This talk is dedicated to the memory of Jürgen Moser, who gave me very good advice at crucial times.

The bispectral problem as a source of non-commutative algebras of differential operators with matrix coefficients

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Many years ago while working in X-ray tomography I landed accidentally in the paradise of INTEGRAL SYSTEMS. Made many friends.

More recently
Quantum walks, Recurrence properties for discrete time unitary evolutions, TOPOLOGICAL INSULATORS.
Mainly joint work with Luis Velazquez, but also with him and J. Bourgain, C. Cedzich, C. Stahl, A. Werner, R. Werner, J. Wilkening.

Plan for the talk:

1. From SIGNAL PROCESSING to the scalar BISPECTRAL PROBLEM in a few slides.

2. From scalar to matrix valued: a new formulation of the bispectral problem

3. An explicit example: Spin Calogero-Moser, and the resulting non-commutative algebra. Here is where I need help; I can describe the algebra(s) in question but I do not have a nice description for them. Actually these algebras come with "time parameters".

4. Good algebraic properties with important numerical consequences.

5. Matrix valued eigenvalues: some "email exchanges" with Leonhard Euler.

Back around 1950 the "explicit" spectral analysis of the integral operator K acting in $L^2(A)$

$$(Kf)(x) = \int_{-T}^{T} \frac{\sin(\mathcal{W}(x-y))}{x-y} f(y)\,dy, \qquad x \in A.$$

with

$$A = [-T, T], \qquad B = [-\mathcal{W}, \mathcal{W}],$$

became important in signal processing.

The problem originates with C. Shannon, a magical solution can be found in joint papers by: David Slepian, Henry Landau and Henry Pollak. (Bell labs,1960's)

They found that the operator

$$(Df)(x) = ((T^2 - x^2)f'(x))' - \mathcal{W}^2 x^2 f$$

has simple spectrum and an appropriate selfadjoint extension of D commutes with K

$$KD = DK$$

This is a very useful fact: one computes (numerically) the eigenfunctions of D and this is a stable and efficient process.

Trying to ignore this ALGEBRAIC miracle only produces numerical GARBAGE, as I will show later in a finite dimensional caricature of this situation. We have an algebraic miracle with important practical consequences.

Is there a curve $P(K, D) = 0$??

Around 1975 I was interested in X-Ray Tomography and needed something similar in a different situation.

Notice that the kernel of K is obtained by integrating the product $e^{izx}\ e^{-izy}$ in $[-\mathcal{W}, \mathcal{W}]$ and one gets an integral operator acting on $L^2(A)$. This operator is compact, its spectrum accumulates at the origin and this is the reason for the numerical instability in computing its eigenfunctions. This is not the only practical problem: dealing with differential operators is much easier than dealing with integral ones, and besides, the spectrum of D is well separated.

$$k(x,y) = \frac{\sin(\mathcal{W}(x-y))}{x-y} = \int_{-\mathcal{W}}^{\mathcal{W}} e^{izx} e^{-izy}\, dz$$

Looking for an extension of the very useful property

$$KD = DK$$

to other situations I asked the question

If $\phi(x,z)$ are eigenfunctions of some arbitrary L

$$L\left(x, \frac{d}{dx}\right)\phi(x,z) \equiv (-D^2 + V(x))\phi(x,z) = z^2\phi(x,z)$$

and you construct a new kernel $k(x,y)$ by means of

$$k(x,y) = \int_{-\mathcal{W}}^{\mathcal{W}} \phi(x,z)\bar{\phi}(y,z)dz$$

when (i.e. for which V(x), besides V=0) will the resulting integral operator K acting on $L^2(A)$ allow for a commuting differential operator D??

A bit of experimentation led me to see the possible relevance of a different property of the eigenfunctions $\phi(x,z)$, and I started asking the question

Find all nontrivial instances where a function $\varphi(x,z)$ satisfies

$$L\left(x, \frac{d}{dx}\right)\varphi(x,z) \equiv (-D^2 + V(x))\varphi(x,z) = z^2\varphi(x,z)$$

as well as

$$B\left(z, \frac{d}{dz}\right)\phi(x,z) \equiv \left(\sum_{i=0}^{M} b_i(z)\left(\frac{d}{dz}\right)^i\right)\phi(x,z) = \Theta(x)\phi(x,z).$$

All the functions $V(x), b_i(z), \Theta(x)$ are, in principle, arbitrary except for smoothness assumptions. Notice that here M is arbitrary (finite).

The problem in this generality was posed and solved with Hans Duistermaat in a paper that appeared in 1986. We started around 1980.

This field is full of miracles and having run into Hans back around 1980 is one of the most important ones for me.

One finds in the paper with Duistermaat a proof of the equivalence between the **bispectral property** above and the **purely algebraic relation**

$$ad^{M+1}(L)(\Theta) = 0$$

where $ad(X)(Y) = [X, Y] = XY - YX$ is the usual commutator of the operators X and Y.

The complete solution is given as follows:

Theorem *If $M = 2$, then $V(x)$ is (except for translation) either c/x^2 or ax, i.e. we have a Bessel or an Airy case. If $M > 2$, there are two families of solutions*

a) *L is obtained from $L_0 = -D^2$ by a finite number of Darboux transformations ($L = AA^* \to \tilde{L} = A^*A$). In this case V is a rational solution of* the Korteweg deVries hierarchy of equations.

b) *L is obtained from $L_0 = -D^2 + \frac{1}{4x^2}$ after a finite number of "rational" Darboux transformations.*

Here D stand for the usual derivative.

Rank 1 and rank 2 respectively.

There are many other characterizations of these L , for instance in terms of "monodromy". This will be mentioned in the case of the second family later on.

It was later observed by Magri and Zubelli that in the case of the second family we are dealing with rational solutions of the Virasoro equations (i.e. master symmetries of KdV).

Observe that the “trivial cases” when $M = 2$ are self-dual in the sense that since the eigenfunctions $f(x,z)$ are functions either of the product xz or of the sum $x+z$, one gets B by replacing z for x in L.

The *bispectral involution* introduced by G. Wilson around 1993 shows how this can be adapted in the "higher order cases". Having attracted the attention of G. Wilson to this bispectral game was another important miracle for me.

For the $V(x)$ in the KdV family we have

$$V(x) = \sum_{p \in \mathcal{P}} \frac{\nu_p(\nu_p + 1)}{(x - p)^2}$$

with $\mathcal{P}$ a finite subset of $\mathbb{C}$, and $\nu_p \in \mathbb{Z}_{>0}$ for $p \in \mathcal{P}$ being such that

$$\sum_{\substack{q \in \mathcal{P} \\ q \neq p}} \frac{\nu_q(\nu_q + 1)}{(q - p)^{2j+1}} = 0 \text{ for } 1 \leq j \leq \nu_p \text{ and each } p \in \mathcal{P}.$$

One can also write

$$V(x) = -2 \left(\frac{\theta'(x)}{\theta(x)} \right)' ,$$

where $\theta(x) = \theta_\nu(x)$ is a polynomial defined by a recursive formula in ν. We can take

$$\theta(x) = \prod_p (x - p)^{\frac{1}{2}\nu_p(\nu_p + 1)} .$$

For these potentials we have the following characterization of the algebra of differential operators in the spectral variable
The eigenfunction $\phi_\infty^\pm(x,z)$ *satisfies an equation of the form* $B^\pm(z,\partial_z)\phi_\infty^\pm(x,z) = \Theta(x)\phi_\infty^\pm(x,z)$ *if and only if the polynomial* Θ *has the property that*

$$\Theta^{(2j-1)}(p) = 0 \text{ for all } 1 \le j \le \nu_p, \text{ for each pole } p \in \mathbb{C} \text{ of } V.$$

This condition clearly gives an algebra of polynomials.

I will return to the question of describing the **algebra** for a fixed eigenfunction later on.

For the $V(x)$ in the even family we have

After a suitable translation in the x-variable either $V(x) = \frac{c}{x^2}$ *(Bessel) or*

$$V(x) = \frac{\ell^2 - \frac{1}{4}}{x^2} + \sum_{p \in \mathcal{P}} \frac{\nu_p(\nu_p + 1)}{(x - p)^2} \ ,$$

where $\ell \in \mathbb{Z}_{\geq 0}$, $\nu_p \in \mathbb{Z}_{>0}$, $\nu_{-p} = \nu_p$ *and* $\mathcal{P}$ *is a finite subset of* $\mathbb{C} \backslash \{0\}$, *symmetric around* 0.

Furthermore, all eigenfunctions of $-\partial_x^2 + V(x)$ *are* **single-valued** *around all the poles* $p \neq 0$ *of* V. *This last property is equivalent to*

$$\frac{\ell^2 - \frac{1}{4}}{p^{2j+1}} + \sum_{q \in \mathcal{P} \backslash \{p\}} \frac{\nu_q(\nu_q + 1)}{(p - q)^{2j+1}} = 0 \text{ for } 1 \leq j \leq \nu_p, \text{ all } p \in \mathcal{P}.$$

The fact that we have rational solutions of the KP hierarchy puts us in contact with Calogero-Moser systems.

This ends my quick review of the **scalar** bispectral problem.

$$L\phi(x,z) = z^2\phi(x,z)$$

and

$$B\phi(x,z) = \Theta(x)\phi(x,z)$$

One may wonder what is the relation between solutions of the bispectral problem and instances when the commuting property

$$KD = DK$$

holds. This is the property that really matters in discussing "time-and-band limiting" as in the classical case of Slepian, Landau and Pollak.

The fact that bispectrality implies this desirable commutativity in all the cases when I have tried can be seen in the following publications when the eigenfunction $\Phi(x, z)$ and the coefficients of the differential operators L and B are **matrix valued** too.

F. Alberto Grünbaum, Ines Pacharoni and Ignacio Zurrian, *Time and band limiting for matrix valued functions, an example* SIGMA 11 (2015) 044.

A more ambitious case, showing that this property survives even after an application of the Darboux process appears in a new paper joint with Mirta Castro.

My plan now is to re-formulate the bispectral problem in a way suitable to this matrix situation, and to spend the rest of the time on an example and its non-commutativity algebra.
This formulation appears in the context of MATRIX valued orthogonal polynomials (M. G. Krein) in

M.M. Castro and F.A. Grünbaum, The algebra of differential operators associated to a given family of matrix valued orthogonal polynomials: five instructive examples. Int. Math. Res. Not., 27(2):1–33, 2006.

F.A. Grünbaum and J. Tirao, The algebra of differential operators associated to a weight matrix Integral Equations Operator Theory, 58(4):449–475, 2007.

Feeling that having both variables x, z as continuous ones should be easier I decided to leave orthogonal polynomials in peace and go back to differential operators on each of these variables.

In the case when both operators are differential ones this formulation of the (matrix) bispectral problem appears in two papers, one by C. Boyallian and J. Liberati (2008) and one by M. Bergvelt, M. Gekhtman and A. Kasman (2009).

More recently it appears in

F. Alberto Grünbaum , Some noncommutative matrix algebras arising in the bispectral problem SIGMA 10(9) 078 (2014).

It also appeared earlier (and I should have mentioned this in the paper above) in some widely circulated private notes from 2009 by GEORGE WILSON which I got in 2013. I feel very bad about receiving his notes and not reading them.

Consider a situation of the following kind

$$\mathcal{L}\Phi(x,z) = \Phi(x,z)F(z) \qquad \Phi(x,z)\mathcal{B} = \Theta(x)\Phi(x,z) \qquad (1)$$

for non-constant **matrix valued** functions $F(z)$ and $\Theta(x)$. The differential operators have matrix valued coefficients and they act on the matrix valued eigenfunction $\Phi(x,z)$.

We now use G. Wilson's approach to Spin Calogero-Moser systems (Gibbons-Hermsen).

Given matrices X and Z both in $C^{n\times n}$ and matrices A and B with A, B^t in $C^{r\times n}$ satisfying the condition

$$[X, Z] - I = BA$$

we consider the matrix valued Baker-Akhiezer function $\Phi(x, t_2, z)$ given by

$$\Phi(x, t_2, z) = e^{xz+2t_2z^2}(I_r + A(xI + 2t_2Z - X)^{-1}(zI_n - Z)^{-1}B)$$

We choose in our **example** below, $n = 3, r = 2$ and specified X, Z, A, B.

One can allow for more "time parameters" $t_2, t_3\ldots$ as will be seen in some of the examples displayed below.

It turns out that this gives rise to a matrix valued bispectral situation as described above.

$$\mathcal{L}\Phi = \Phi F(z) \qquad \Phi\mathcal{B} = \Theta(x)\Phi \tag{2}$$

The question I want to raise is : for this FIXED Φ, what is the algebra of all possible $\Theta(x)$?? This is an algebra of 2×2 matrix valued polynomials. Same question for the algebra of all possible $F(z)$.

Here t_i are "time parameters" and we should write $\Theta(x, t_2, t_3, ...)$, $F(z, t_2, t_3, ...)$.

One could also talk about the corresponding non-commutative algebras of differential operators $\mathcal{L}$ and $\mathcal{B}$.

The algebra of the $F(z)$.

One can see that, for the $\Phi(x, z, t_2, t_3, ..)$ above, the algebra of all $F(z, t_2, t_3, ...)$ such that for some $\mathcal{L}$ one has

$$\mathcal{L}\Phi = \Phi F(z)$$

is given by polynomials in z of the form

$$\begin{pmatrix} a & 0 \\ b-a & b \end{pmatrix} + \begin{pmatrix} c & c \\ a-b-c & -c \end{pmatrix} z + \begin{pmatrix} a-b-c & c+a-b \\ d & e \end{pmatrix} (z^2)/2$$

plus a polynomial of the form $z^3 P(z)$ where $P(z)$ is an arbitrary 2×2 matrix-valued polynomial and all the quantities a, b, c, d, e are arbitrary.
It is not hard to see that this forms an algebra for which a nice description in terms of generators and relations remains a challenge.

Looking now at the other algebra, for each $M = 3, 4, ...$ we have four linearly independent operators $\mathcal{B}$ of order M-or equivalently matrix valued polynomials $\Theta(x)$ of degree M. For $M = 2$ there is only a two dimensional space of such operators. There is no operator of order one and for order zero we only have multiples of the identity.

As a vector space our algebra of polynomials $\Theta(x)$ has, for each fixed value of $t_2, t_3, \ldots$, a basis described as follows:

Three special matrices, namely

$$I, \begin{pmatrix} -s & 1 \\ x^2+s & (x-2)x \end{pmatrix}, \begin{pmatrix} \frac{x(2x-1)+s(4s-1)}{2} & \frac{x(2x-3)-2(2s+1)}{2} \\ -\frac{s(4x+3+4s)}{2} & \frac{3x}{2} \end{pmatrix}$$

Here we are using $s = 2t_2 + 3t_3 + 4t_4 + \ldots$.

and then

for each integer d greater than 2 we have four matrices, each one of them of the forms

$$\begin{pmatrix} x^d + p_{1,1,d}(x) & p_{1,2,d}(x) \\ p_{2,1,d}(x) & p_{2,2,d}(x) \end{pmatrix}$$

$$\begin{pmatrix} q_{1,1,d}(x) & x^d + q_{1,2,d}(x) \\ q_{2,1,d}(x) & q_{2,2,d}(x) \end{pmatrix}$$

$$\begin{pmatrix} r_{1,1,d}(x) & r_{1,2,d}(x) \\ x^d + r_{2,1,d}(x) & r_{2,2,d}(x) \end{pmatrix}$$

$$\begin{pmatrix} t_{1,1,d}(x) & t_{1,2,d}(x) \\ t_{2,1,d}(x) & x^d + t_{2,2,d}(x) \end{pmatrix}$$

where $p_{i,j,d}(x), q_{i,j,d}(x), r_{i,j,d}(x), t_{i,j,d}(x)$ is a specific polynomial in x of degree not higher than 2. The specific form of these lower degree polynomials depends on the choice of the first three matrices we made earlier.

Some of these are displayed below. For $d = 3$ we have

$$\begin{pmatrix} x^3 - \frac{x(3x+2)+2s(3x-4)}{2} & x-2 \\ s(2x-3) & -\frac{3(x-2)x}{2} \end{pmatrix}$$

$$\begin{pmatrix} \frac{5x^2+4sx-x+s(8s-5)}{2} & x^3 - \frac{(7x+2)+2s(x+3)}{2} \\ -s\left(\frac{10x+8s+3}{2}\right) & \frac{x(3(x+1)+2s)}{2} \end{pmatrix}$$

$$\begin{pmatrix} x-2s & 2 \\ x^3 - s(x-3) & x(2x-3) \end{pmatrix}$$

$$\begin{pmatrix} -\frac{x-s}{2} & \frac{x-2}{2} \\ \frac{s(2x-3)}{2} & x^3 - \frac{3x(2x-1)}{2} \end{pmatrix}$$

and for $d = 4$ we have

$$\begin{pmatrix} x^4 - \frac{x^2(7+4s)+x(5+8s)-19s}{2} & \frac{5(x-2)}{2} \\ \frac{5s(2s-3)}{2} & -\frac{x(x(2s+9)-4s-15}{2} \end{pmatrix}$$

$$\begin{pmatrix} \frac{4x^2(s+3)+s(9x-16)+s^2(15+4s)}{2} & x^4 - \frac{x(16+5s)+2s(2s+7)}{2} \\ -\frac{s(4x(s+6)+s(4s+19))}{2} & \frac{x(2x(s+6)+5s}{2} \end{pmatrix}$$

$$\begin{pmatrix} \frac{x(x+4)-2s(s-4)}{2} & s+4 \\ x^4 - s(x-s-6) & \frac{x(x(2s+9)-4(s+3))}{2} \end{pmatrix}$$

$$\begin{pmatrix} -\frac{x(x+3)-5s}{2} & \frac{3(x-2)}{2} \\ \frac{3s(2x-3)}{2} & x^4 - \frac{x(x(2s+15)-4s-9}{2} \end{pmatrix}$$

We now illustrate the fact that we have an algebra. We list the elements of the basis (as a vector space) in an appropriate fashion. The three matrices given earlier are called $E_1 = I, E_2, E_3$. The next four matrices (which have x^3 in one of their entries) are called E_4, E_5, E_6, E_7....the next four (which have x^4 in one of their entries are called E_8, E_9, E_{10}, E_{11} etc.....

With this ordering we can see that, for instance:

$$E_2^2 = sE_1 + E_3 - 2E_6 - 4E_7 + E_{10} + E_{11}$$

$$E_2 E_3 = -s(s+1)E_1 + 3sE_2 - sE_3 - (4s+1)/2E_6 + E_{10} + E_{11}$$

Conclusion:

We have a few instances of (pairs of) non-commutative algebras of differential operators with a common eigenfunction. They come from bispectral situations and a rich structure behind them, Matrix KP, Spin Calogero-Moser.

Are there any useful "algebraic-geometric" objects that can be associated to these algebras?

A more general question: what about situations with only one non-commutative algebra with a common eigenfunction?

My examples are rank one, but in the scalar case there are also rank two examples in the paper with Duistermaat. There should be such examples here too. Master symmetries of KP....what replaces Calogero Moser?

Some numerical illustration, or "good algebraic properties" with important numerical consequences.

Here we display the results of some numerical computations. This should make clear the importance of having found, as above, a matrix such as D for a given K.

If our task is to compute the eigenvectors of K we can use the QR algorithm as implemented in LAPACK. The results are recorded below, where we denote by X the matrix of its eigenvectors (normalized and given as columns of X) and by Λ the diagonal matrix of eigenvalues. The matrix X is (displayed here with few digits)

$$X = \begin{pmatrix} 0.046 & -.031 & 0.294 & -.953 & -0.099 & -.0406 \\ -.0424 & -.0353 & -.241 & -.0756 & .852 & -.429 \\ .214 & -.170 & .649 & -.216 & .0981 & -.654 \\ -.195 & -.189 & -.589 & -.185 & .494 & 0.569 \\ .706 & -.649 & -.215 & 0.0105 & -.0312 & .174 \\ -.642 & -.714 & .196 & .053 & -.0883 & -.171 \end{pmatrix}$$

The question is: should we trust the result produced by this high quality numerical package?
One could be quite satisfied by observing that the difference

$$KX - X\Lambda$$

is indeed very small. On the other hand LAPACK reports for eigenvalues of M , with appropriate rounding-off, the values $1.0, 1.0, 1.0, 1.0, 1.0, 1.0$. This should be a red flag.

Recall that the eigenvectors of D should agree (up to order) with those of K. If we denote the matrix made up of the normalized eigenvectors of D by Y we get

$$Y = \begin{pmatrix} .641 & .227 & .0318 & -.674 & .280 & -.046 \\ .688 & .247 & .035 & .628 & -.258 & .042 \\ -.229 & .613 & .170 & .280 & .645 & -.214 \\ -0.245 & .667 & .187 & -.260 & -.593 & .195 \\ 0.028 & -.172 & .649 & -.040 & -.214 & -.706 \\ .030 & -.187 & 0.714 & .037 & .197 & .642 \end{pmatrix}$$

For the eigenvalues of D , LAPACK returns the values 6.46314, 6.55601, 6.63761, -5.61601, -5.54541, -5.4863 a reasonably **spread out** spectrum.

If we compute the matrix of inner products given by

$$Y^T X$$

we expect to have the identity matrix up to some permutation and possibly some signs due to the normalization of the eigenvectors which are the columns of X and Y. In our case we get for the moduli of the entries of $Y^T X$ the matrix

$$\begin{pmatrix} 4.65e^{-7} & 1.71e^{-4} & .019 & .667 & .671 & .021 \\ 2.83e^{-6} & 9.61e^{-4} & .013 & 0.234 & .2244 & .9602 \\ 9.85e^{-6} & .9999 & 7.801e^{-4} & 2.385e^{-4} & 4.395e^{-7} & 8.81e^{-4} \\ 7.49e^{-7} & 1.61e^{-4} & 8.569e^{-4} & 0.707 & .707 & .278 \\ 3.93e^{-6} & 7.89e^{-4} & .9997 & .0147 & .007 & .0115 \\ 1.0 & 9.84e^{-6} & 3.89e^{-6} & 9.61e^{-7} & 1.46e^{-6} & 2.55e^{-6} \end{pmatrix}$$

Observe that some of the entries of this matrix are indeed very close to the theoretically correct values, while others are terribly off. The reason is that there are a few eigenvalues of K that are just too close together. This produces numerical instability in the computation of the corresponding eigenvectors. On the other hand all the eigenvalues of D are nicely separated, and the corresponding eigenvectors can be trusted.

Some papers dealing with non-commutative algebras of differential operators with a common eigenfunction.

M.M. Castro and F.A. Grünbaum, The algebra of differential operators associated to a given family of matrix valued orthogonal polynomials: five instructive examples. Int. Math. Res. Not., 27(2):1–33, 2006.

F.A. Grünbaum and J. Tirao, The algebra of differential operators associated to a weight matrix Integral Equations Operator Theory, 58(4):449–475, 2007.

J.Tirao, The algebra of differential operators associated to a weight matrix: a first example. Polcino Milies, César (ed.), Groups, algebras and applications. XVIII Latin American algebra colloquium, São Pedro, Brazil, August 3–8, 2009. Proceedings. Providence, RI: American Mathematical Society (AMS). Contemporary Mathematics 537, 291-324 (2011), 2011.

Zurrian,I. "The Algebra of Differential Operators for a Gegenbauer Weight Matrix" arXiv: arXiv:1505.03321

Motivation for the new formulation.

Consider a field $U = U(t, x)$ whose time evolution is given by the action of a differential operator with matrix coefficients $\mathcal{T}$ acting on U. From physical/geometric considerations this is given by the action of a differential operator $\mathcal{S}$ (once again with matrix coefficients) in the spatial variables x.

To guarantee commutativity of these two actions we agree that operators in t act on the left and operators in x act on the right.

The evolution equation is given by

$$\mathcal{T}U(t, x) = U(t, x)\mathcal{S}$$

If we try solutions of the form

$$U(t,x) = Psi(t)Phi(x)$$

we get

$$\mathcal{T}Psi(t)Phi(x) = Psi(t)Phi(x)\mathcal{S}$$

and then we get the identity

$$Psi(t)^{-1}\mathcal{T}Psi(t) = Phi(x)\mathcal{S}Phi(x)^{-1}$$

Since this needs to be a constant matrix Λ we get

$$\mathcal{T}Psi(t) = Psi(t)\Lambda$$

and

$$Phi(x)\mathcal{S} = \Lambda Phi(x)$$

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