Integrability in Algebra,
Geometry and Physics: New Trends
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On Witten - Kontsevich tau-function and its generalizations

Boris DUBROVIN
SISSA, Trieste

Plan

- KdV hierarchy and Witten - Kontsevich tau-function

- Generating series for Witten - Kontsevich correlators

- Airy-type functions associated with simple Lie algebras

- Applications to Witten and Fan - Jarvis - Ruan invariants of moduli spaces

Based on joint works with Marco Bertola and Di Yang

arXiv:1504.06452, 1508.03750

KdV hierarchy: system of PDEs for $u = u(x, t_0, t_1, t_2, \dots)$

$$
\begin{aligned}
u_{t_0} &= u_x \\
u_{t_1} &= u\,u_x + \frac{1}{12} u_{xxx} \\
u_{t_2} &= \frac{u^2}{2!} u_x + \frac{1}{12}\left(2u_x u_{xx} + u\,u_{xxx}\right) + \frac{1}{240} u^V \\
\cdots & \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots\cdots
\end{aligned}
$$

$$2u_{t_k} \equiv L_{t_k} = [A_k, L], \quad L = \partial_x^2 + 2u, \quad A_k = \frac{1}{(2k+1)!!}\left(L^{k+\frac{1}{2}}\right)_+$$

Integrability $\qquad (u_{t_i})_{t_j} = (u_{t_j})_{t_i}$

$\Rightarrow\ \exists$ common solution for given initial data

$$u_0(x) = u(x, 0, 0, 0, \dots)$$

Useful: initial wave function $\qquad \psi_0'' + 2u_0(x)\psi_0 = \lambda\,\psi_0$

The Witten - Kontsevich solution: $u_0(x) = x$

Claim (W-K): the *tau-function* of the solution

$$\tau = \tau(t_0, t_1, t_2, \dots), \quad \partial_x^2 \log \tau = u(t_0, t_1, t_2, \dots), \quad (x = t_0)$$

has the form

$$\log \tau(\mathbf{t}) = \sum_{g \geq 0} F_g(\mathbf{t})$$

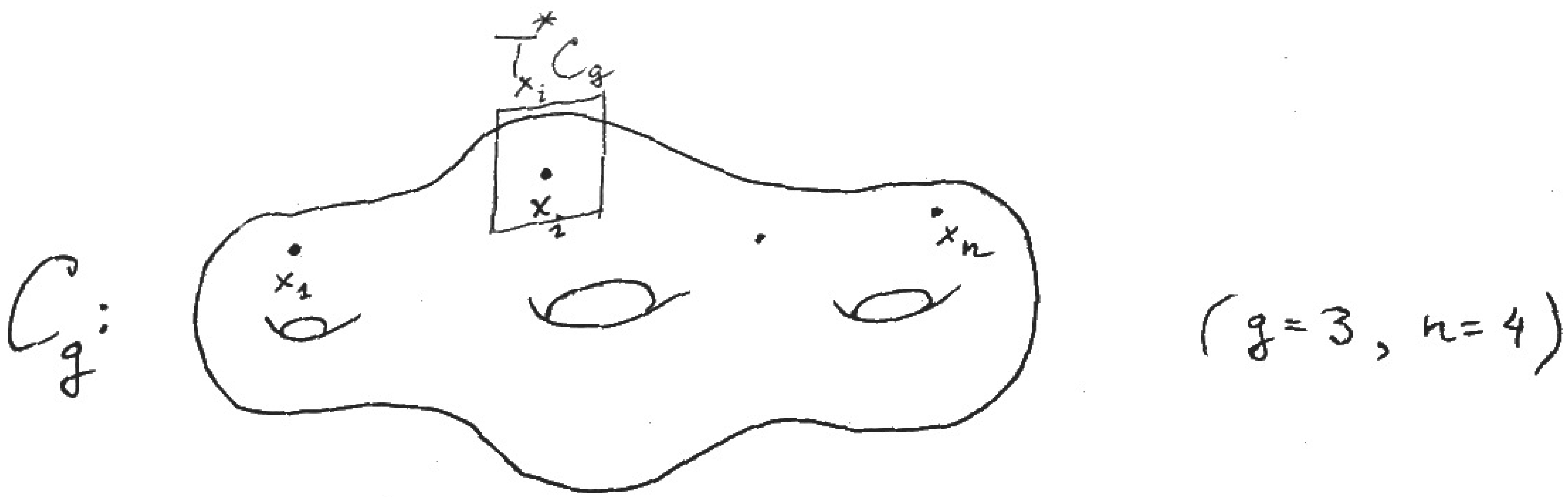
where

$$F_g(\mathbf{t}) = \sum_n \frac{1}{n!} \sum t_{p_1} \dots t_{p_n} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{p_1} \dots \psi_n^{p_n}$$

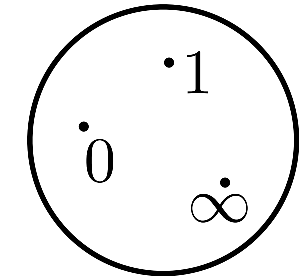
integral over moduli space of stable algebraic curves

Deligne - Mumford moduli spaces of stable algebraic curves

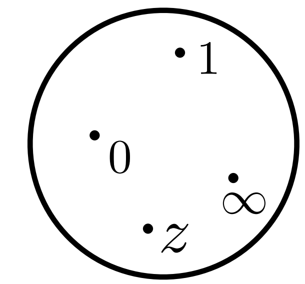
$$\overline{\mathcal{M}}_{g,n} = \{(C_g, x_1, \ldots, x_n)\} / \sim$$



$\overline{\mathcal{M}}_{0,3} = \mathrm{pt}$



$\overline{\mathcal{M}}_{0,4} = \mathbf{P}^1$



$\overline{\mathcal{M}}_{1,1} =$ {elliptic curves}

Tautological line bundles

$$\begin{array}{c} \mathcal{L}_i \\ T^*_{x_i} C_g \big\downarrow \\ \overline{\mathcal{M}}_{g,n} \end{array}$$

$$\psi_i := c_1(\mathcal{L}_i) \in H^2\left(\overline{\mathcal{M}}_{g,n}\right), \quad i = 1, \ldots, n$$

In physics literature (Witten *et al.*):

the tau-function = partition function of 2D quantum gravity

$$\log \tau(\mathbf{t}) = \left\langle e^{\sum_{i\geq 0} t_i \tau_i} \right\rangle$$

$\tau_0 = 1, \tau_1, \tau_2, \ldots$ observables

time variables of KdV hierarchy = coupling constants

Correlators

$$\langle \tau_{p_1} \ldots \tau_{p_n} \rangle = \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{p_1} \ldots \psi_n^{p_n} = \frac{\partial^n \log \tau(\mathbf{t})}{\partial t_{p_1} \ldots \partial t_{p_n}} |_{\mathbf{t}=0}$$

nonzero only if $\quad p_1 + \cdots + p_n = 3g - 3 + n$

Goal: a simple algorithm for computing the correlators

Previously:

- V.Kac, A.Schwarz (1991): from Virasoro constraints for tau-function to a differential equation in spectral parameter for the wave function

- Kontsevich “matrix Airy function”

- Recursion using genus expansion + Virasoro constraints (T.Eguchi et al. 1994)

- A.Okounkov generating functions 2000

- Topological recursion by L.Chekhov, B.Eynard, N.Orantin 2007

- A.Alexandrov cut-and-join formula 2010

Thm (M.Bertola, B.D., D.Yang 2015)

$$\sum_{k_1,\dots,k_n=0}^{\infty} \langle \tau_{k_1} \dots \tau_{k_n} \rangle \frac{(2k_1+1)!!}{\lambda_1^{k_1+1}} \dots \frac{(2k_n+1)!!}{\lambda_n^{k_n+1}}$$

$$= -\frac{1}{n} \sum_{r\in S_n} \frac{\operatorname{Tr} M(\lambda_{r_1}) \cdots M(\lambda_{r_n})}{\prod_{j=1}^{n} (\lambda_{r_j} - \lambda_{r_{j+1}})} - \delta_{n,2} \frac{\lambda_1 + \lambda_2}{(\lambda_1 - \lambda_2)^2}, \quad n \geq 2$$

where

$$M(\lambda) = \frac{1}{2} \begin{pmatrix} -\sum_{g=1}^{\infty} \frac{(6g-5)!!}{24^{g-1}\cdot(g-1)!} \lambda^{-3g+2} & -2\sum_{g=0}^{\infty} \frac{(6g-1)!!}{24^g \cdot g!} \lambda^{-3g} \\ 2\sum_{g=0}^{\infty} \frac{6g+1}{6g-1} \frac{(6g-1)!!}{24^g \cdot g!} \lambda^{-3g+1} & \sum_{g=1}^{\infty} \frac{(6g-5)!!}{24^{g-1}\cdot(g-1)!} \lambda^{-3g+2} \end{pmatrix}$$

(related to Faber - Zagier series). Observe (Faber; Okounkov):

$$\sum_{k=0}^{\infty} \langle \tau_k \rangle \frac{(2k+1)!!}{\lambda^{k+1}} = \sum_{g=1}^{\infty} \frac{(6g-3)!!}{24^g \cdot g!} \lambda^{-3g+1}$$

Main Lemma.

Let $\tau(\mathbf{t})$ be the tau-function of a solution $u(\mathbf{t})$

Define $\mathbf{t}$-dependent generating functions

$$F_n(z_1, \dots, z_n; \mathbf{t}) := \sum_{k_1,\dots,k_n=0}^{\infty} \langle\langle \tau_{k_1} \dots \tau_{k_n} \rangle\rangle(\mathbf{t}) \, \frac{(2k_1+1)!!}{z_1^{2k_1+2}} \cdots \frac{(2k_n+1)!!}{z_n^{2k_n+2}}$$

where $$\langle\langle \tau_{k_1} \tau_{k_2} \dots \tau_{k_n} \rangle\rangle(\mathbf{t}) := \frac{\partial^n \log \tau}{\partial t_{k_1} \dots \partial t_{k_n}}(\mathbf{t}), \quad n \geq 1$$

$$\psi(z; \mathbf{t}) = \frac{\tau(\mathbf{t} - [z^{-1}])}{\tau(\mathbf{t})} \mathrm{e}^{\vartheta(z;\mathbf{t})}, \quad \vartheta(z; \mathbf{t}) := \sum_{j=0}^{\infty} t_j \frac{z^{2j+1}}{(2j+1)!!}, \quad [z^{-1}] = \left(\frac{1}{z}, \frac{3!!}{z^3}, \frac{5!!}{z^5}, \dots \right)$$

is the wave function and $\psi^*(z; \mathbf{t}) = \psi(-z; \mathbf{t})$ the dual wave function

$(z = \sqrt{\lambda})$

Put $\Psi(z;\mathbf{t}) = \begin{pmatrix} \psi(z;\mathbf{t}) & \psi^*(z;\mathbf{t}) \\ -\psi_x(z;\mathbf{t}) & -\psi^*_x(z;\mathbf{t}) \end{pmatrix}$

$$M(z;\mathbf{t}) := \Psi(z;\mathbf{t})\,\sigma_3\,\Psi^{-1}(z;\mathbf{t}) = \frac{1}{2}\begin{pmatrix} -R_x & -2R \\ R_{xx} - 2(z^2 - 2\,u)R & R_x \end{pmatrix}$$

Here $R = R(z;\mathbf{t}) = \psi(z;\mathbf{t})\psi^*(z;\mathbf{t})$ the “resolvent”

of the Lax operator $L = \partial_x^2 + 2u$

$$F_1(z;\mathbf{t}) = \frac{1}{2}\mathrm{Tr}\left(\Psi^{-1}(z)\Psi_z(z)\sigma_3\right) - \vartheta_z(z)$$

$$F_n(z_1,\ldots,z_n;\mathbf{t}) = -\frac{1}{n}\sum_{r\in S_n}\frac{\mathrm{Tr}\,M(z_{r_1})\cdots M(z_{r_n})}{\prod_{j=1}^{n}(z_{r_j}^2 - z_{r_{j+1}}^2)} - \delta_{n,2}\frac{z_1^2+z_2^2}{(z_1^2-z_2^2)^2}, \quad n\geq 2$$

Proof (for n=1) uses Hirota bilinear relations and (for n≥2) the equations for time dependence of wave function

$$\psi_{t_n} = A_n \psi$$

n-th equation of the KdV hierarchy $\quad L_{t_n} = [A_n, L]$

$$L\,\psi = z^2 \psi \quad \Rightarrow \quad A_n \psi = a_n(x,z)\psi + b_n(x,z)\psi', \quad a_n = -\frac{1}{2} b_n'$$

Generating function $\nabla(z) := \sum_{k=0}^{\infty} \frac{(2k+1)!!}{z^{2k+2}} \frac{\partial}{\partial t_k}$

Proposition (B.D. 1975) $\nabla(z)\psi(w) = \frac{R(z)\psi_x(w) - \frac{1}{2} R_x(z)\psi(w)}{z^2 - w^2}$

$$R = 1 + \mathcal{O}\left(\frac{1}{z}\right) \quad \text{satisfies} \quad R\,R_{xx} - \frac{1}{2} R_x^2 + (4\,u - 2\,z^2)\,R^2 = -2\,z^2$$

$$\text{or} \quad \frac{1}{4} R''' + (2u - z^2)\,R' + u'\,R = 0$$

Setting $\mathrm{t} = 0$

one obtains logarithmic derivatives of the tau-function

in terms of initial wave function $\psi_0(z;x)$

$$\psi_0''(z;x) + 2u_0(x)\psi_0(z;x) = \lambda\,\psi_0(z;x), \quad \lambda = z^2$$

For the W - K solution arrive at Airy equation

$$\psi'' + 2x\,\psi = z^2\psi$$

so

$$\psi_0 = a(z)\mathrm{Ai}\left(2^{-\frac{2}{3}}(z^2 - 2\,x)\right)$$

$a(z)$ a normalizing factor

Determine the normalizing factor using Kac - Schwarz operator

$$S = \frac{1}{z}\,\partial_z - \frac{1}{2z^2} - z$$

Proposition (Kac - Schwarz 1991). String equation

$$\sum_{k=0}^{\infty} t_{k+1}\,\frac{\partial \tau}{\partial t_k} + \frac{t_0^2}{2}\,\tau = \frac{\partial \tau}{\partial t_0}$$

implies $S\,\psi = -\partial_x \psi$

hence $a(z) = \sqrt{2\pi z}\, e^{\frac{z^3}{3}}\, 2^{\frac{1}{3}}$

Generalization (M.Bertola, B.D., D.Yang, in progress)

$\mathfrak{g}$ a simple Lie algebra of rank ℓ

$\mathfrak{h} \subset \mathfrak{g}$ Cartan subalgebra

$\alpha_1, \dots \alpha_\ell \in \mathfrak{h}^*$ basis of simple roots

$\Delta \subset \mathfrak{h}^*$ the set of all roots

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha, \quad [a, x] = \alpha(a) x \quad \text{for} \quad x \in \mathfrak{g}_\alpha, \quad a \in \mathfrak{h}$$

root decomposition

$E_i \in \mathfrak{g}_{\alpha_i}, \quad F_i \in \mathfrak{g}_{-\alpha_i}$ and $H_i = [E_i, F_i]$ the set of Weyl generators of $\mathfrak{g}$

$$(E_i | F_i) = \frac{2}{(\alpha_i | \alpha_i)}, \quad i = 1, \dots, \ell, \quad \text{where} \quad (a \,|\, b) = \frac{1}{2h^\vee} \mathrm{tr}\,(\mathrm{ad}\, a \cdot \mathrm{ad}\, b) \quad \text{Killing form}$$

Loop algebra $L(\mathfrak{g}) = \mathfrak{g} \otimes \mathbb{C}\left[\lambda, \lambda^{-1}\right]$

Denote

$$e_0 = E_0 \otimes \lambda, \quad f_0 = F_0 \otimes \lambda^{-1}, \quad e_i = E_i \otimes 1, \quad f_i = F_i \otimes 1, \quad i = 1, \dots, \ell$$

$$E_0 \in \mathfrak{g}_{-\theta}, \quad F_0 \in \mathfrak{g}_{\theta}, \quad \theta = \text{the highest root}$$

Put

$$\Lambda = e_0 + e_1 + \cdots + e_\ell$$

and consider differential equation

$$\boxed{M' = [M, \Lambda]}$$

for a $\mathfrak{g}$-valued function $M = M(\lambda)$

Example. $\mathfrak{g} = sl_2(\mathbb{C})$

$$\Lambda = \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix}$$

for $M = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ obtain

$$a = \frac{b'}{2}, \quad c = -\frac{b''}{2} + \lambda\, b,$$

b satisfies $-\frac{1}{2}\, b''' + 2\lambda\, b' + b = 0$

Choice of a solution?

Rewrite the main equation $M' = [M, \Lambda]$ in the form

$$\begin{pmatrix} a' \\ b' \\ c' \end{pmatrix} = \begin{pmatrix} 0 & \lambda & -1 \\ 2 & 0 & 0 \\ -2\lambda & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

system of linear ODEs with polynomial coefficients

Singular point at infinity. Eigenvalues of the matrix of coefficients are $0, \ \pm 2\sqrt{\lambda}$

regular at $\lambda = \infty$ (→ 0)

exponential behaviour $\sim e^{\pm \frac{4}{3}\lambda^{3/2}}$ (→ $\pm 2\sqrt{\lambda}$)

$$M = \lambda^{-1/2}\left[\Lambda(\lambda) - \frac{1}{4\lambda}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \frac{7}{32\lambda^2}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \dots\right]$$

the above series $M(\lambda)$

General simple Lie algebra: regular solutions to $M' = [M, \Lambda]$

Eigenvectors with zero eigenvalue = centralizer of Λ

Principal gradation on the Kac - Moody Lie algebra

$$\deg e_i = 1,\ \deg f_i = -1,\ i = 0, 1, \dots, \ell, \quad \deg \lambda = h$$

Exponents $\quad m_1 = 1 < m_2 \leq \cdots \leq m_{\ell-1} < m_\ell = h - 1$

$h\ =\ $ Coxeter number. $\qquad$ Set $\quad E = \bigcup_{\alpha, k} m_\alpha + h\, k$

Proposition (Kac 1978) $\quad \mathrm{Ker}\, \mathrm{ad}\, \Lambda = \bigoplus_{j \in E} \mathbb{C}\, \Lambda_j$

Normalization $\quad \Lambda_{m_\alpha + h\, k} = \lambda^k \Lambda_{m_\alpha}, \quad \left(\Lambda_{m_\alpha} \,|\, \Lambda_{m_\beta}\right) = h\, \lambda\, \delta_{\alpha+\beta, \ell+1}$

Thm. (BDY, to appear) There exists canonical basis in the space of regular solutions to $M' = [M, \Lambda]$ of the form

$$M_\alpha(\lambda) = \lambda^{-\frac{m_\alpha}{h}} \left[\Lambda_{m_\alpha} + \sum_{k=1}^{\infty} M_{\alpha,k} \right], \quad M_{\alpha,k} \in L(\mathfrak{g}), \quad \deg M_{\alpha,k} = m_\alpha - (h+1)\, k$$

$$\alpha = 1, \dots, \ell$$

Useful: Kac - Wakimoto gradation operator

$$\mathrm{gr} := h\, \lambda \frac{d}{d\lambda} + \mathrm{ad}\, \rho^\vee$$

$$\rho^\vee \in \mathfrak{h} \quad \text{Weyl covector}, \quad \alpha_i\left(\rho^\vee\right) = 1, \quad i = 1, \dots, \ell$$

$$\deg X(\lambda) = j \quad \Leftrightarrow \mathrm{gr}\left(X(\lambda)\right) = j\, X(\lambda)$$

Thm (*ibid.*) There exists a unique solution to the Drinfeld - Sokolov hierarchy of type $\mathfrak{g}$ such that

$$\sum_{k_1,\dots,k_n} \frac{\langle \tau_{\alpha_1,k_1} \dots \tau_{\alpha_n,k_n} \rangle}{\lambda_1^{k_1+\frac{m_{\alpha_1}}{h}+1} \dots \lambda_n^{k_n+\frac{m_{\alpha_n}}{h}+1}}$$

$$= -\frac{1}{n} \sum_{s \in S_n} \frac{\mathrm{T}\left[M_{\alpha_1}(\lambda_{s_1}), \dots, M_{\alpha_n}(\lambda_{s_n})\right]}{\prod_{j=1}^{n} \left(\lambda_{s_j} - \lambda_{s_{j+1}}\right)}$$

$$-\delta_{n,2}\delta_{\alpha_1+\alpha_2,\ell+1}\lambda_1^{-\frac{m_{\alpha_1}}{h}}\lambda_2^{-\frac{m_{\alpha_2}}{h}} \frac{m_{\alpha_1}\lambda_1 + m_{\alpha_2}\lambda_2}{(\lambda_1 - \lambda_2)^2}$$

for arbitrary indices $1 \leq \alpha_1, \dots, \alpha_n \leq \ell, \quad n \geq 2$

Here $\mathrm{T}[x_1, x_2, \dots, x_n] := \mathrm{Tr}\,(\mathrm{ad}\, x_1 \cdot \mathrm{ad}\, x_2 \dots \mathrm{ad}\, x_n)$

Reminder: Drinfeld - Sokolov hierarchy, A_ℓ-type

Lax operator $L=\partial_x^{\ell+1}+u_1(x)\partial_x^{\ell-1}+\dots+u_\ell(x)$

Exponents $1,\,2,\dots,\ell.$ Coxeter number $h=\ell+1$

The hierarchy $\dfrac{\partial L}{\partial t_k^\alpha}=\left[A_{\alpha,k},L\right],\quad \alpha=1,\dots,\ell,\quad k=0,\,1,\,\dots$

$$A_{\alpha,k}=c_{\alpha,k}\left(L^{k+\frac{\alpha}{\ell+1}}\right)_+$$ for a normalizing constant $c_{\alpha,k}$

$$\Lambda=\begin{pmatrix} 0 & 1 & 0 & \dots & 0\\ 0 & 0 & 1 & \dots & 0\\ 0 & & \dots & \dots & 0\\ 0 & & \dots & \dots & 1\\ \lambda & 0 & 0 & \dots & 0\end{pmatrix},\quad \Lambda_k=\Lambda^k,\quad k=1,\dots,\ell$$

Simply-laced case, applications to the E.Witten and H.Fan, T.Jarvis, Y.Ruan invariants of moduli spaces $\overline{\mathcal{M}}_{g,n}$ ("quantum singularity theory")

Claim (*ibid.*) For a simple Lie algebra of the ADE type log of the above tau-function is the generating function of the Witten and Fan-Jarvis-Ruan invariants

Normalization: $M' = \kappa\,[M, \Lambda]\,, \quad \kappa = (-h)^{-\frac{h}{2}}$

Proved for A-type using C.Faber, S.Shadrin, D.Zvonkine proof of the Witten's conjecture (2010)

A_ℓ case: moduli space of r-spin structures on $\overline{\mathcal{M}}_{g,n}$, $\quad \ell = r-1$

Fix $1 \leq \alpha_1, \ldots, \alpha_n \leq r-1$ s.t. $\alpha_1 + \cdots + \alpha_n - n - (2g-2) = m\, r$

Choose a line bundle $\mathcal{L}$ over curve C with marked points $x_1, \ldots, x_n$ s.t.

$$\mathcal{L}^{\otimes r} = K_C \otimes \mathcal{O}\left((1-\alpha_1)x_1\right) \otimes \cdots \otimes \mathcal{O}\left((1-\alpha_n)x_n\right)$$

$\Rightarrow$ moduli space $\overline{\mathcal{M}}^{1/r}_{g,n}(\alpha_1, \ldots, \alpha_n)$ $\quad$ (covering of $\overline{\mathcal{M}}_{g,n}$)

Witten class $c_W(\alpha_1, \ldots, \alpha_n) \in H^{2(m+g-1)}\left(\overline{\mathcal{M}}_{g,n}\right)$

Define $\quad \langle \tau_{\alpha_1,k_1} \ldots \tau_{\alpha_n,k_n} \rangle := \displaystyle\int_{\overline{\mathcal{M}}_{g,n}} c_W(\alpha_1, \ldots, \alpha_n) \psi_1^{k_1} \ldots \psi_n^{k_n}$

Witten conjecture, proved by C.Faber, S.Shadrin, D.Zvonkine 2010

$$\tau(\mathbf{t}) = \exp\Big(\sum_{n=0}^{\infty}\frac{1}{n!}\sum_{k_1,\dots,k_n\geq 0}\ \sum_{\alpha_1,\dots,\alpha_n=1}^{r-1}\langle\tau_{\alpha_1,k_1}\cdots\tau_{\alpha_n,k_n}\rangle\, t_{k_1}^{\alpha_1}\cdots t_{k_n}^{\alpha_n}\Big)$$

is tau-function of a particular solution to the A_{r-1} hierarchy

Generalization for D and E cases proved by Fan, Jarvis, Ruan

Non simply-laced case: curves with symmetry (S.-Q.Liu, Y.Ruan, Y.Zhang 2014), like folding of Dynkin diagrams. To be done

С Днем Рожденья, Саша!

Many happy returns of the day!