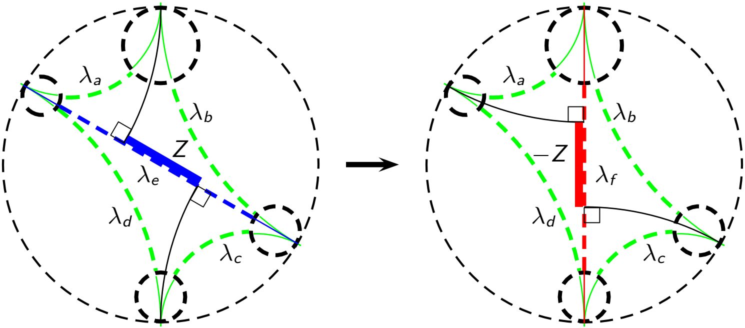
$$G_\mathfrak{a} = \mathrm{Tr}[KX_{\pi_1}LX_{Z_\alpha}\cdots X_{Z_r}LF_{\omega_i}LX_{Z_r}\cdots X_{Z_\beta}RX_{\pi_2}],$$

where $X_{Z_\alpha}$ and $X_{\pi_j}$ are the edge matrices $X_A = \begin{bmatrix} 0 & -e^{A/2} \\ e^{-A/2} & 0 \end{bmatrix}$; $A$ is either $Z_\alpha$ or $\pi_j$, the turn matrices are the same as before, $R = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$, $L = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$, and the new matrix

$$K = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}.$$

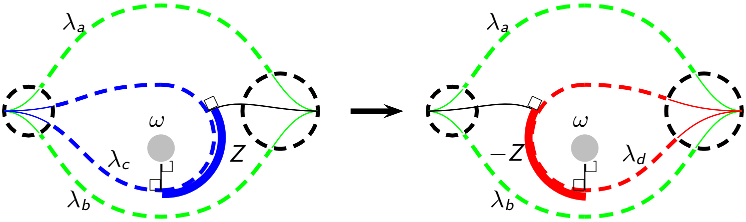
**Relation between shear coordinates and $\lambda$-lengths**

The very important remark is that thus defined arcs are nothing but $\lambda$-lengths on the corresponding bordered Riemann surface [S.Fomin, M.Shapiro, D.Thurston]; for inner edges:




$e^Z = \dfrac{\lambda_b\lambda_d}{\lambda_a\lambda_c}$; **mutation:** $\lambda_f = \dfrac{\lambda_a\lambda_c + \lambda_b\lambda_d}{\lambda_e}$

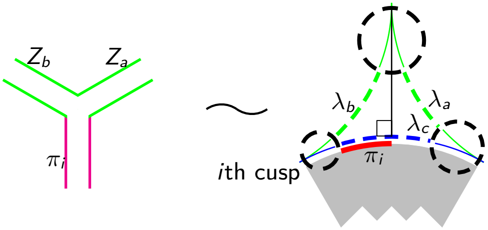
for loops:




$e^Z = \frac{\lambda_b}{\lambda_a}$; **mutation:** $\lambda_d = \frac{\lambda_a^2 + \lambda_b^2 + \omega\lambda_a\lambda_b}{\lambda_c}$ [Ch.,Shapiro]

# Combinatorial description of $\mathfrak{T}^H_{g,s,n}$

for decorated cusps:




$$e^{\pi_i} = \frac{\lambda_c \lambda_b}{\lambda_a};$$

**no mutation of border arcs are allowed**

## Lemma

*For any $\Sigma_{g,s,n}$ with $n > 0$ we have a complete lamination comprising exactly $6g - 6 + 3s + 2n - s_\omega$ arcs $\lambda_r$, where $s_\omega$ is the number of momogons or holes/orbifold points without bordered cusps (with the corresponding parameters $\omega_i$). The shear coordinates $\{Z_\alpha, \pi_j, \omega_i\}$ on $\Gamma_{g,s,n}$ dual to this set of arcs are in 1-1 correspondence with $\{\lambda_r, \omega_i\}$.*

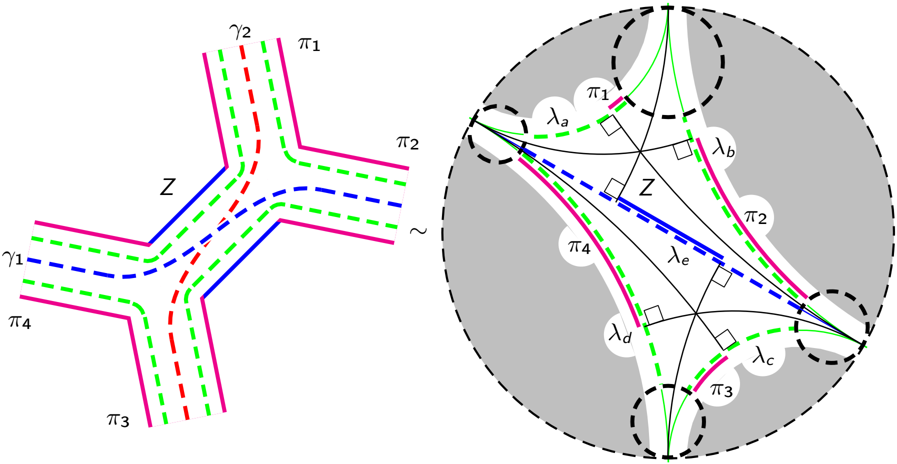
**Laurent property and positivity**

arcs= cluster varieties

complete GLs of arcs=seeds

Given a seed, we have the set of $Z_\alpha$ and $\pi_j$ expressed as monomials of $\lambda_r$; any other arc, or cluster variety of any other seed, is a positive Laurent polynomial of the shear coordinates thus being a Laurent polynomial with positive integer coefficients of the original $\lambda_r$. We thus obtain the positivity and Laurent property for $\lambda$-lengths.

# Combinatorial description of $\mathfrak{T}^H_{g,s,n}$. Example: $\Sigma_{0,1,4}$




$$G_{\gamma_1} = e^{\pi_1/2+\pi_3/2+Z/2} + e^{\pi_1/2+\pi_3/2-Z/2}, \quad G_{\gamma_2} = e^{\pi_2/2+\pi_4/2+Z/2}$$

**Theorem**

*In the coordinates $Z_\alpha$, $\pi_j$ of $\mathfrak{T}^H_{g,s,n}$ on any fixed spine $\Gamma_{g,s,n}$ corresponding to a surface with at least one bordered cusp, the Weil–Petersson bracket $B_{\mathbf{WP}}$ reads*

$$\{f(\mathbf{Y}), g(\mathbf{Y})\} = \sum_{\substack{\text{3-valent} \\ \text{vertices } \alpha = 1}}^{4g+2s+|\delta|-4} \sum_{i=1}^{3 \mod 3} \left( \frac{\partial f}{\partial Y_{\alpha_i}} \frac{\partial g}{\partial Y_{\alpha_{i+1}}} - \frac{\partial g}{\partial Y_{\alpha_i}} \frac{\partial f}{\partial Y_{\alpha_{i+1}}} \right),$$

*where the sum ranges all three-valent vertices of a graph that are not adjacent to loops and $\alpha_i$ are the labels of the cyclically (clockwise) ordered ($\alpha_4 \equiv \alpha_1$) edges $Y_\alpha$, which are either $Z_\beta$ or $\pi_j$, incident to the vertex with the label $\alpha$. This bracket*

(1) *is equivariant w.r.t. the morphisms generated by flips (mutations) of inner edges and by flips (mutations) of edges adjacent to loops;*

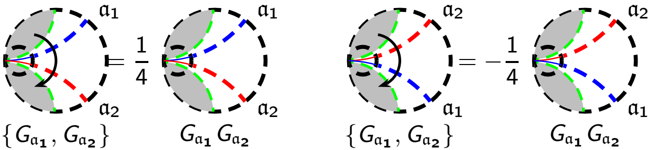
(2) *gives rise to the* Goldman bracket *on the space of GLs.*

*The Casimirs are $\sum_{\alpha \in I} Z_\alpha$ where the sums range (with proper multiplicities) edges bounding a cusped hole (labeled $I$). The coefficients $\omega_i$ of monogons are central by construction.*

# Poisson and quantum cluster algebras for $\mathfrak{T}^H_{g,s,n}$

**Lemma**

*The Poisson brackets on $Z_\alpha$, $\pi_j$ induce homogeneous Goldman-type Poisson relations for arcs ($\lambda$-lengths) constituting a GL.*




We evaluate these Goldman brackets for all four combinations of ends of two arcs (ends at different cusps Poisson commute); for instance, in the case where all four ends are at the same cusp, and the both ends of $\mathfrak{a}_1$ are to the right of both ends of $\mathfrak{a}_2$ (provided these arcs has no intersections inside the Riemann surface), the total bracket will be $\{G_{\mathfrak{a}_1}, G_{\mathfrak{a}_2}\} = G_{\mathfrak{a}_1} G_{\mathfrak{a}_2}$.

## Quantization

$$\{Z_\alpha, \pi_j\} \to \{Z^\hbar_\alpha, \pi^\hbar_j\}(= Y^\hbar_j): \ [Y^\hbar_i, Y^\hbar_j] = 2\pi i\hbar\{Y_i, Y_j\}; \ [Z^\hbar_\alpha]^* = Z^\hbar_\alpha, \ [\pi^\hbar_j]^* = \pi^\hbar_j$$

### Definition

We define the **quantum cluster variable** corresponding to an arc to be

$$G^\hbar_{\mathfrak{a}} = \mathrm{Tr}[KX_{\pi_1}LX_{Z_\alpha}\cdots X_{Z_r}LF_{\omega_i}LX_{Z_r}\cdots X_{Z_\beta}RX_{\pi_2}],$$

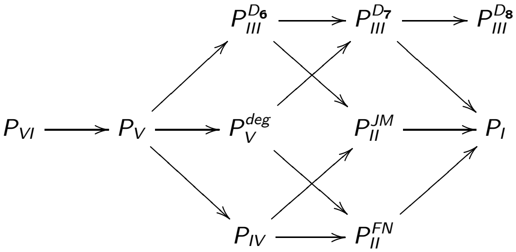
where $X_{Z_\alpha}$ and $X_{\pi_j}$ are the edge matrices with quantum entries and the quantum ordering is the natural ordering in the product. The matrices $R \to q^{1/4}R$, $L \to q^{-1/4}L$, $K \to K$, and $F_\omega \to F_\omega$.

### Theorem

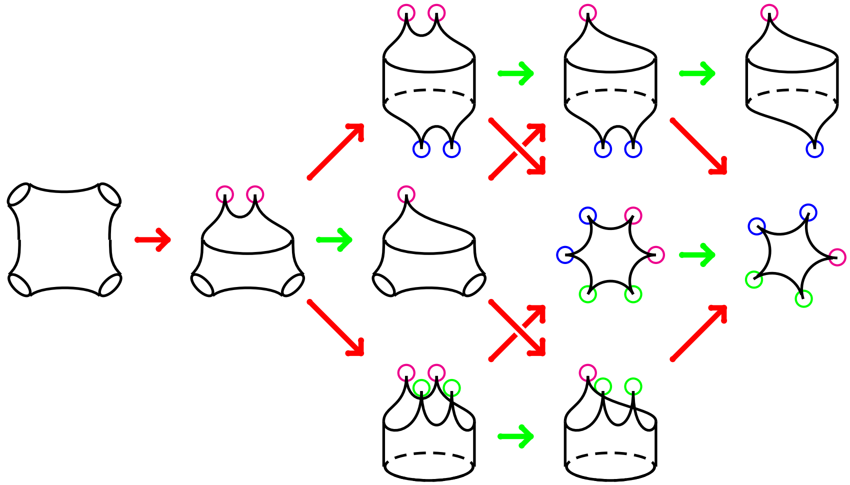
*The thus defined quantum cluster variables are invariant under the quantum MCG transformation and satisfy the quantum skein relations. In particular, the quantum arcs from the same GL (or quantum variables from the same seed) satisfy q-commutation relations.*

We thus identify our quantum arcs with quantum cluster algebras of Berenstein and Zelevinsky.

# Confluences for Painlevé eqs. (LCh, Mazzocco, Roubtsov in preparation)



## Confluences for Painlevé eqs. (LCh, Mazzocco, Roubtsov in preparation)



THANK YOU!

**Vth International Workshop on Combinatorics of Moduli Spaces, Hurwitz Numbers, and Cohomological Field Theories**

*Organized by Laboratoire Poncelet, Steklov Mathematical Institute, and Higher School of Economics, Moscow*

Key speakers include: J. Andersen (Aarhus), G. Borot (MPIM), B. Dubrovin$^{\star}$ (SISSA), B. Eynard (Saclay), V. Fock (Strasbourg), S. Fomin (UMich), R. Kashaev (Geneve), Melissa Liu (Columbia), A. Marshakov (Moscow), M. Mazzocco (L-boro), P. Norbury (Melbourne), R. Penner$^{\star}$ (Caltech), V. Roubtsov (Angers), S. Shadrin (UvA), M. Shapiro (MSU), A. Zabrodin (Moscow), Don Zagier (MPIM), P. Zograf (St.Pb.), D. Zvonkine (ParisVI)

- Cluster algebras and partitions of Riemann surfaces
- Topological recursion in problems of Hurwitz numbers, Frobenius manifolds, and cohomological field theories
- Quantum spectral curves and integrable systems: D2→D3