Orthogonal polynomials and Normal matrix models

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(joint work with F. Balogh and D. Merzi and report on F. Balogh and D. Merzi. Equilibrium measures for a class of potentials with discrete rotational symmetries. Constr. Approx., 2015)

Integrability in algebra, geometry and physics: new trends. In honour of Alexander Veselov 60's birthday

Summary

1. Motivation: study asymptotics of orthogonal polynomials related to normal matrix models
2. Distribution of eigenvalues of normal matrix models for the external potentials

$$V(z) = |z|^{2m} - tz^d - \bar{t}\bar{z}^d \qquad 0 \leq d \leq 2m,\ d, m \in \mathbb{N},\ \ t \in \mathbb{C}$$

3. Pointwise asymptotics of the orthogonal polynomials $p_n(z)$ characterized by the orthogonality relations

$$\int_{\mathbb{C}} p_n(z)\bar{z}^k e^{-N\left(|z|^{2d} - tz^d - \bar{t}\bar{z}^d\right)} dA(z) = 0 \quad k = 0, 1, \ldots, n-1$$

   in the scaling limit

$$n \to \infty, \quad N \to \infty, \quad \frac{n}{N} \to T$$

4. Limiting distribution of the zeros of $p_n(z)$
5. Relations between the limiting zeros distribution of $p_n(z)$ and the eigenvalue distribution of the normal matrix model.

Normal matrix models

Algebraic variety of $n \times n$ complex normal matrices:

$$\mathcal{N}_n := \{M \in \mathrm{Mat}_{n\times n}(\mathbb{C}) \; : \; MM^\dagger = M^\dagger M\}.$$

External potential:

$$V\colon \mathbb{C} \to \mathbb{R}, \quad V(z) = \Phi(z\bar{z}) - \mathcal{P}(z) - \overline{\mathcal{P}(z)},$$

with $V(z) > c \log |z|^2$ as $|z| \to \infty$ and $\mathcal{P}(z)$ polynomials.
Probability density on normal matrices:

$$M \mapsto \frac{1}{\tilde{\mathcal{Z}}_n} \exp(-N \,\mathrm{Tr}\,(V(M)))dM\,.$$

Invariance under unitary conjugation $\Rightarrow$ joint probability density on eigenvalues:

$$\mathcal{P}_n(\lambda_1, \ldots, \lambda_n) = \frac{1}{\mathcal{Z}_n} \prod_{1\leq k<l\leq n} |\lambda_k - \lambda_l|^2 e^{-N\sum_{k=1}^n V(\lambda_k)}$$

(with respect to the area measure dA in $\mathbb{C}$).
Partition function:

$$\mathcal{Z}_n := \int_{\mathbb{C}^n} \prod_{1\leq k<l\leq n} |\lambda_k - \lambda_l|^2 e^{-N\sum_{k=1}^n V(\lambda_k)} dA(\lambda_1).dA(\lambda_2)\cdots dA(\lambda_n)$$

Orthogonal polynomials associated to a normal matrix model

Monic orthogonal polynomial of degree n for the measure $e^{-NV(z)}dA(z)$:

$$\begin{cases} \int_{\mathbb{C}} p_n(z)\bar{z}^k e^{-NV(z)} dA(z) = 0, & k = 0, \dots, n-1 \\ p_n(z) = z^n + \text{lower order terms} \end{cases}$$

Norming constants:

$$h_n = \int_{\mathbb{C}} |p_n(z)|^2 e^{-NV(z)} dA(z) \qquad n = 0, 1, \dots$$

Christoffel-Darboux-type reproducing kernel:

$$K_n(z, w) = e^{-\frac{N}{2}V(z) - \frac{N}{2}V(w)} \sum_{k=0}^{n-1} \frac{1}{h_k} p_k(z)\overline{p_k(w)}$$

No three terms recurrence relation for OP and Christoffel-Darboux identity

The role of orthogonal polynomials in normal matrix models

- Joint probability density as a determinant (Gaudin–Mehta):

$$\frac{1}{\mathcal{Z}_n}\prod_{i<j}|\lambda_i-\lambda_j|^2 e^{-N\sum_i V(\lambda_i)} = \det_{1\le i,j\le n}(K_n(\lambda_i,\lambda_j))$$

- Partition function in terms of norming constants:

$$\mathcal{Z}_n = n!\det_{0\le k,l\le n-1}\left(\int_{\mathbb{C}} z^k\bar{z}^l e^{-NV(z)}dA(z)\right) = n!\prod_{k=0}^{n-1} h_k\ .$$

- Expected number of eigenvalues in a set $B\subseteq\mathbb{C}$ (density of states):

$$\mathbb{E}(\#\{\text{eigenvalues of } M \text{ in } B\}) = \frac{1}{n}\int_B K_n(z,z)dA(z)$$

Goal. Determine the behaviour of relevant quantities as $n\to\infty$, in particular, the behaviour of the orthogonal polynomial $p_n(z)$ for every $z\in\mathbb{C}$, and the limiting distribution of the zeros of $p_n(z)$.

Distribution of eigenvalues of the normal matrix model

Coulomb gas interpretation:

$$\mathcal{P}_n(\lambda_1,\dots,\lambda_n) = \frac{1}{\mathcal{Z}_n}\prod_{1\le k<l\le n}|\lambda_k-\lambda_l|^2 e^{-N\sum_{k=1}^n V(\lambda_k,\bar\lambda_k)} = \frac{1}{\mathcal{Z}_n}\exp\left(-n^2\mathcal{I}_n(\lambda_1,\dots,\lambda_n)\right)$$

where

$$\mathcal{I}_n(\lambda_1,\dots,\lambda_n) := \frac{1}{n^2}\sum_{i\neq j}\log\frac{1}{|\lambda_i-\lambda_j|} + \frac{N}{n}\sum_{k=1}^n \frac{V(\lambda_k,\bar\lambda_k)}{n}$$

The probability density is maximal when $\mathcal{I}_n(\cdots)$ is minimal (*Fekete points*):

$$\sigma_n := \frac{1}{n}\sum_{k=1}^n \delta_{z_k^\star}\ , \qquad (z_1^\star,\dots,z_n^\star) \text{ is an optimal configuration}$$

Scaling for the asymptotics:

$$n\to\infty\ , N\to\infty\ , \frac{N}{n}\to\frac{1}{T} \qquad T>0\ .$$

In the continuum limit one obtains the variational problem:

$$\begin{cases} \mathcal{I}(\mu) := \iint \log\frac{1}{|z-w|}d\mu(z)d\mu(w) + \frac{1}{T}\int V(z)d\mu(z) \to \text{MIN}. \\ \mu\ge 0\ , \quad \mu(D)=1, \quad \text{supp}\mu = D \end{cases}$$

Logarithmic energy problem

$$\begin{cases} \mathcal{I}(\mu) := \iint \log \dfrac{1}{|z-w|} d\mu(z)d\mu(w) + \dfrac{1}{T}\displaystyle\int V(z)d\mu(z) \to \text{MIN.} \\ \mu \geq 0 \ , \quad \mu(D) = 1 \end{cases}$$

Theorem (Frostman, Saff-Totik, Elbau-Felder)

For every lower semi-continuos potential $V(z) = \Phi(\bar{z}z) - \mathcal{P}(z) - \overline{\mathcal{P}(z)}$, bounded from below, and such that $V(z) - \log|z|^2 \to \infty$ as $|z| \to \infty$, the electrostatic energy functional $\mathcal{I}(\mu)$ has a unique minimizer μ_V with support D. Moreover if $\Phi(s)$ is C^2 and $(s\Phi')'$ is positive and integrable near zero, then

$$d\mu_V(z) = \frac{1}{2\pi}\chi_D(z)\Delta V(z)dA(z)$$

where χ_D is the characteristics function of the domain D.

The measure μ_V is called the equilibrium measure for V.
If V is real analytic then the boundary ∂D is a finite union of analytic arcs with at most a finite number of singularities (Hedenmalm, Makarov).

Timeline

1965	*Ginibre*	$V(z) = \|z\|^2$
1984	*Girko*	circular law
1986	*Girko*	elliptic law
1994	*Di Francesco, Gaudin, Itzykson, Lesage*	$V(z) = \|z\|^2 - tz^2 - \bar{t}\bar{z}^2$, Hermite polynomials
1998	*Chau, Zaboronsky*	general normal matrix model and Toda lattice
2000-2013	*Mineev-Weinstein, Wiegmann, Zabrodin,*	integrable structure in conformal maps
2001	Kostov, Krichever, M-W W. and Z.	τ function for analytic curves
2002	*Akemann*	$V(z) = \frac{1}{1-\tau^2}\left(\|z\|^2 - \tau\left(z^2 + \bar{z}^2\right)\right) + (2a+1)\log\frac{1}{\|z\|}$ Laguerre polynomials
2004	*Hedenmalm, Makarov*	equilibrium measure
2005	*Elbau, Felder*	$V(z) = \|z\|^2 + P(z) + \overline{P(z)}$ with a cut-off and polynomial curves
2006	*Teodorescu, Wiegmann, Zabrodin*	$V(z) = \|z\|^2 + c\log\frac{1}{\|z-a\|}$, $V(z) = \|z\|^2 + t(z^3 + \bar{z}^3)$
2007	*Etingof, Ma*	$V(z) = \Phi(z\bar{z}) - P(z) - \overline{P(z)}$, equilibrium measure (precritical)
	Its, Takhtajan	$\bar{\partial}$-problem
2008	*Ameur, Hedenmalm, Makarov*	asymptotics of C-D kernel, linear statistics
	Elbau	polynomial curves
2009-2010	*Lee-Peng Teo*	Integrable dynamics for a pair of (f, g) of univalent functions
2011	*Bleher, Kuijlaars*	cubic potential, asymptotic of orthogonal polynomials
	Ameur, Hedenmalm, Makarov	fluctuations and Gaussian free field
2012	*Balogh, Bertola, Lee, McLaughlin*	$V(z) = \|z\|^2 + c\log\frac{1}{\|z-a\|}$
	Balogh, Merzi	$V(z) = \|z\|^{2n} + tz^d + \bar{t}\bar{z}^d$ equilibrium measure
2014	*Kuijlaars, López-Garcia*	$V(z) = \|z\|^2 + tz^k + \bar{t}\bar{z}^k$, asymptotic of orth. polynomials
	Kuijlaars, Tovbis	critical cubic case
	Ameur, Kang, Makarov	edge scaling limits
2015	*Leble, Serfaty*	Asymptotic expansion of the partition function
	S.Y. Lee, Riser	Fine asymptotic of eigenvalues:Ellipse case
	Natanzon, Zabrodin	Hurwitz numbers and conformal dynamics

There is a connection between the limiting distribution $d\nu$ of the zeros of OPs and the limiting distribution $d\mu$ of the eigenvalues of the associated matrix models

Hermitian matrix models

Theorem I	**Theorem II**	**Theorem III**
Christoffel-Darboux density	Fekete points	zeroes of OPs
$\frac{1}{n}K_{n,N}(x,x)dx \longrightarrow d\mu$	$d\sigma_{n,N} \xrightarrow{w*} d\mu$	$d\nu_{n,N} \longrightarrow d\mu$
as $n, N \to \infty, N/n \to T$.	as $n, N \to \infty, N/n \to T$.	as $n, N \to \infty, N/n \to T$.
(Johannson, 98)	(Saff-Totik, '97)	(J.' 98, DKMVZ '99)

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Hermitian matrix models

Theorem I
Christoffel-Darboux density

$$\frac{1}{n}K_{n,N}(x,x)dx \longrightarrow d\mu$$

as $n, N \to \infty, N/n \to T$.
(Johannson, 98)

Theorem II
Fekete points

$$d\sigma_{n,N} \xrightarrow{w*} d\mu$$

as $n, N \to \infty, N/n \to T$.
(Saff-Totik, '97)

Theorem III
zeroes of OPs

$$d\nu_{n,N} \longrightarrow d\mu$$

as $n, N \to \infty, N/n \to T$.
(J.' 98, DKMVZ '99)

Normal matrix models

Theorem I
Christoffel–Darboux density

$$\frac{1}{n}K_{n,N}(z,\bar{z})dA(z) \xrightarrow{w*} d\mu$$

$n, N \to \infty, N/n \to T$.
(Hedenmalm-Makarov '04, Elbau-Felder, '05)

Theorem II
Fekete points

$$d\sigma_{n,N} \xrightarrow{w*} d\mu$$

$n, N \to \infty, N/n \to T$.
(Saff-Totik, '97)

Conjecture
zeroes of OP: $d\nu_{n,N} \to d\nu$

$$\iint_D \frac{d\mu(s)}{z-s} = \int_\Gamma \frac{d\nu(s)}{z-s}$$

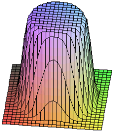
(No general proof yet, several examples)

The limiting distribution $d\nu$ of the zeros of orthogonal polynomials satisfies

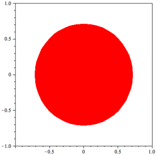
$$\iint_D \frac{d\mu(s)}{z-s} = \int_\Gamma \frac{d\nu(s)}{z-s}, \quad z \in \mathbb{C} \backslash D$$

Elbau-Felder, Balogh-Bertola- S.Y.Lee-McLaughlin, Balogh-Harnad, Bleher-Kuijlaars, Kuijlaars-Lopez Garcia...

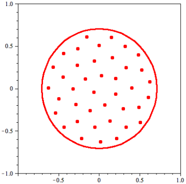
The simplest example: $V(z) = \frac{1}{2}|z|^2$ (Ginibre,1965).



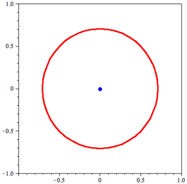
C-D density



equilibrium support



Fekete points

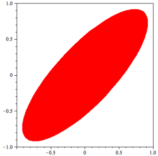


zeroes of OPs

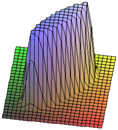
$P_n(z) = z^n$ (by rotational symmetry)

The second simplest example: $V(z) = \frac{1}{2}\left[|z|^2 - \mathrm{Re}(rz^2)\right]$

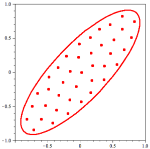
(Di Francesco-Gaudin-Itzykson-Lesage, 1994)



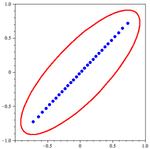
equilibrium support



Christoffel-Darboux density



Fekete points



zeroes of OPs

$$P_n(z) \propto H_n\left(\sqrt{\frac{1-|r|^2}{r}}\sqrt{N}z\right), \quad \text{for } r \text{ real, } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ with } a = \sqrt{\frac{1+r}{1-r}},\ b = 1/a$$

$$\int_E \frac{dA(s)}{z-s} = \frac{1}{2i}\int_{\partial E} \frac{\bar{s}ds}{z-s} = \frac{1}{2i}\int_{\partial E} \frac{ds}{z-s}\left[\frac{a^2+b^2}{c^2}s - \frac{2ab}{c}\sqrt{s^2-c^2}\right] = \frac{2ab}{c^2}\int_{-c}^{c} \frac{\sqrt{c^2-s^2}}{z-s}ds$$

where $\frac{2}{\pi c^2}\sqrt{s^2-c^2}ds$ is the Wigner semi-circle law for the zeros of OP and $\frac{dA}{\pi ab}$ is the eigenvalues distribution.

Potential with discrete rotational symmetry (F. Balogh, D. Merzi)

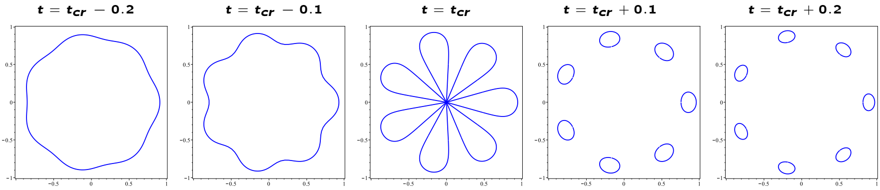
Consider the potential

$$V(z,\bar{z}) = |z|^{2m} - tz^{d} - \bar{t}\bar{z}^{d}, \quad 2m \geq d.$$

Notice that V(z) is invariant for $z \to e^{\frac{2\pi ik}{d}} z$. The distribution of the eigenvalues of the associated normal matrix model is given in term of the *equilibrium measure*:

$$d\mu(z) = \frac{1}{4\pi}\chi_D(z)\Delta V(z)dA(z)$$

where the domain D is determined explicitly.
To determine the domain D, the concept of singularity correspondence for conformal map is used (after Richardson 1972, Gustafsson 1983, Etingof - Ma 2007). For example for $m = 9$ and $d = 7$ the domain D is depicted below for several values of t



Symmetry reduction for the equilibrium measure

Folded measure:

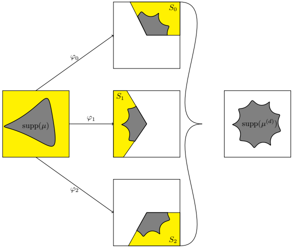
$$S_k = \left\{ z \in \mathbb{C} : \frac{2\pi k}{d} \leq \arg(z) < \frac{2\pi(k+1)}{d} \right\}$$

$$\varphi_k : \mathbb{C} \to S_k \ , \quad \varphi_k(re^{i\theta}) = r^{\frac{1}{d}} e^{\frac{i\theta}{d}} e^{\frac{2\pi i k}{d}}$$

$$\mu_k^{(d)}(B) = \mu\left(\varphi_k^{-1}(B \cap S_k)\right)$$

$$(k = 0, \ldots, d-1)$$

$$\mu^{(d)} = \frac{1}{d} \sum_{k=0}^{d-1} \mu_k^{(d)}$$



Folding out μ to $\mu^{(d)}$

Lemma

If $V(z)$ can be written in terms of the potential Q as

$$V(z) = \frac{1}{d} Q(z^d)$$

then their equilibrium measures are related by

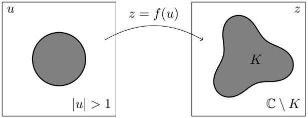
$$\mu_V = \mu_Q^{(d)}.$$

Symmetry reduction and conformal map

Symmetry reduction for the potential:

$$Q(z) = \frac{d}{T}\left(|z|^{2m/d} - tz - \overline{tz}\right)$$

- Density of μ_Q:
$$d\mu_Q(z) = \frac{1}{4\pi}\Delta Q(z)\cdot\chi_K(z)dA(z)$$
where $K \subset \mathbb{C}$ is the support of μ_Q
- Support of μ_Q: *simply connected* for all values of t !



$$f_Q\colon \{u\colon |u| > 1\} \to \mathbb{C}\setminus K$$

$$0 \in K: \quad f_Q(u) = ru\left(1 + \frac{a_1}{u} + \dots\right) \qquad u\to\infty$$

$$0 \notin K: \quad f_Q(u) = r(u + a_0)\left(1 + \frac{a_1}{u} + \dots\right) \qquad u\to\infty$$

Critical value of the parameter t:

$$t_{cr} = \frac{m}{d}\left(\frac{T}{2m-d}\right)^{\frac{2m-d}{2m}} .$$

For $|t| < t_{cr}$ the domain D is simply connected while for $|t| > t_{cr}$ the domain D is multiple connected.

Theorem (Balogh-Merzi)

There exist parameters $r(t) > 0$ and $|\alpha(t)| < 1$ so that for $|t| < t_{cr}$ the exterior uniformizing map of the domain K associated to the potential Q is

$$f_Q(u) = r(t)u\left(1 - \frac{\alpha(t)}{u}\right)^{\frac{d}{m}} .$$

For $|t| > t_{cr}$ such map is given by

$$f_Q(u) = r(t)\left(u - \frac{1}{\bar{\alpha}(t)}\right)\left(1 - \frac{\alpha(t)}{u}\right)^{\frac{d}{m}-1} .$$

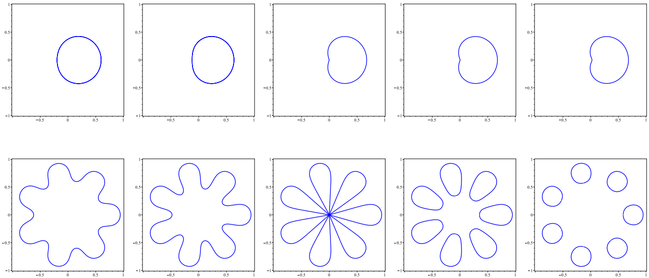
For $|t| < t_{cr}$, the folded conformal mapping for $V(z) = \dfrac{1}{d}Q(z^d)$ is

$$f_V(u) = \left(f_Q(u^d)\right)^{\frac{1}{d}} .$$

For $|t| > t_{cr}$, $f_V(u)$ is a parametrisation of the multiply connected domain but it is not a conformal map.

The support K of the equilibrium measure for the potential Q and the support D of the equilibrium measure for the potential

$$V(z) = \frac{1}{d} Q(z^d) = |z|^{2m} - tz^d - \bar{t}\bar{z}^d, \quad m = 9, \ d = 7.$$

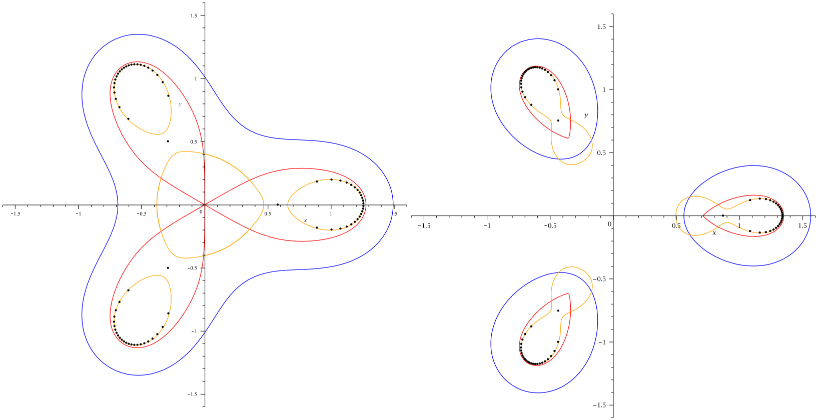


Point-wise asymptotic of orthogonal polynomials

We are considering the monic orthogonal polynomials characterized by:

$$\int_{\mathbb{C}} p_n(z)\bar{z}^j e^{-N\left(|z|^{2d}-tz^d-\bar{t}\bar{z}^d\right)} dA(z) = 0, \qquad j = 0, 1, \ldots, n-1.$$

Goal: determine the asymptotic of orthogonal polynomials $p_n(z)$ as $n \to \infty$ and $N \to \infty$ with $N/n \to T$. (Work in progress G.-Balogh-Merzi).



$$\int_{\mathbb{C}} p_n(z)\bar{z}^j e^{-NV(z)} dA(z) = 0, \qquad j = 0, 1, \ldots, n-1$$

with $V(z) = |z|^{2d} - tz^d - \bar{t}\bar{z}^d$.

Theorem

Let $d\nu(z)$ be the limiting distribution of the zeros of the orthogonal polynomials $p_n(z)$ as $n \to \infty$, $N \to \infty$ such that $N/n \to T$ and let the contour Γ be the support of such measure. Then the equation

$$T \int_{\Gamma} \frac{d\nu(s)}{z - s} = \partial_z V(z)$$

defines the boundary ∂D of a domain D that contains Γ. The domain D coincides with the support of the measure $d\mu$ that describes the limiting distribution of the eigenvalues of the matrix models. Furthermore for $z \notin D$ it follows that

$$\int\int_D \frac{d\mu(s)}{z - s} = \int_{\Gamma} \frac{d\nu(s)}{z - s}$$

Remark: an equivalent result had been obtained in several other papers, for different potentials $V(z)$. The above theorem hold when the domain D is simply connected or multiply connected.

The domains D and the curve Γ

The domain D, support of the eigenvalue distribution is given by

$$\partial D : \quad z^d \bar{z}^d - (tz^d + \bar{t}\bar{z}^d) + |t|^2 - t_{cr}^2 = 0$$

and the curve Γ that describes the location of the zeros of the OP is given by

$$\Gamma : \quad \left| (\bar{t} - z^d) \exp\left(\frac{z^d t}{t_{cr}^2}\right) \right| = \begin{cases} |t| & \text{precritical case } |t| < t_{cr} \\ t_{cr}^2/|t| & \text{postcritical case } |t| > t_{cr} \end{cases}$$

Remark

The Riemann surface

$$z^d \xi^d - (tz^d + \bar{t}\xi^d) + |t|^2 - t_{cr}^2 = 0$$

has genus $g = (d-1)^2$ for $|t| > t_c$ and $|t| < t_{cr}$ and genus $(d-1)(d-2)/2$ for $t = t_{cr}$. The surface has an anti-holomorphic involution $(z, \xi) \to (\bar{\xi}, \bar{z})$. It has only one real oval for $|t| < t_{cr}$ that describes the support of the eigenvalues, while for $|t| > t_{cr}$ it has d real ovals.

Asymptotic of orthogonal polynomials

These OPs inherit the $\mathbb{Z}_d$ symmetry from the potential:

$$p_n\left(e^{\frac{2\pi i}{d}} z\right) = e^{\frac{2\pi i n}{d}} p_n(z)$$

That is, there exists a monic polynomial $q_k^{(l)}$ of degree k such that

$$p_n(z) = z^l q_k^{(l)}(z^d),$$

where $q_k^{(l)}$ is a monic polynomial of degree k and $n = kd + l$, with $0 \le l \le d-1$. Hence the OPs $\{p_n(z)\}_{n=0}^{\infty}$ can be split into d subsequences $\{q_k^{(l)}(u)\}_{k=0}^{\infty}$, which satisfy

$$\int_{\mathbb{C}} q_k^{(l)}(u)\bar{u}^j |u|^{2\frac{l-d+1}{d}} e^{V(u,\bar{u})} dA(u) = 0, \qquad j = 0, \ldots, k-1.$$

This symmetry reduction allows us to consider the OPs w.r.t. the weight

$$|u|^{-2\gamma} e^{-N(|u|^2 - tu - \bar{t}\bar{u})} dA(u), \qquad \gamma := \frac{d-l-1}{d} \in [0,1).$$

From 2D to 1D

It is possible to reduce the 2D integral to a contour integral on a curve ($\bar{\partial}$ problem).

Theorem

For any polynomial $q(z)$ the following integral identity holds:

$$\int_{\mathbb{C}} q(u)\bar{u}^j |u|^{-2\gamma} e^{-N\left(|u|^2 - tu - \overline{tu}\right)} dA(u)$$
$$= \frac{\pi\Gamma(j-\gamma+1)}{N^{j-\gamma+1}} \frac{1}{2\pi i} \oint_{\tilde{\Sigma}} q(u) \frac{e^{Ntu}}{(u-\bar{t})^{j+1}} \left(1 - \frac{\bar{t}}{u}\right)^{\gamma} du \, ,$$

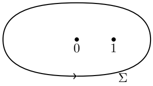
where $\tilde{\Sigma}$ is a positively oriented simple closed loop enclosing $z = 0$ and $z = \bar{t}$.

With the linear change of coordinates $u = -\bar{t}(\lambda - 1)$ one gets

$$\frac{1}{2\pi i} \oint_{\Sigma} \pi_k(\lambda)\lambda^j \frac{e^{-N|t|^2\lambda}}{\lambda^k} \left(\frac{\lambda}{\lambda - 1}\right)^{\gamma} d\lambda = 0 \qquad j = 0, 1, \dots, k-1$$

where

$$\pi_k(\lambda) := \frac{(-1)^k}{\bar{t}^k} q_k(-\bar{t}(\lambda - 1))$$



Riemann-Hilbert problem for orthogonal polynomials

$$\frac{1}{2\pi i}\oint_{\Sigma}\pi_k(\lambda)\lambda^j \underbrace{\frac{e^{-N|t|^2\lambda}}{\lambda^k}\left(\frac{\lambda}{\lambda-1}\right)^{\gamma}}_{w_k(\lambda)} d\lambda = 0 \qquad j=0,1,\ldots,k-1,$$

We define the weight function

$$w_k(\lambda) := e^{-kV_k(\lambda)}\left(1-\frac{1}{\lambda}\right)^{-\gamma}, \quad \text{where} \quad V_k(\lambda) = \frac{N|t|^2}{k}\lambda + \log(\lambda)\ .$$

Fokas-Its-Kitaev Riemann-Hilbert problem

For a 2×2 matrix $Y(\lambda)$

- $Y(\lambda)$ is analytic in $\mathbb{C}\setminus\Sigma$
- the jump on Σ
$$Y_+(\lambda) = Y_-(\lambda)\begin{pmatrix}1 & w_k(\lambda)\\ 0 & 1\end{pmatrix},$$
- large z boundary behaviour:
$$Y(\lambda) = \left(I + \mathcal{O}\left(\frac{1}{\lambda}\right)\right)\lambda^{k\sigma_3}, \quad \lambda\to\infty$$

characterizes the orthogonal polynomial $\pi_k(\lambda)$ uniquely.

Unique solution encoding $\pi_k(\lambda)$:

$$Y(\lambda) = \begin{pmatrix} \pi_k(\lambda) & \frac{1}{2\pi i}\int_\Sigma \frac{\pi_k(t)w_k(t)dt}{t-\lambda} \\ r_{k-1}(\lambda) & \frac{1}{2\pi i}\int_\Sigma \frac{r_{k-1}(t)w_k(t)dt}{t-\lambda} \end{pmatrix} \quad (1)$$

so in order to obtain the large k asymptotics of $\pi_k(\lambda)$ one just need to know the asymtotics of $Y_{11}(\lambda)$.

g-function and asymptotic analysis

Following Deift-Zhou steepest descent method introduce a function $g(\lambda)$

$$g(\lambda) = \int_\Gamma \log(\lambda - s) d\nu(s)$$

for an unknown measure ν and a contour Γ homotopically equivalent to Σ. Then transform $Y(\lambda) \to U(\lambda)$

$$U(\lambda) = e^{-k(\ell/2)\sigma_3} Y(\lambda) \left(1 - \frac{1}{\lambda}\right)^{-\frac{\gamma}{2}\sigma_3} e^{-kg(\lambda)\sigma_3} e^{k(\ell/2)\sigma_3} \qquad \lambda \in \mathbb{C} \setminus (\Gamma \cup [0,1])$$

- $U(\lambda)$ is analytic in $\mathbb{C} \setminus (\Gamma \cup [0,1])$
- the jumps:

$$U_+(\lambda) = U_-(\lambda) \begin{cases} \begin{pmatrix} e^{-k(g_+ - g_-)} & e^{k(g_+ + g_- - \ell - V_k(\lambda))} \\ 0 & e^{-k(g_+ - g_-)} \end{pmatrix} & \text{on } \Gamma \\ e^{-\gamma \pi i \sigma_3} & \text{on} (0,1) \ . \end{cases}$$

- large λ boundary behaviour:

$$U(\lambda) = \left(I + \mathcal{O}\left(\frac{1}{\lambda}\right)\right), \quad \lambda \to \infty \ .$$

Asymptotic distribution of the zeros: Szëgo type curves

The zero distribution of the OP is determined by the conditions

$$g_+ + g_- - \ell - V_k = 0, \quad \mathrm{Re}(g_+ - g_-) = 0, \quad \text{on } \Gamma \tag{2}$$

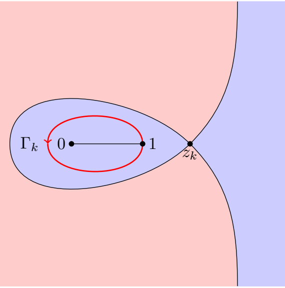
$$z_k = \frac{k}{N|t|^2} \longrightarrow z_\infty = \frac{t_{cr}^2}{|t|^2}$$

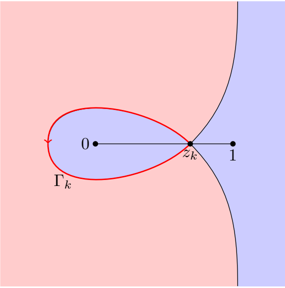
Pre-critical case: $z_\infty > 1$

$$\Gamma_k : \quad \left|\lambda e^{\frac{1-\lambda}{z_k}}\right| = 1$$

Post-critical case: $z_\infty < 1$

$$\Gamma_k : \quad \left|\lambda e^{\frac{1-\lambda}{z_k}}\right| = z_k$$

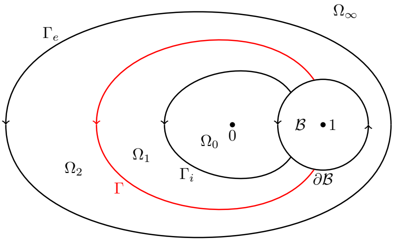




The conditions (2) determine a family of a-priori probability measure ν_k supported along each of the level curves.

Asymptotics for $\pi_k(\lambda)$: precritical case

$$\pi_k(\lambda) = e^{kg(\lambda)} \left(1 - \frac{1}{\lambda}\right)^{\frac{\gamma}{2}} [U_k(\lambda)]_{11}$$



The exterior region Ω_∞

$$\pi_k(\lambda) = \lambda^k \left(1 - \frac{1}{\lambda}\right)^{\gamma} \left(1 + \mathcal{O}\left(\frac{1}{k^{2+\gamma}}\right)\right)$$

The interior region $\Omega_0 \setminus \mathcal{B}$

$$\pi_k(\lambda) = \frac{c}{k^{1+\gamma}} \left(1 - \frac{1}{z_k}\right)^{-\gamma-1} \frac{e^{k\left(\frac{\lambda-1}{z_k}\right)}}{\lambda - 1} \left(1 + \mathcal{O}\left(\frac{1}{k}\right)\right)$$

The interesting region $\Omega_1 \setminus \mathcal{B}$

$$\pi_k(\lambda) = \lambda^k \left(1 - \frac{1}{\lambda}\right)^{\gamma} + \frac{e^{kg(\lambda)}}{k^{1+\gamma}} \left[\frac{c}{\lambda - 1} \left(1 - \frac{1}{z_k}\right)^{-\gamma-1} + \mathcal{O}\left(\frac{1}{k}\right)\right]$$

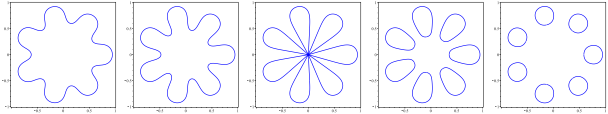
The other interesting region $z \in \Omega_2 \setminus \mathcal{B}$:

$$\pi_k(\lambda) = \lambda^k \left(1 - \frac{1}{\lambda}\right)^{\gamma} - \frac{e^{kg(\lambda)}}{k^{1+\gamma}} \left[\frac{c}{\lambda - 1} \left(1 - \frac{1}{z_k}\right)^{-\gamma-1} e^{-k\phi(\lambda)} + \mathcal{O}\left(\frac{1}{k^{2+\gamma}}\right)\right]$$

Post-critical case (sketch)

The geometry of the problem in the post-critical case is pretty different:

- The level curve Γ is passing through $z_k \in (0, 1)$ so the jumps contours are quite different
- The subleading order asymptotic of the orthogonal polynomials is obtained using parabolic cylinder functions
- Critical transition at $|t| = t_{cr}$ (Painlevé IV ?) [Similar asymptotics in Kuijlaars–Dai, 2009]



Happy Birthday Sasha

A $\bar{\partial}$-problem

Orthogonality relations

$$\int_{\mathbb{C}} q(u)\bar{u}^k |u|^{-2\gamma} e^{-N(|u|^2 - tu - \overline{tu})} dA(u) = 0, \tag{3}$$

$\bar{\partial}$-problem

$$\partial_{\bar{u}} \chi_k(u, \bar{u}) = \bar{u}^k |u|^{-2\gamma} e^{-N\left(|u|^2 - tu - \overline{tu}\right)} \tag{4}$$

Contour integral solution:

$$\begin{aligned}
\chi_k(u, \bar{u}) &= u^{-\gamma} e^{Ntu} \int_0^{\bar{u}} s^{k-\gamma} e^{-Nus + N\bar{t}s} ds \\
&= \frac{1}{N^{k-\gamma+1}} \left(1 - \frac{\bar{t}}{u}\right)^{\gamma} e^{Ntu} \int_0^{N\bar{u}(u-\bar{t})} r^{k-\gamma} e^{-r} dr \\
&= \frac{1}{N^{k-\gamma+1}} \left(1 - \frac{\bar{t}}{u}\right)^{\gamma} e^{Ntu} \left[\Gamma(k-\gamma+1) - \int_{N\bar{u}(u-\bar{t})}^{\infty} r^{k-\gamma} e^{-r} dr\right] \\
&= \frac{\Gamma(k-\gamma+1)}{N^{k-\gamma+1}} \left(1 - \frac{\bar{t}}{u}\right)^{\gamma} e^{Ntu} \left[1 - \mathcal{O}\left(e^{-N\bar{u}(u-\bar{t})}\right)\right] \quad |u| \to \infty
\end{aligned}$$

Applying Stokes' Theorem

For a polynomial $q(z)$

$$d\left[q(u)\chi_k(u,\overline{u})du\right] = q(u)\partial_{\overline{u}}\chi(u,\overline{u})d\overline{u}\wedge du\ , \tag{5}$$

$$\begin{aligned}\int_{\mathbb{C}} q(u)\overline{u}^k|u|^{-2\gamma}e^{-N(|u|^2-tu-\overline{tu})}dA(u) &= \lim_{R\to\infty}\int_{|u|\le R} q(u)\overline{u}^k|u|^{-2\gamma}e^{-N(|u|^2-tu-\overline{tu})}dA(u)\\ &= \frac{1}{2i}\lim_{R\to\infty}\int_{|u|\le R} q(u)\overline{u}^k|u|^{-2\gamma}e^{-N(|u|^2-tu-\overline{tu})}d\overline{u}\wedge d\\ &= \frac{1}{2i}\lim_{R\to\infty}\oint_{|u|=R} q(u)\chi_k(u,\overline{u})du\\ &= \frac{1}{2i}\lim_{R\to\infty}\oint_{|u|=R} q(u)\left[G_k(u)-\mathcal{O}\left(e^{-\overline{u}(u-\overline{t})}\right)\right]du\\ &= \frac{1}{2i}\oint_{|z|=R_0} q(u)G_k(u)du\end{aligned}$$

where R_0 is sufficiently large and

$$G_k(u) = \frac{\Gamma(k-\gamma+1)}{N^{k-\gamma+1}}\left(1-\frac{\overline{t}}{u}\right)^{\gamma}e^{Ntu} \tag{6}$$

(does not depend on $\overline{u}$, single-valued on $\mathbb{C}\setminus[0,\overline{t}]$)

Thanks for the attention!