Double affine symmetric group action
on the toric network
and
generalized disctete Toda lattice

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Happy Birthday, Sasha!

Plan

§1 Background: Discrete Toda lattice

§2 Toric network and Double affine symmetric group

§3 Algebro geometrical study of the network

based on a joint work with
Thomas Lam (Michigan) and Pavlo Pylyavskyy (Minnesota)
arXiv:1504.03448

§1 Background: Discrete Toda lattice

• Discrete Toda lattice [Hirota et al 1993]

$$\begin{cases} q_i^{t+1} + w_{i-1}^{t+1} = w_i^t + q_i^t \\ w_i^{t+1} q_i^{t+1} = q_{i+1}^t w_i^t \end{cases} \qquad q_i^t, w_i^t \in \mathbb{C} \;\; (i, t \in \mathbb{Z})$$

$$\overset{\delta \to 0}{\longrightarrow} \;\; \frac{d^2}{dt^2} x_i = e^{x_{i+1} - x_i} - e^{x_i - x_{i-1}} \quad \text{via} \; \begin{cases} q_i^t = 1 + \delta \frac{d}{dt} x_i \\ w_i^t = \delta^2 e^{x_{i+1} - x_i} \end{cases}$$

• n-periodic case

a birational map τ on $\mathcal{M} = \{q := (q_i, w_i)_{i \in \mathbb{Z}/n\mathbb{Z}}\} \simeq \mathbb{C}^{2n}$:

$$\tau : (q_i, w_i)_i \overset{\pi}{\mapsto} (w_i, q_{i+1})_i \overset{R}{\mapsto} (q_i', w_i')_i$$

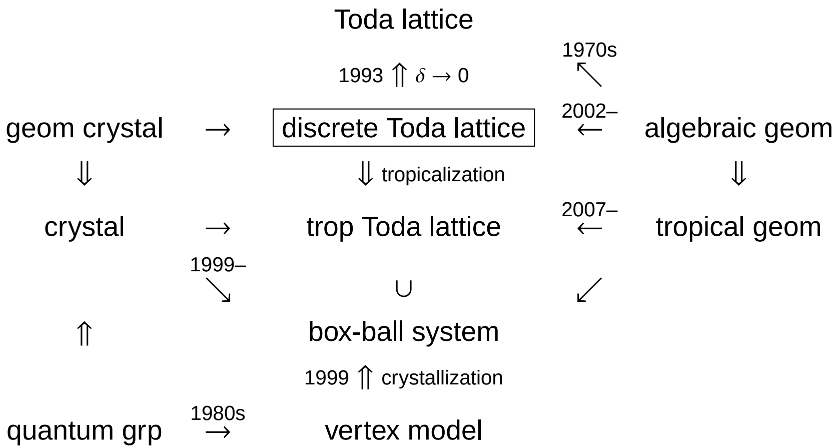
$R \curvearrowright \mathbb{C}^n \times \mathbb{C}^n$: geometric R matrix (of A_{n-1}-type), substruction free

• tropicalization: $(\mathbb{C}, +, \times) \to (\mathbb{R}, \min, +)$

$R^* \curvearrowright \mathbb{R}^n \times \mathbb{R}^n$

$R^*|_{\mathbb{Z}^n_{\geq 0} \times \mathbb{Z}^n_{\geq 0}}$: combinatorial R matrix (for A_{n-1}-crystal)

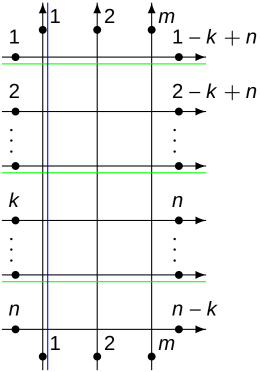
• Related math



§2 Toric network and Double affine symmetric group

- Toric network [Lam-Pylyvavskyy 2008-]

$n, m, k \in \mathbb{Z}_{>0}$, $1 \le k \le n$, $N := \gcd(n, k)$



$(n, m, k) = (6, 3, 4)$

· Phase space

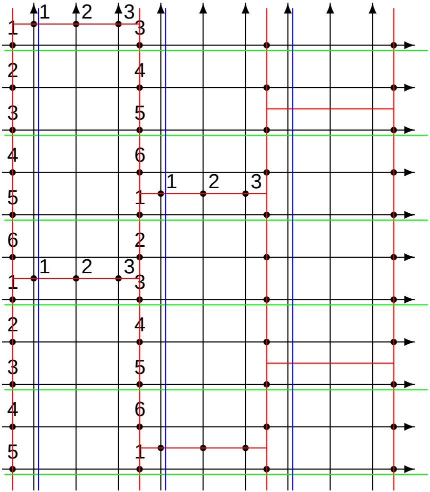
$$\mathcal{M} \simeq \mathbb{C}^{mn} \ni (q_{i,j})_{1 \le i \le m, 1 \le j \le n}$$

$$q_{i,j+n} = q_{i,j}, \; q_{i+m,j} = q_{i,j-k}$$

· $m + N$ simple closed curves

$$\mathbf{q}_i := (q_{i,j})_{1 \le j \le n} \quad i = 1, \ldots, m$$

$$\tilde{\mathbf{q}}_j := (q_{i,j})_{1 \le i \le \frac{mn}{N}} \quad j = 1, \ldots, N$$



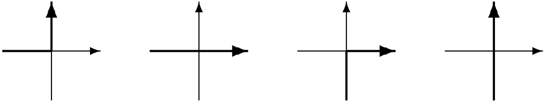
1 2 3
1 3
2 4
3 5
4 6
5 1
1 2 3
6 2
1 2 3
1 3
2 4
3 5
4 6
5 1

· Highway path

a path $\mathbf{p}$ from a source i to a sink j has $\mathrm{wt}(\mathbf{p}) = x^{n_{\mathbf{p}}} \prod_{p\in\mathbf{p}} \mathrm{wt}(p)$;

$n_{\mathbf{p}} :=$ (a number $\mathbf{p}$ crosses x-lines)

p:



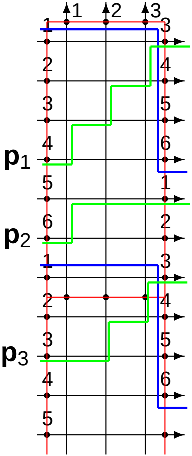
wt(p): 1 $\qquad q_{ij} \qquad$ 1 $\qquad$ 0

(Ex) $(n, m, k) = (6, 3, 4)$

$$\mathrm{wt}(\mathbf{p}_1) = x \cdot 1^6$$

$$\mathrm{wt}(\mathbf{p}_2) = 1^2 \cdot q_{2,5} \cdot q_{3,5}$$

$$\mathrm{wt}(\mathbf{p}_3) = x \cdot q_{1,3} \cdot 1^4$$



Lemma [Lam-Pylyavskyy 13]

Define the Lax matrix $L(x) := Q_1(x)\cdots Q_m(x)P(x)^k \in \mathrm{Mat}_n(\mathbb{C})$;

$$Q_i(x) = \begin{pmatrix} q_{i1} & 0 & 0 & x \\ 1 & q_{i2} & 0 & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & 0 & 1 & q_{in} \end{pmatrix} \quad P(x) = \begin{pmatrix} 0 & 0 & 0 & x \\ 1 & 0 & 0 & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Then $L(x)_{i,j} = \sum_{\mathbf{p}:\text{highway path } i\to j} \mathrm{wt}(\mathbf{p}).$

- R-matrix action

$$R \curvearrowright \mathbb{C}^s \times \mathbb{C}^s;\ (\mathbf{a},\mathbf{b}) := (a_i, b_i)_{1\le i\le s} \mapsto (b_i \frac{P_i}{P_{i-1}}, a_i \frac{P_{i-1}}{P_i})_{1\le i\le s}$$

$$\text{where } P_i := P_i(\mathbf{a},\mathbf{b}) = \sum_{j=0}^{s-1} \prod_{l=1}^{j} b_{l+i} \prod_{l=j+2}^{s} a_{l+i},\ i \in \mathbb{Z}/s\mathbb{Z}$$

(Ex) $R \curvearrowright \mathbb{C}^3 \times \mathbb{C}^3; (\mathbf{a},\mathbf{b}) \mapsto (\mathbf{a}',\mathbf{b}')$

$$a_1' = b_1 \frac{a_3a_1 + b_2a_1 + b_2b_3}{a_2a_3 + b_1a_3 + b_1b_2}, \quad b_1' = a_1 \frac{a_2a_3 + b_1a_3 + b_1b_2}{a_3a_1 + b_2a_1 + b_2b_3}$$

Remark

$$R \circ R = \mathrm{id}_{\mathbb{C}^s \times \mathbb{C}^s}$$

$$R_1R_2R_1 = R_2R_1R_2 \quad \curvearrowright \mathbb{C}^s \times \mathbb{C}^s \times \mathbb{C}^s$$

$$R_1(\mathbf{a},\mathbf{b},\mathbf{c}) := (R(\mathbf{a},\mathbf{b}),\mathbf{c}),\ \ R_2(\mathbf{a},\mathbf{b},\mathbf{c}) := (\mathbf{a}, R(\mathbf{b},\mathbf{c}))$$

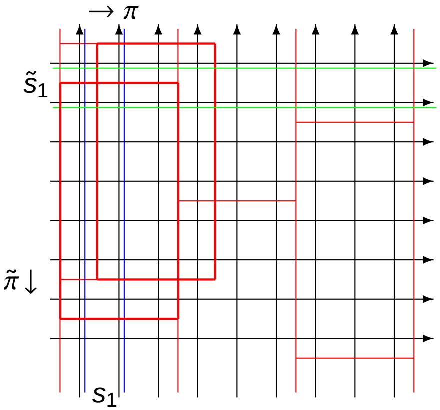
- Affine symmetric grp action on $\mathcal{M}$

$$W := < \pi, s_1, \ldots, s_{m-1} >, \quad \widetilde{W} := < \tilde{\pi}, \tilde{s}_1, \ldots, \tilde{s}_{N-1} >$$

$$s_i^2 = \mathrm{id}, \; s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \; \pi s_{i+1} = s_i \pi \;\; \text{etc}$$

$$s_i(\mathbf{q}_1, \ldots, \mathbf{q}_m) = (\cdots, R(\mathbf{q}_i, \mathbf{q}_{i+1}), \cdots), \quad \pi(\mathbf{q}_i) = (\mathbf{q}_{i+1})$$

$$\tilde{s}_i(\tilde{\mathbf{q}}_1, \ldots, \tilde{\mathbf{q}}_N) = (\cdots, R(\tilde{\mathbf{q}}_i, \tilde{\mathbf{q}}_{i+1}), \cdots), \quad \tilde{\pi}(\tilde{\mathbf{q}}_j) = (\tilde{\mathbf{q}}_{j+1})$$



Commuting subgrp of $W \times \tilde{W}$:

$$e_u := (s_u \cdots s_{m-1})(s_{u-1} \cdots s_{m-2}) \cdots (s_1 \cdots s_{m-u})\pi^u \quad u = 1, \ldots, m$$
$$\tilde{e}_u := (\tilde{s}_u \cdots \tilde{s}_{N-1})(\tilde{s}_{u-1} \cdots \tilde{s}_{N-2}) \cdots (\tilde{s}_1 \cdots \tilde{s}_{N-u})\tilde{\pi}^u \quad u = 1, \ldots, N$$

$\rightsquigarrow$ $\mathbb{Z}^m \times \mathbb{Z}^N$ action on $\mathcal{M}$

Remark

$(n, m, k) = (n, 2, n-1)$: discrete Toda lattice ($\tau = e_1$)

$(n, m, 0)$: $W(A^{(1)}_{m-1} \times A^{(1)}_{n-1})$-symmetric system

[Kajiwara-Noumi-Yamada 2002]

Prop

$W \times \widetilde{W}$-action induces the action on the Lax matrix $L(x) = Q_1(x) \cdots Q_m(x) P(x)^k$ as

$$
\begin{cases}
s_u^*(L(x)) = L(x) \\
\pi^*(L(x)) = Q_1(x)^{-1} L(x) Q_1(x) = Q_2(x) \cdots Q_m(x) P(x)^k Q_1(x) \\
\tilde{s}_u^*(L(x)) = B_u^{-1} L(x) B_u \ (B_u : \text{almost diagonal}) \\
\tilde{\pi}^*(L(x)) = P(x) L(x) P(x)^{-1}
\end{cases}
$$

In particular, $\mathrm{Det}(L(x) - y)$ is invariant under the action.

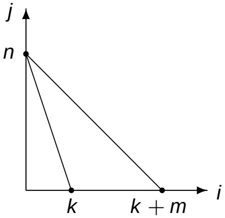
§3 Algebro geometrical study of the network

• Spectral curve

Recall $L(x) := Q_1(x) \cdots Q_m(x) P(x)^k \in \mathrm{Mat}_n(\mathcal{M})$.

$\psi : \mathcal{M} \to V \subset \mathbb{C}[x,y];\ q \mapsto L(x) \mapsto \mathrm{Det}(L(x) - y\mathbb{I}) =: f$

$V := \left\{ f = \sum_{i,j} f_{ji} x^i y^j \in \mathbb{C}[x,y] \text{ having Newton polygon } \swarrow \right\}$



$f \in V$ is generic

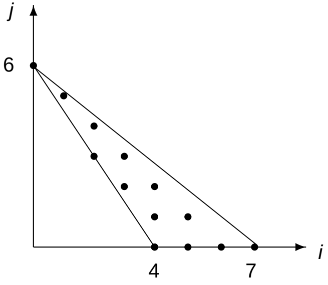
$\iff$ $C_f =$ (normalization of $\overline{\{f = 0\}}$ at $(0,0)$) is smooth

with
- P : a point at ∞
- O_j $(1 \le j \le N)$: N points over $(0,0)$
- A_i $(1 \le i \le m)$: m points of $(*, 0)$

(Ex) $(n,m,k) = (6,3,4)$, $N = 2$

$$L(x) = Q_1(x)Q_2(x)Q_3(x)P(x)^4$$

$$f = (y^6 + x^7) + y^5 x\, f_{5,1} + y^4 x^2 f_{4,2} + y^3(x^2 f_{3,2} + x^3 f_{3,3})$$
$$+ y^2(x^3 f_{2,3} + x^4 f_{2,4}) + y(x^4 f_{1,4} + x^5 f_{1,5}) + x^4 f_{0,4} + x^5 f_{0,5} + x^6 f_{0,6}$$



$\mathrm{genus}(C_f) = 7$

$A_1, A_2, A_3, O_1, O_2, P \in C_f$

$\mathbb{Z}^3 \times \mathbb{Z}^2$ action

$$\begin{cases} e_1 = s_1 s_2 \pi,\ e_2 = s_2 s_1 \pi^2,\ e_3 = \pi^3 \\ \tilde{e}_1 = \tilde{s}_1 \tilde{\pi},\ \tilde{e}_2 = \tilde{\pi}^2 \end{cases}$$

Thm [I-Lam-Pylyavskyy 2015]

(1) The $W \times \widetilde{W}$ action preserves the fiber $\psi^{-1}(f)$ of $\psi : \mathcal{M} \to V$.
(isolevel set)

(2) Fix a generic $f \in V$. We have an embedding (eigenvector map)

$$\phi : \psi^{-1}(f) \hookrightarrow \mathrm{Pic}^g(C_f) \times S_f \times R_O \times R_A =: \mathcal{J},$$

where $g :=$ (genus of C_f), $M := \gcd(n, m+k)$
$S_f := \left\{(c_1, \cdots, c_M) \in (\mathbb{C}^*)^M ; \prod_i c_i = (\text{a sum of some } f_{ji})\right\}$
$R_O := \{$orderings of $(O_1, \ldots, O_N)\}$
$R_A := \{$ordering of $(A_1, \ldots, A_m)\}$.

(Cf) [van Moerbeke-Mumford 1979]

$(\phi : \psi^{-1}(f) \hookrightarrow \mathrm{Pic}^g(C_f) \times S_f \times R_O \times R_A =: \mathcal{J})$

(3) For $q \in \psi^{-1}(f)$; $\phi(q) = ([D], c = (c_1, \ldots, c_M), O, A) \in \mathcal{J}$, the $\mathbb{Z}^m \times \mathbb{Z}^N$ action on $\mathcal{J}$ is described as

$$\phi(e_u(q)) = ([D - uP + A_1 + \cdots + A_u], \sigma^{-u}(c), O, A)$$
$$\phi(\tilde{e}_u(q)) = ([D + uP - O_1 - \cdots - O_u], \sigma^{u}(c), O, A),$$

where $\sigma(c) = (c_M, c_1, \ldots, c_{M-1})$. (linearization on $\mathrm{Pic}^g(C_f)$)

(4) When $N = 1$, ϕ^{-1} is explicitly given by Riemann theta function:

$$q_{i,j} = \mathrm{const} \cdot a_i \cdot c_{i+j-1} \frac{\theta_{j,i}(D)\, \theta_{j-1,i-1}(D)}{\theta_{j-1,i}(D)\, \theta_{j,i-1}(D)}.$$

(Cf) [Iwao 2008-10]

Key

$W \times \widetilde{W}$ induces

$$\pi^* : ([D], c, O, A) \mapsto ([D + A_1 - P], \sigma^{-1}(c), O, (A_2, \ldots, A_m, A_1))$$
$$s_u^* : ([D], c, O, A) \mapsto ([D], c, O, (\ldots, A_{u+1}, A_u, \ldots))$$
$$\tilde{\pi}^* : ([D], c, O, A) \mapsto ([D - O_1 + P], \sigma(c), A, (O_N, O_1, \ldots, O_{N-1}))$$
$$\tilde{s}_u^* : ([D], c, O, A) \mapsto ([D], c, O, (\ldots, O_{u+1}, O_u, \ldots))$$

$$
\begin{array}{ccccc}
 & & \text{Cluster integrable system} & & \\
 & & \vdots & \overset{\text{2013–}}{\nwarrow} & \\
\text{geom crystal} & \rightarrow & (n,m,k)\text{-Toda lattice} & \leftarrow & \text{algebraic geom} \\
\Downarrow & & \Downarrow \text{ tropicalization} & & \Downarrow \\
\text{crystal} & \rightarrow & \text{trop Toda lattice} & \overset{\text{2007–}}{\leftarrow} & \text{tropical geom} \\
 & \overset{\text{1999–}}{\searrow} & \cup & \swarrow & \\
\Uparrow & & \text{box-ball system} & & \Uparrow \\
 & & \text{1999} \Uparrow \text{ crystallization} & & \\
\text{quantum grp} & \overset{\text{1980s}}{\rightarrow} & \text{vertex model} & \leftarrow & \text{??}
\end{array}
$$

Thank you!