ON THE PHASE SHIFT IN THE KUZMAK-WHITHAM ANSATZ

Sergey Dobrokhotov
A.Ishlinskiy Institute for Problems in Mechanics of
Russian Academy of Sciences
&
Moscow Institute of Physics and Technology

joint work with D.Minenkov

**INTEGRABILITY IN ALGEBRA, GEOMETRY AND PHYSICS:
NEW TRANDS
A. VESELOV'S 60 TH BIRTHDAY**

Switzerland   13-17 July, 2015

The Kuzmak-Whitham anzats = a class of special asymptotic solutions

Linear plane wave:

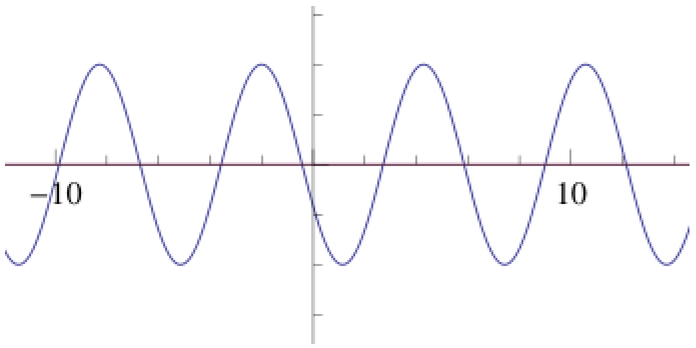
$$u = A\cos(k \cdot x' - \omega t' + \Phi)$$

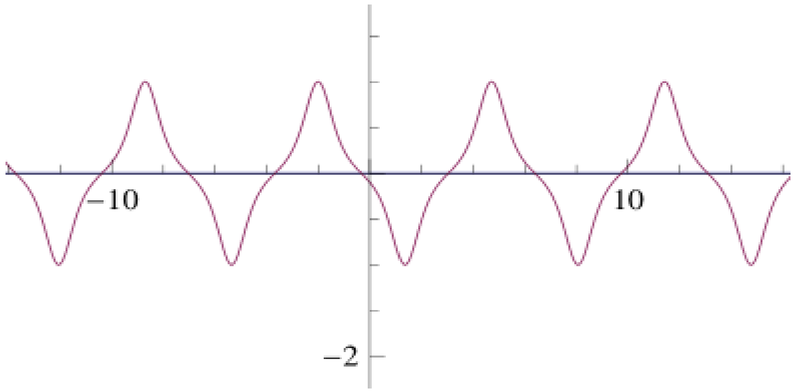
Nonlinear plane wave (for instance a cnoidal wave of the KdV equation):

$$u = X(k \cdot x' - \omega t' + \Phi, E)$$

$\omega, k, A, E = (E_0, E_1, \dots E_l), \Phi$ are parameters,

$X(\theta, E)$ is 2π-periodic function on θ





Modulated wave with slightly varying parameters:

$\omega, k, A, E, \Phi \to \omega(\varepsilon x', \varepsilon t'), k(\varepsilon x', \varepsilon t'), A(\varepsilon x', \varepsilon t'), E(\varepsilon x', \varepsilon t'), \Phi(\varepsilon x', \varepsilon t')$

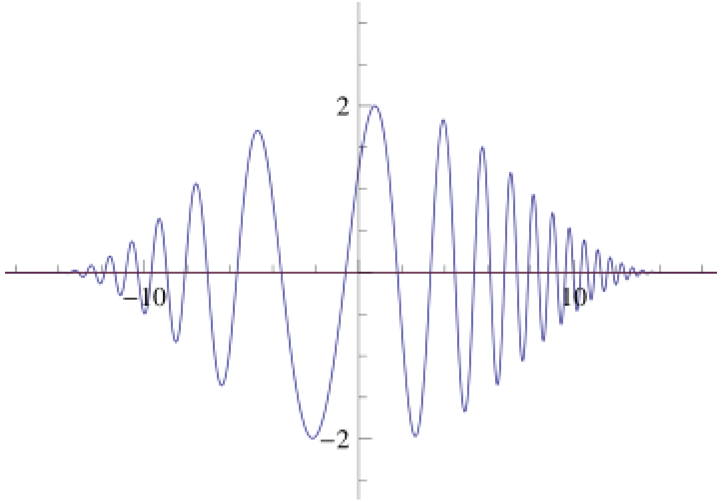
$\varepsilon > 0$ is a **small parameter**.

We put $x = \varepsilon x', t = \varepsilon t'$ and introduce the **phase**

$$S(x,t) = k(x,t) \cdot x - \omega t.$$

This gives **Kuzmak-Whitham anzats for one-phase (formal) asymptotics**:

$$u(x,t,\varepsilon) = \mathcal{X}\Big(\frac{S(x,t)}{\varepsilon} + \Phi(x,t), x, t, \varepsilon\Big) =$$
$$X(\frac{S(x,t)}{\varepsilon} + \Phi(x,t), E(x,t), x, t) + O(\varepsilon),$$
$$\mathcal{X}(\theta, x, t, \varepsilon) = X(\theta, x, t) + \varepsilon X_1(\theta, x, t) + \varepsilon^2 X_2(\theta, x, t)$$

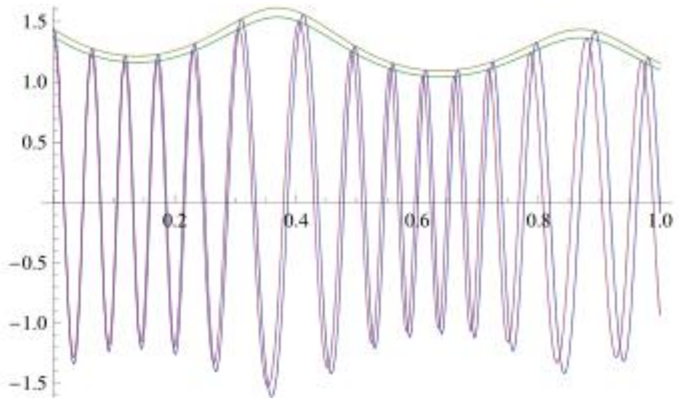


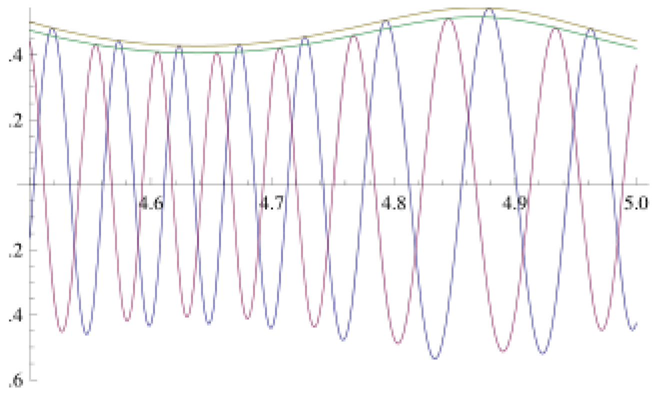
$\Phi(x,t)$ is the **phase shift**,
$X(\frac{S(x,t)}{\varepsilon} + \Phi(x,t), E(x,t), x, t)$ is the **leading term** of the asymptotics, $X_j(\theta, x, t) = X_j(\theta + 2\pi, x, t)$ are corrections.

To construct the leading term of (formal) asymptotics one should find the **"structure function"** $X(\theta, E)$ (or sometimes $X(\theta, E, x, t)$), the **phase** $S(x,t)$, the slow varying **"parameters"** $E(x,t)$ and the **phase shift** $\Phi(x,t)$.

Under some assumptions the Whitham method allows one to obtain the equations for X, $S(x,t)$ and $E(x,t)$. The equations for $E(x,t)$ (or sometimes for $S(x,t)$ and $E(x,t)$) are known as **Whitham equations**.

The influence of the phase shift: the same envelope but different phase of oscillations





KUZMAK (1959)

WHITHAM (1963)

LUKE (1966)

MASLOV (1969)

DOBROKHOTOV-MASLOV (1980)

FLASHKA-FOREST-MCLAUGHLIN (1980)

DUBROVIN-NOVIKOV (1984)

Neishtadt (1984)

TSAREV (1985)

HABERMAN, BOURLAND (1988)

KRICHEVER (1988)

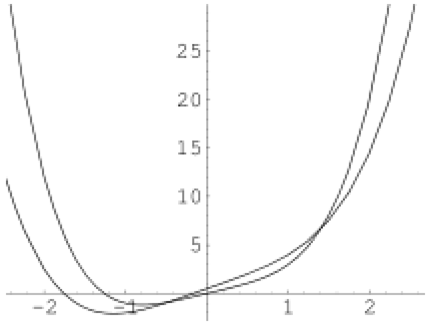
COLE, KEVORKIAN (1996)

IL'IN (1999)

....

MALTSEV (2008)

THE MAIN EXAMPLES OF EQUATIONS



The nonlinear Klein-Gordon equation

$$\varepsilon^2(u_{tt} - c^2(x,t)\triangle u) + V_u(u,x,t,\varepsilon) = 0,$$
$$u(x,t,\varepsilon)|_{t=0} = u^0(x,h), \quad hu_t(x,t,\varepsilon)|_{t=0} = v^0(x,\varepsilon)$$

Here $V(u,x,t,\varepsilon)$, $c(x,t)$, $u^0(x)$ и $v^0(x)$ $\in$ $\mathbb{R}$ –are smooth functions; $t \in \mathbb{R}$, $x \in \mathbb{R}^n$, $0 < \varepsilon \ll 1$ is a small parameter. The function $V(u,x,t,\varepsilon)$ for each (x,t) has a form of a potential well which guarantees the existence of rapidly oscillating solutions.

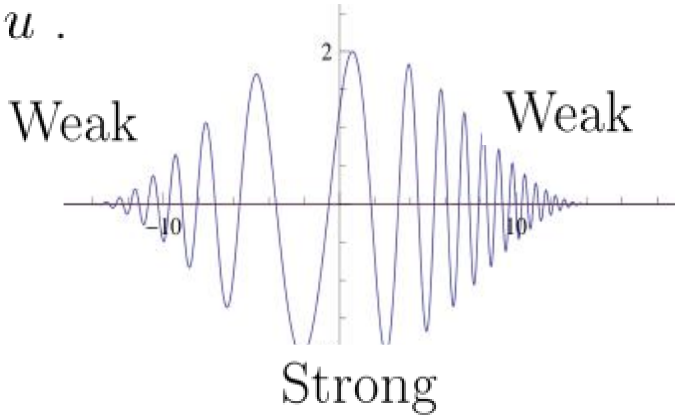
Examples:

(1) $V = V(u,x,t,\varepsilon) = \frac{1}{2}q^2(x,t)u^2 + \lambda V_1(u,x,t), V_1(u,x,t) = O(u^3)$ (2) $V = q^2(x,t)\cosh u$, (3) $V = q^2(x,t)\cos u$.

"Artificial" classification.

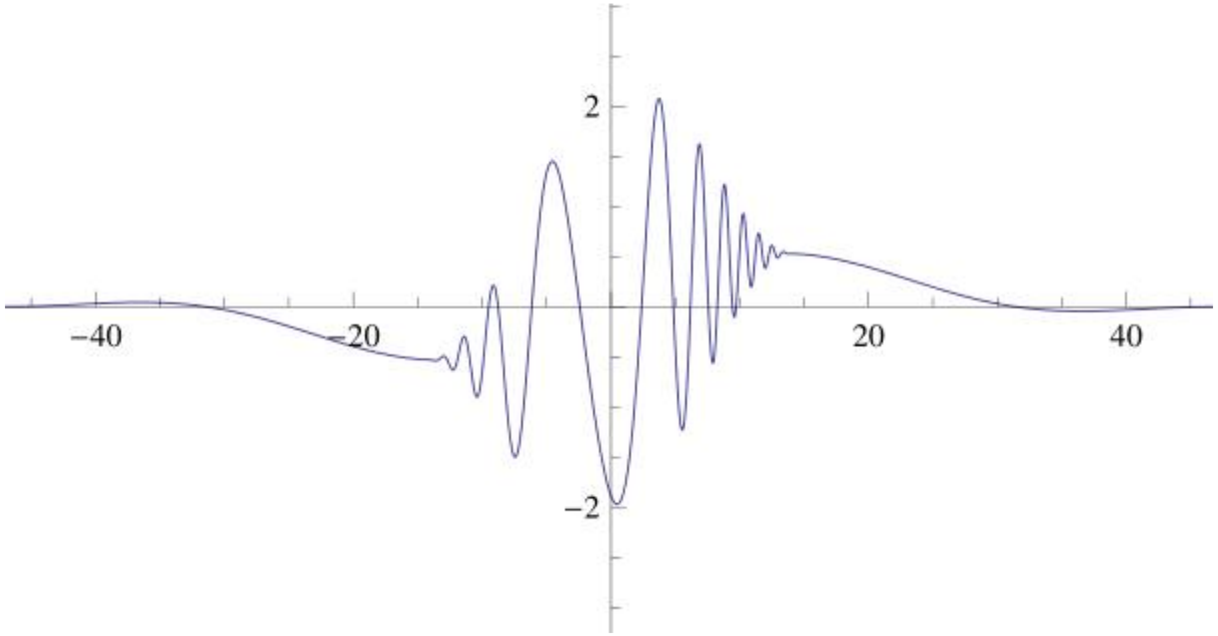
Weak nonlinearity: $\lambda = O(\varepsilon)$, $X = A\cos\theta$.

Strong nonlinearity: $\lambda = O(1)$, $X = X(\theta, E, x, t)$



The special Cauchy problem for the Korteveg-de Vries equation with small dispersion:

$$u_t - 6uu_x + h^2 u_{xxx} = 0,\ u(x,t,h)|_{t=0} = u^0(x,h),$$



The problem: the Whitham equations are of the **first order** with respect to time t and very often has good "geometrical" structure. The equation for the phase shift is of the **first order** with respect to time t for **weak** nonlinear cases and is of the **second order** with respect to time t for the strong nonlinear cases. There is **no** uniform passage from strong to weak nonlinear case. Also the equation for the phase shift has not good structure.

The main result is that one can **to eliminate the phase shift** from the final asymptotic formulas and "play" only with the Whitham equations.

The Korteveg-de Vries equation:

$$u_t - 6uu_x + \varepsilon^2 u_{xxx} = 0, \quad x \in \mathbb{R}, \quad t \geq 0,$$

The periodic cnoidal wave: "one gap solution" with the ends of the bands $E_0 < E_1 \leq E_2$

$$X(\theta, E) = C(E) - 2\Big(k(E)\frac{\partial}{\partial\theta}\Big)^2 \ln\Theta\big(\theta \,\big|B(E)\big),$$

$$\Theta(\theta|B) = \sum_{l\in\mathbb{Z}} e^{\frac{B}{2}l^2} e^{il\theta},$$

$$C(E) = E_0 + E_1 + E_2 - 2\int_{E_2}^{E_1} \frac{zdz}{R(z)} \Big/ \int_{E_2}^{E_1} \frac{dz}{R(z)},$$

$$B(E) = -2\pi \int_{E_1}^{E_0} \frac{dz}{|R(z)|} \Big/ \int_{E_2}^{E_1} \frac{dz}{R(z)} < 0,$$

$$R^2(z, E) = (z - E_0)(z - E_1)(z - E_2)$$

$$k(E) = -2\int_{E_2}^{E_1} \Omega_1, \quad \Omega(E) = -24\int_{E_2}^{E_1} \Omega_2,$$

For the Kuzmak-Witham anzats we have:

$$E = E(x,t),$$

$$dS = \Omega(E(x,t)dt + k(E(x,t))dx, \quad S|_{t=0} = S^0(x)$$

$$u = X(\frac{S(x,t)}{h} + \Phi(x,t), E(x,t)) + hX_1(\frac{S(x,t)}{h} + \Phi(x,t), x, t) + O(h^2),$$

PLUS **the Whitham equations for** $E(x,t)$:

$$\frac{\partial}{\partial t}E_l = -\Lambda_l(E)\frac{\partial}{\partial x}E_l, \quad l = 0,1,2,$$

$$\text{here} \quad \Lambda_l(E) = 6\Big(E_0 + E_1 + E_2 - 2E_l - 2\frac{\int_{E_1}^{E_0}(E_l - x)\frac{zdz}{|R(z)|}}{\int_{E_1}^{E_0}(E_l - z)\frac{dz}{|R(z)|}}\Big),$$

$$E|_{t=0} = E^0(x)$$

PLUS **the second order equation (for strong nonlinear case) for the phase shift**

$$\alpha_1[S,E]\Phi_{tt} + \alpha_2[S,E]\Phi_t + \alpha_3[S,E] = 0.$$

PROPOSITION for the KdV and nonlinear Klein-Gordon equations and the CONJECTURE for many others.

For any Whitham-type solution one can always correct the initial data for the Whitham equations

$$E|_{t=0} = E^0(x) \to E|_{t=0} = E^0(x) + \varepsilon E^1(x)$$

which change

$$S(x,t) \to \tilde{S}(x,t,h) = S(x,t) + \varepsilon\Phi(x,t) + O(\varepsilon^2),$$
$$E(x,t) \to \tilde{E}(x,t,\varepsilon) = E(x,t) + O(\varepsilon),$$

and reconstruct $X_1 \to \tilde{X}_1$ in such a way that **the phase shift disappear**

$$u = X(\frac{\tilde{S}(x,t,\varepsilon)}{\varepsilon}, \tilde{E}(x,t,\varepsilon))) + \varepsilon\widetilde{X}_1(\frac{\tilde{S}(x,t)}{\varepsilon}, x, t) + O(\varepsilon^2).$$

This representation is uniform with respect to a passage from weak to strong nonlinear case.

Results for Nonlinear Klein-Gordon :

$$u(x,t,h) = X\Big(\frac{S(x,t)}{\varepsilon} + \Phi(x,t), I(x,t), x, t\Big) + \varepsilon X_1\Big(\frac{S(x,t)}{\varepsilon} + \Phi(x,t), x, t\Big) + O(\varepsilon^2).$$

Ordinary differential equation for $X(\theta, E, x, t)$

$$\mathbf{\Omega}^2(x,t)X_{\theta\theta} + V_u(X,x,t) = 0,$$
$$\mathbf{\Omega}^2(x,t) = S_t^2 - c^2(x,t)(\nabla S)^2.$$

Hence $X(\theta, E, x, t)$:

$$\theta = \pm\mathbf{\Omega}(x,t)\int_{u_{\min}}^{X(\theta,x,t)} \frac{dz}{\sqrt{2(E(x,t) - V(z,x,t))}},$$

here $E(x,t)$ is the “constant of integration”, “+” is for $\theta \geq 0$ and “−”is for $\theta \leq 0$; $u_{\min}(x,t), u_{\max}(x,t), u_{\min} < u_{\max}$ are roots of the equation $E(x,t) = V(u,x,t)$.

Along with E we introduce the action type "parameter":

$$I(x,t) = \frac{1}{\pi}\int_{u_{\min}}^{u_{\max}} \sqrt{2(E(x,t) - V(z,x,t))}dz.$$

Then $E(x,t) = E(I(x,t),x,t)$ and $X(\theta,x,t) = \mathcal{X}(\theta, I(x,t),x,t)$. 2π –Periodicity of X with respect to θ means

$$\mathbf{\Omega}(x,t) = \Omega(I(x,t),x,t) \equiv \pi / \int_{u_{\min}}^{u_{\max}} \frac{dz}{\sqrt{2(E(I,x,t) - V(z,x,t))}}.$$

Together with the first orthogonality condition this gives Whitham type equations for the phase S and "parameter" I (or E):

$$S_t^2 - c^2(x,t)(\nabla S)^2 = \Omega(I,x,t),$$

$$\frac{\partial}{\partial t}\Big[\frac{I}{\Omega(I,x,t)}S_t\Big] - c^2(x,t)\Big\langle \nabla, \frac{I}{\Omega(I,x,t)}\nabla S\Big\rangle = 0,$$

The equation for the phase shift $\Phi(x,t)$

1) **Weak nonlinearity** $\Omega_I(I,x,t)\equiv 0$

$$S_t\Phi_t - c^2(x,t)\langle\nabla S,\nabla\Phi\rangle = 0,\quad \Phi(x,t)|_{t=0} = \Phi^0(x).$$

2) **Strong nonlinearity** $\Omega_I(I,x,t)\neq 0)$.

$$\frac{\partial}{\partial t}\Big[\frac{I}{\Omega}\Phi_t\Big] - c^2\Big\langle\nabla,\frac{I}{\Omega}\nabla\Phi\Big\rangle +$$

$$\frac{\partial}{\partial t}\Big[(\frac{1}{\Omega}-\frac{I\Omega_I}{\Omega^2})I_1S_t\Big] - c^2\Big\langle\nabla,(\frac{1}{\Omega}-\frac{I\Omega_I}{\Omega^2})I_1\nabla S\Big\rangle = 0,$$

$$\text{here}\quad I_1\equiv\frac{S_t\Phi_t - c^2\langle\nabla S,\nabla\Phi\rangle}{\Omega\Omega_I},$$

$$S(x,t)|_{t=0} = S^0(x),\quad I(x,t)|_{t=0} = I^0(x),\quad \Phi(x,t)|_{t=0} = \Phi^0(x)$$

$$\Phi_t(x,t)|_{t=0} = \Omega_I^0(x)I_1^0(x),$$

$$\text{where}\quad I_1^0(x)\equiv -\frac{1}{(\Omega^0(x))^2}\int_0^{\Phi^0(x)} V_t(X(\theta,x,0)) - \overline{V_t}(x,0)d\theta,$$

$$\overline{V_t}(x,0)(x,t)\equiv\frac{1}{2\pi}\int_0^{2\pi} V_t(X(\theta,x,t),x,t)d\theta.$$

The uniform representation (Almost nothing necessary to do!)

$$S(x,t) \to \tilde{S}(x,t,\varepsilon) = S(x,t) + \varepsilon\Phi(x,t) + O(\varepsilon^2),$$
$$I(x,t) \to \tilde{I}(x,t,\varepsilon) = I(x,t) + \varepsilon I_1(x,t) + O(\varepsilon^2),$$

$\tilde{S}(x,t,h), \tilde{I}(x,t,\varepsilon)$ is the solution to the **same Whitham system** with corrected initial data

$$\tilde{S}(x,t,\varepsilon)|_{t=0} = S^0(x) + \varepsilon\Phi^0(x),$$
$$\tilde{I}(x,t,\varepsilon)|_{t=0} = I^0(x) + \varepsilon I_1^0(x),$$

then

$$u(x,t,h) = X\big(\frac{\tilde{S}(x,t,\varepsilon)}{\varepsilon}, \tilde{I}(x,t,\varepsilon), x, t\big) + O(\varepsilon)$$

The phase shift disappears!

The “stability” of Kusmak-Whitham asymptotics:

We see that if for $t = 0$ one changes in prescribed “one phase” class the initial data by $O(\varepsilon)$ the leading term can be changed by $O(1)$! But **it does mean** that the Kusmak-Whitham asymptotics are unstable in **Lyapunov sense**. It just means that possible changes of the initial data should be $O(\varepsilon^{1+\gamma}), \gamma > 0$.

Kuzmak-Whitham asymptotics for the anharmonic oscillator

$$\frac{d^2u}{dt'^2} + f(u, \varepsilon t') = 0, \quad (t = \varepsilon t') \quad \Longleftrightarrow \quad \frac{d^2u}{dt^2} + f(u,t) = 0$$

$$\Updownarrow$$

$$\dot{u} = p, \quad \dot{p} = -V_u(u,t) \equiv -f(u,t), \quad u|_{t=0} = u^0, \quad p|_{t=0} = p^0.$$

Kuzmak -Whitham - nonlinear WKB ansatz:

$$x = \mathcal{X}(\frac{S(t)}{\varepsilon}, t, \varepsilon), \qquad t = \varepsilon t, \qquad \theta = \frac{S(t)}{\varepsilon},$$

$$\mathcal{X}(\theta, t, \varepsilon) = X(\theta, t) + \varepsilon X_1(\theta, t) + \varepsilon^2 X_2(\theta, t) \cdots +,$$

Here $X_k(\theta, t)$ are $2\pi-$ periodic smooth functions of θ

Luke scheme:

We have for $\mathcal{X}(\theta, t, \varepsilon)$

$$S_t^2 \frac{\partial^2 \mathcal{X}}{\partial \theta^2} + f(\mathcal{X}, t, \varepsilon) + \varepsilon(2S_t \frac{\partial}{\partial t} + S_{tt}) \frac{\partial \mathcal{X}}{\partial \theta} + \varepsilon^2 \frac{\partial^2 \mathcal{X}}{\partial t^2} = 0,$$

Let $\quad f(u,t,\varepsilon) = f^0(u,t) + \varepsilon f^1(u,t) + \varepsilon^2 f^2(u,t) \dots$

$$\Downarrow$$

$$S_t^2 \frac{\partial^2 X}{\partial \theta^2} + f^0(X,t) = 0,$$

$$(S_t^2 \frac{\partial^2 X_k}{\partial \theta^2} + f_u^0(X,t))X_k = \mathcal{F}_k,$$

For $k = 1, 2$:

$$\mathcal{F}_1 = -(2S_t \frac{\partial}{\partial t} + S_{tt}) \frac{\partial X}{\partial \theta} - f^1(X,t),$$

$$\mathcal{F}_2 = -(2S_t \frac{\partial}{\partial t} + S_{tt}) \frac{\partial X_1}{\partial \theta} - \frac{1}{2} f_{uu}(X,t) X_1^2 - \frac{\partial^2 X}{\partial t^2} - \frac{\partial f^1}{\partial u}(X,t) X_1 - f^2(X,t).$$

The zero approximation

The solution:

$$X = Z(\theta + \phi, E, t)$$

$$\frac{S_t^2}{2}\left(\frac{\partial Z}{\partial \theta}\right)^2 + V(Z, t, 0) = E \Longleftrightarrow \int_{z_-}^{Z} \frac{S_t dz}{\sqrt{2(E - V(z, t, 0)}} = \theta + \phi,$$

Here $E(t)$, $\phi(t)$ are "constant" of integration

$$+$$

The dispersion relation (2π–periodicity of X):

$$S_t = \Omega(E, t) \equiv \frac{2\pi}{\oint \frac{dz}{\sqrt{2(E - V(z,t))}}}$$

Two different cases:

1) **Weak nonlinear case:** $V(u,t,0) = \frac{\omega^2(t)u^2}{2}$

$$S_t = \omega(t), \quad X = \sqrt{E(t)}\cos(\theta + \phi(t))$$

$$\mathcal{L} = \omega^2(\frac{d^2}{d\theta^2} + 1) \Longrightarrow \mathrm{Ker}\mathcal{L} = \{\sin(\theta+\phi), \cos(\theta+\phi)\} \Longrightarrow$$

$Dim\mathcal{L} = 2 \Longrightarrow$ **2 conditions:**

$$\int_0^{2\pi} \mathcal{F}_k \cos(\theta+\phi)\, d\theta = 0, \qquad \int_0^{2\pi} \mathcal{F}_k \sin(\theta+\phi)\, d\theta = 0$$

$$\Updownarrow$$

$k = 1$ **2 equations for** $E(t)$ **and the phase shift** $\phi(t)$**:**

$$\frac{d}{dt}(\omega(t)E) = 0, \quad \frac{d\phi}{dt} = \frac{1}{2\pi E}\int_0^{2\pi} f^1(Y_0)Y_0\, d\theta$$

Initial conditions: $\sqrt{E(0)}\cos\phi(0) = u^0, \quad \sqrt{E(0)}\omega(0)\sin\phi(0) = -p^0$

2) **Strong nonlinear case:**

$$S_t = \Omega(E,t)$$

$$\mathcal{L} = S_t^2 \frac{d^2}{d\theta^2} + f_u^0(X,t) \Longrightarrow$$

$$\mathrm{Ker}\mathcal{L} \quad (\text{among} \quad 2\pi - \text{periodic functions} \quad) = \{\frac{\partial \mathrm{X}}{\partial \theta}\} \Longrightarrow$$

$\mathrm{Dim}\mathcal{L}$ (among $2\pi-$periodic functions) = 1(!!!) $\Longrightarrow$ **1 condition:**

$$\int_0^{2\pi} \mathcal{F}_k \frac{\partial X}{\partial \theta}\, d\theta = 0$$

$$\Downarrow$$

$k = 1$ **the equation for** $E(t)$

$$\frac{dI(E,t)}{dt} = 0, \quad I = \frac{1}{2\pi} \oint \sqrt{2(E - V(z,t))}\, dz$$

I **is an** ***adiabatic*** **invariant**,

also $$X_1 = X_1^{part} + C_1(t) \frac{\partial X}{\partial \phi}$$

C_1 is the "constant" of integration of the 1-st approximation

The equation for the phase shift $\phi(t)$

$k=2$ **the equation for** $\phi(t)$

$$\int_0^{2\pi} \mathcal{F}_2 \frac{\partial X}{\partial \theta}\, d\theta = 0$$

$$\Downarrow$$

$$\frac{d}{dt}\Big(\frac{1}{\Omega_I}\frac{d\phi}{dt}\Big) = 0, \qquad \Omega_I = \Omega_E E_I = \frac{1}{2}\frac{\partial \Omega^2}{\partial E}$$

$$\Downarrow$$

$$\frac{d\phi}{dt} = \Omega_I \triangle I, \quad \phi = \triangle I \int_0^t \Omega_I\, dt + \phi^0, \quad \triangle I = \mathrm{const}, \quad \phi^0 = \mathrm{const}$$

The problem: the **jump** of Ker$\boldsymbol{\mathcal{L}}$ $\Longrightarrow$ there is no simple passage from a weak nonlinear case to a strong nonlinear case.

Again we change (S, I) by $(\tilde{S}, \tilde{I})$:

$$\tilde{S} = \int_0^t \Omega(\tilde{I}(\varepsilon), \tau) d\tau, \quad \tilde{I} = I - \frac{\varepsilon}{\Omega(I,0)} \int_0^{\phi^0} \big(V_\tau(X(\theta, I, 0), 0) - \overline{V}_t(0)\big)\, d\theta$$

and "kill" the phase shift.

Averaging in action-angle variables

$$\dot{p} = -V_x, \quad \dot{u} = p, \qquad H = \frac{p^2}{2} + V(u,t)$$

Let us fix t and choose $X(0,E,t) = z^-, \quad P(0,E,t) = 0$

$$\Downarrow$$

$$p = P(\theta,E,t)|_{\theta=\Omega t+\phi}, \quad u = X(\theta,E,t)|_{\theta=\Omega t+\phi}, \quad H = E$$

are $T--$periodic solutions with the period

$$T = 2\pi/\Omega = \oint \frac{dz}{\sqrt{2(E-V(z,t))}} \Longleftrightarrow \Omega = 2\pi/\oint \frac{dz}{\sqrt{2(E-V(z,t))}}$$

$$P(\theta+2\pi,E,t) = P(\theta,E,t), \quad X(\theta+2\pi,E,t) = X(\theta,E,t)$$

and

$$X(-\theta,E,t) = X(\theta,E,t), \quad P(-\theta,E,t) = -P(\theta,E,t)$$

Canonical transform:

The action:

$$I = \frac{1}{2\pi}\int_0^{2\pi} P\ dX = \frac{1}{2\pi}\oint dz\sqrt{2(E - V(z,t))}$$

$$\Downarrow$$

(I,θ) are the action-angle variables:

$$E = E(I,t) \quad \Longrightarrow p = P(\theta, E(I), t), \quad x = X(\theta, E(I), t)$$

$$H = H(I,\theta,t) \equiv E(I,t) + \varepsilon H_1(I,\theta,t)$$

$$H_1 = -P\frac{\partial X}{\partial t} + \frac{\partial}{\partial t}(\int_0^\theta PdX) \Longrightarrow$$

We have the problem known in classical averaging theory like problem with **one rapid phase**. One can use standard averaging process based on sequences of changes of the variable $\Longrightarrow$ **Neishtadt** results based on ideas of Poincare, Bogoljubov, Kolmogorov, Arnold etc.

The defect of the Nejshtadt scheme for small amplitudes (weak nonlinear case): there is no analyticity and even smoothness of **generating** functions for small action I because
$X \sim \sqrt{\frac{2I}{\omega}} \cos\theta, \qquad P \sim -\sqrt{2I\omega} \cos\theta.$
One can use the variables P, X, but this problem could take place. In averaging theory this problem is known like the averaging near singular manifolds (Gelfreich, Lerman 2002, Bruening, Dobrokhotov, Poterjakhin, 2001).

The simple analysis shows that the question could be formulated as the following problem: consider on 2-D plane with coordinates (y_1, y_2) and polar coordinates $\rho = \sqrt{y_1^2 + y_2^2}, \varphi$ the equation for the function $z(y_1, y_2)$:

$$\frac{\partial z}{\partial \varphi} = F(y_1, y_2),$$

where $F(y_1, y_2)$ is the analytical function in some neighborhood of the origin and

$$\int_0^{2\pi} F(\rho\cos\varphi, \rho\sin\varphi)d\varphi = 0.$$

Is it possible to present integral representation for the analytical solution $z(y_1, y_2)$?

The answer (Bruening,Dobrokhotov,Poterjakhin)

$$z(y_1, y_2) = \int^{\varphi} F(\rho \cos \varphi, \rho \sin \varphi) d\varphi \equiv$$

$$\frac{1}{2} \int_0^{\varphi} F(\rho \cos \varphi, \rho \sin \varphi) d\varphi + \frac{1}{2} \int_{\pi}^{\varphi} F(\rho \cos \varphi, \rho \sin \varphi) d\varphi + C(\rho^2).$$

If now we understand the integrals in Neishtadt work in this sense then they are working uniformly for weak and strong nonlinear cases.

S.Yu.Dobrokhotov, D.S.Minenkov , On various averaging methods for a nonlinear oscillator with slow time-dependent potential and a nonconservative perturbation, Regul. Chaotic Dyn., 2010, 15 (2-3), pp. 285-299.

S. Yu. Dobrokhotov, D. S. Minenkov, Remark on the phase shift in the Kuzmak–Whitham ansatz, Theor. Math. Phys., 166:3 (2011), 350–365

The problem which is good to study:

(1) To prove the **Proposition** about the phase shift in the Whitham averaging method for equations considered by I.Krichever

(2) To prove the **Proposition** about the phase shift in the Whitham averaging method for Navier-Stokes equations and nondifferential like Benjamin-Ono equation and Toda lattice

(3) To study numerically the interaction of two wave trains etc

(4) To analyze from this point of view the perturbed soliton formulas obtained by Grimshaw, Maslov and Omel'janov

THANK YOU FOR YOUR ATTENTION!

DEAR SASHA, HAPPY BIRTHDAY TO YOU AND MANY HAPPY RETURNS!

GIOVANNI, MARTA, MISHA, OLEG, YURI, THANKS A LOT FOR ORGANISATION OF THIS WONDERFUL CONFERENCE!