Connection problem for the tau-function of the sine-Gordon reduction of the third Painlevé equation

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Setting of the problem

The Painlevé III:

$$u_{xx} + \frac{1}{x}u_x + \sin u = 0, \quad x > 0,$$

The Cauchy data

$$u(x) = \alpha \ln x + \beta + O\left(x^{2-|\Im\alpha|}\right), \quad x \to 0,$$

$$\alpha \in \mathbb{C}, \quad |\Im\alpha| < 2 \quad \beta \in \mathbb{C}$$

The Hamiltonian structure.

$$H = -x\cos u + \frac{v^2}{2x}, \quad \Omega = dv \wedge du, \quad \{v, u\} = 1,$$

$$PIII \quad \Longleftrightarrow \quad u_x = \frac{\partial H}{\partial v}, \;\; v_x = -\frac{\partial H}{\partial u} \quad (v = xu_x)$$

The tau-function:

$$\frac{d\ln\tau}{dx} = -\frac{1}{4}H$$

(M. Jimbo, T. Miwa, K. Ueno; K. Okamoto, 1980)

Large and small x behavior of the tau-function:

$$\tau(x) = C_0 x^{-\frac{\alpha^2}{8}}(1 + o(1)), \quad x \to 0,$$

and

$$\tau(x) = C_\infty x^{\nu^2} e^{\frac{x^2}{8} + 2\nu x}(1 + o(1)), \quad x \to \infty.$$

Here,

$$|\Im \nu| < \frac{1}{2}, \quad \nu \equiv \nu(\alpha, \beta) \quad \text{is known}$$

The question:

$$\frac{C_\infty}{C_0}(\alpha, \beta) = ?$$

History of the "constant problem"

- The Strong Szegő Theorem (Szegő; Onsager & Kaufman, 1952, special PVI)
- The constant in the scaling theory of the Ising model (Tracy, 1991, special PIII)
- Random matrices, Dyson's constant (Dyson, 1976; Krasovsky, 2004; Ehrhardt, 2006, special PV)
- Toeplitz determinants and Fedholm determinants arising in statistical mechanics and random matrices (Widom; Basor & Widom; Basor& Tracy; Budylin&Buslaev; Deift, Krasovsky, Zhou, I; Baik, Buckingham, DiFranco)

The answer

Define,

$$\sigma := \frac{1}{4} + \frac{i}{8}\alpha, \quad \text{and} \quad \eta := \frac{1}{4} + \frac{1}{4\pi}\Big(\beta + \alpha \ln 8\Big) + \frac{i}{2\pi} \ln \frac{\Gamma\left(\frac{1}{2} - \frac{i\alpha}{4}\right)}{\Gamma\left(\frac{1}{2} + \frac{i\alpha}{4}\right)},$$

$$0 < \Re\sigma < \frac{1}{2} \quad \sin 2\pi\eta \neq 0, \quad \left|\arg \frac{\sin 2\pi\eta}{\sin 2\pi\sigma}\right| < \frac{\pi}{2}$$

Then,

$$\nu = \frac{1}{\pi} \ln \frac{\sin 2\pi\eta}{\sin 2\pi\sigma},$$

and

$$\frac{C_\infty}{C_0} = \frac{2^{\frac{3}{2}} e^{-i\frac{\pi}{4}}}{\pi (G(\frac{1}{2}))^4} (2\pi)^{i\nu} 2^{2\nu^2+\sigma^2 24-12\sigma} e^{2\pi i(\eta^2-2\sigma\eta-\sigma^2+2\eta-\sigma)}$$

$$\times \frac{\Gamma(1-2\sigma)}{\Gamma(2\sigma)} \left(\frac{G(1+i\nu)G(1+2\sigma)G(1-2\sigma)}{G(\frac{1+i\nu}{2}-\sigma-\eta)} \right)^2$$

$$\times \left(\frac{G(1+\sigma+\eta+\frac{1-i\nu}{2})G(\frac{1-i\nu}{2}-\sigma-\eta)}{G(1+\sigma+\eta+\frac{1+i\nu}{2})} \right)^2,$$

$G(z)$ - Barnes G -function

O. LIzovyy, Yu. Tykhyy, A. I. - conjecture, 2014;
A. Prokhorov, A. I. - proven, 2015

The Riemann-Hilbert representation of PIII

The PIII Riemann-Hilbert problem:

- $\Psi(\lambda)$ is analytic in $\mathbb{C} \setminus \{\Gamma\}$
- $\Psi_+(\lambda) = \Psi_-(\lambda)S$
- $\Psi(\lambda) = \left(I + O\left(\frac{1}{\lambda}\right)\right)e^{-\frac{ix^2\lambda}{16}\sigma_3}, \quad \lambda \to \infty,$
- $\Psi(\lambda) = P_0\left(I + O(\lambda)\right)e^{-\frac{i}{\lambda}\sigma_3}, \quad \lambda \to 0.$

The RH representation for the third Painlevé transcendent:

$$u(x) = 2\arccos(P_0)_{11}$$

(Flaschka-Newell, 1980)

S-matrices:

$$S_1^{(\infty)} = S_2^{(0)} = \begin{pmatrix} 1 & 0 \\ p+q & 1 \end{pmatrix},$$

$$S_2^{(\infty)} = S_1^{(0)} = \begin{pmatrix} 1 & p+q \\ 0 & 1 \end{pmatrix},$$

$$E = \frac{1}{\sqrt{1+pq}} \begin{pmatrix} 1 & p \\ -q & 1 \end{pmatrix},$$

$$p, q \in \mathbb{C}, \quad 1 + pq \neq 0,$$

σ, η -parametrization of the RH data:

$$p := -i\frac{\sin 2\pi(\sigma + \eta)}{\sin 2\pi\eta},$$

$$q := i\frac{\sin 2\pi(\sigma - \eta)}{\sin 2\pi\eta}.$$

Important:

(η, σ) - canonical coordinates. In fact,

$$\Omega = 32\pi i d\eta \wedge d\sigma$$

This parametrization of the PIII Riemann-Hilbert data was introduced by M. Jimbo in 1982.

Connection formulae for $u(x)$

$$u(x) \equiv u(x; \eta, \sigma),$$

Assume,

$$0 < \Re\sigma < \frac{1}{2} \quad \sin 2\pi\eta \neq 0, \quad \left|\arg \frac{\sin 2\pi\eta}{\sin 2\pi\sigma}\right| < \frac{\pi}{2}$$

Then,

- as $x \to 0$,

$$u(x) = \alpha \ln x + \beta + O\left(x^{2-|\Im\alpha|}\right),$$

where,

$$\alpha = i(2 - 8\sigma),$$

$$\beta = -\pi + 4\pi\eta - i(2 - 8\sigma)\ln 8 - 2i \ln \frac{\Gamma(1 - 2\sigma)}{\Gamma(2\sigma)},$$

M. Jimbo, 1982; V. Novokshenov, 1985

- as $x \to \infty$,

$$u(x) = b_+ e^{ix} x^{i\nu - 1/2}\left(1 + O\left(\frac{1}{x}\right)\right) + b_- e^{-ix} x^{-i\nu - 1/2}\left(1 + O\left(\frac{1}{x}\right)\right)$$

$$+ O\left(x^{3|\Im\nu| - 3/2}\right) \pmod{2\pi},$$

$$\nu = -\frac{1}{4} b_+ b_-, \quad |\Im\nu| < 1/2,$$

where

$$b_{\pm} = -e^{\frac{\pi\nu}{2} \mp \frac{i\pi}{4}} 2^{1 \pm 2i\nu} \frac{1}{\sqrt{2\pi}} \Gamma(1 \mp i\nu) \frac{\sin 2\pi(\sigma \mp \eta)}{\sin 2\pi\eta},$$

$$\nu = \frac{1}{\pi} \ln \frac{\sin 2\pi\eta}{\sin 2\pi\sigma},$$

V. Novokshenov, 1985; V. Novokshenov, A. I., 1986; A. V. Kitaev, 1987

The tau-function. Conformal block approach

(Lizovyy, Yu. Tykhyy, A. I.)

$$\tau(x) \to \tau_m(t):$$

$$\tau\left(2^{-12}x^4\right) = \tau^{1/2}(x)x^{1/4}e^{\frac{iu}{4}}.$$

- The α, β connection formulae $\Longrightarrow$:

$$\tau_m(t) = \text{const}\,\Big[t^{\sigma^2}\left(1 + \frac{t}{2\sigma^2} + \frac{8\sigma^2+1}{4\sigma^2(4\sigma^2-1)}t^2 + \ldots\right)$$

$$-e^{4\pi i\eta}\frac{\Gamma^2(-1-2\sigma)}{\Gamma^2(1+2\sigma)}t^{(\sigma+1)^2}\left(1 + \frac{t}{2(\sigma+1)^2} + \{\sigma\to\sigma+1\} + \ldots\right)$$

$$-e^{-4\pi i\eta}\frac{\Gamma^2(-1+2\sigma)}{\Gamma^2(1-2\sigma)}t^{(\sigma-1)^2}\left(1 + \frac{t}{2(\sigma-1)^2} + \{\sigma\to\sigma-1\} + \ldots\right)$$

$$+\ldots\Big]$$

Chose:

$$\text{const} := \frac{1}{G(1+2\sigma)G(1-2\sigma)},$$

then,

$$\tau_m(t) = \frac{1}{G(1+2\sigma)G(1-2\sigma)} t^{\sigma^2}\left(1 + \frac{t}{2\sigma^2} + \dots\right)$$

$$-e^{4\pi i\eta}\frac{1}{G(1+2\sigma+2)G(1-2\sigma-2)} t^{(\sigma+1)^2}\left(1 + \frac{t}{2(\sigma+1)^2} + \dots\right)$$

$$-e^{-4\pi i\eta}\frac{1}{G(1+2\sigma-2)G(1-2\sigma+2)} t^{(\sigma-1)^2}\left(1 + \frac{t}{2(\sigma-1)^2} + \dots\right)$$

$$+\dots, \quad t \to 0$$

Conjecture 1:

$$\tau_m(t) = \sum_{n\in\mathbb{Z}} e^{4\pi i n\eta} F(\sigma + n, t),$$

$$F(\sigma, t) = \frac{t^{\sigma^2}}{G(1+2\sigma)G(1-2\sigma)} B(\sigma, t)$$

$$B(\sigma, t) = 1 + \sum_{k=1}^{\infty} B_k(\sigma) t^k.$$

- The $b_\pm$ connection formulae $\Longrightarrow$:

$$\tau_m\left(2^{-12}x^4\right) = \text{const } x^{\frac{1}{4}} e^{\frac{x^2}{16}} \left[x^{\frac{\nu^2}{2}} e^{\nu x} \left(1 + \frac{\nu(2\nu^2+1)}{8x} + \dots\right)\right.$$

$$+ \frac{ib_+}{4} x^{\frac{(\nu+i)^2}{2}} e^{(\nu+i)x} \left(1 + \frac{(\nu+i)(2(\nu+i)^2+1)}{8x} + \dots\right)$$

$$+ \frac{ib_-}{4} x^{\frac{(\nu-i)^2}{2}} e^{(\nu-i)x} \left(1 + \frac{(\nu-i)(2(\nu-i)^2+1)}{8x} + \dots\right)$$

$$\left. + \dots \right], \quad x \to \infty,$$

Chose:

$$\text{const} := e^{\frac{i\pi\nu^2}{4}} 2^{\nu^2} (2\pi)^{-\frac{i\nu}{2}} G(1 + i\nu)$$

and take into account that,

$$\frac{ib_\pm}{4} e^{\frac{i\pi\nu^2}{4}} 2^{\nu^2} (2\pi)^{-\frac{i\nu}{2}} G(1 + i\nu)$$

$$= e^{\frac{i\pi(\nu\pm i)^2}{4}} 2^{(\nu\pm i)^2} (2\pi)^{-\frac{i(\nu\pm i)}{2}} G(1 + i(\nu \pm i)) e^{\pm 4\pi i\rho},$$

where

$$e^{4\pi i\rho} := \frac{\sin 2\pi\eta}{\sin 2\pi(\sigma + \eta)}.$$

Conjecture 2:

$$\tau_m\Big(2^{-12}x^4\Big) = \chi(\sigma,\eta)\sum_{n\in\mathbb{Z}} e^{4\pi i\rho}J(\nu+in,x),$$

$$e^{4\pi i\rho} := \frac{\sin 2\pi\eta}{\sin 2\pi(\sigma+\eta)}.$$

$$J(\nu,x) = e^{\frac{i\pi\nu^2}{4}}2^{\nu^2}(2\pi)^{-\frac{i\nu}{2}}G(1+i\nu)x^{\frac{1}{4}+\frac{\nu^2}{2}}e^{\frac{x^2}{16}+\nu x}D(\nu,x),$$

$$D(\nu,x) \sim 1+\sum_{k=1}^{\infty}D_k(\nu)x^{-k},$$

Important:

(ρ, ν) - canonical coordinates. In fact,

$$\Omega = 32\pi d\rho \wedge d\nu$$

Regarding the series at $x = 0$.

- Both the series are convergent
- $F(\sigma, t)$ is the irregular $c = 1$ Virasoro conformal block
- AGT duality $\to$ $F(\sigma, t) =$ the partition function of the $N = 2$ supersymmetric pure SU(2) gauge theory $\to$

$$B(\sigma, t) = \sum_{\lambda,\mu \in \mathbb{Y}} \left(\frac{\dim \lambda \dim \mu}{|\lambda|!|\mu|!} \right)^2 \frac{t^{|\lambda|+|\mu|}}{|b_{\lambda,\mu}(\sigma)|^2},$$

$$b_{\lambda,\mu}(\sigma) = \prod_{(k,l)\in\lambda} (\lambda'_l - k + \mu_k - l + 1 + 2\sigma) \prod_{(k,l)\in\mu} (\mu'_l - k + \lambda_k - l + 1 - 2\sigma)$$

(Nekrasov instanton sum)

O. Gamayun, N. Iorgov, O. Lisovyy, A. Shchechkin, Yu. Tykhyy, J. Teschner

Regarding the series at $x = \infty$.

• *Conjecture*: The Fourier series for $\tau_m\left(2^{-12}x^4\right)$ is convergent. The series for $D(\nu, x)$ is an asymptotic series

• *Great open question*: Conformal block interpretation of $J(\nu, x)$ and the Nekrasov type combinatorial formula for $D(\nu, x)$.

The use of conformal series for evaluating $\chi(\sigma, \eta)$

- **Step 1.**

(σ, η) - canonical coordinates at $x = 0$,

(ν, ρ) - canonical coordinates at $x = \infty$

Let $\mathcal{W}(\sigma, \nu)$ be the **generating function** of the canonical transformation $(\sigma, \eta) \to (\nu, \rho)$, i.e.,

$$d\mathcal{W}(\sigma, \nu) = \eta d\sigma + i\rho d\nu,$$

The functions $\rho(\sigma,\nu)$ and $\eta(\sigma,\nu)$ are known. Indeed,

$$e^{4\pi i\rho} = \frac{\sin 2\pi\eta}{\sin 2\pi(\sigma+\eta)}, \quad e^{\pi\nu} = \frac{\sin 2\pi\eta}{\sin 2\pi\sigma}$$

Hence one can try to evaluate the integral,

$$\mathcal{W}(\sigma,\nu) = \int \eta d\sigma + i\rho d\nu$$

The result of integration:

$$\mathcal{W}(\sigma,\nu) = \frac{1}{8\pi^2}\Big[\mathrm{Li}_2(-e^{2\pi i(\sigma+\eta-i\frac{\nu}{2})}) + \mathrm{Li}_2(-e^{-2\pi i(\sigma+\eta+i\frac{\nu}{2})})$$

$$-4\pi^2\eta^2 + \pi^2\nu^2\Big],$$

$$\mathrm{Li}_2(z) = \int_z^0 \frac{\ln(1-z)}{z}dz.$$

From the classical formula,

$$\mathrm{Li}_2(e^{2\pi z}) = -2\pi i \ln \hat{G}(z) - 2\pi i z \ln \frac{\sin(\pi z)}{\pi} - \pi^2 z(1-z) + \frac{\pi^2}{6},$$

where

$$\hat{G}(z) = \frac{G(1+z)}{G(1-z)},$$

it follows that

$$\mathcal{W}(\sigma,\nu) = \frac{1}{4\pi i} \ln \frac{\hat{G}(\sigma+\eta+\frac{1-i\nu}{2})}{\hat{G}(\sigma+\eta+\frac{1+i\nu}{2})}$$

$+$polynomial of σ, η, ν

• **Step 2.**

Let

$$\chi(\sigma, \eta) \to \chi(\sigma, \nu).$$

Then, conformal series implies that

$$\frac{\chi(\sigma+1, \nu)}{\chi(\sigma, \nu)} = e^{-4\pi i\eta},$$

and

$$\frac{\chi(\sigma, \nu+i)}{\chi(\sigma, \nu)} = e^{4\pi i\rho},$$

Observe that the formal continuous analogs of these recurrence relations are

$$\frac{\partial}{\partial \sigma} \ln \chi(\sigma, \nu) = -4\pi i \eta,$$

and

$$\frac{\partial}{\partial \nu} \ln \chi(\sigma, \nu) = 4\pi \rho,$$

It is conceivable then to expect a relation between $\ln \chi(\sigma, \nu)$ and the generating function $\mathcal{W}(\sigma, \nu)$.

A non-formal observation. Put

$$\mathcal{A}(\sigma,\nu) := \mathcal{W}(\sigma,\nu) - \sigma\eta - i\nu\rho.$$

Then the partial derivatives,

$$4\pi i\frac{\partial}{\partial\sigma}\mathcal{A}(\sigma,\nu) \quad \text{and} \quad 4\pi i\frac{\partial}{\partial\nu}\mathcal{A}(\sigma,\nu)$$

satisfy the **same** recurrence relations as the partial derivatives,

$$\frac{\partial}{\partial\sigma}\ln\chi(\sigma,\nu) \quad \text{and} \quad \frac{\partial}{\partial\nu}\ln\chi(\sigma,\nu).$$

From this observation, one arrives at the following **key** formula,

$$\chi(\sigma,\nu) = (2\pi)^{i\nu}\exp\left\{i\pi\left(\eta^2 - 2\sigma\eta - \sigma^2 + \eta - \sigma - \frac{\nu^2}{4}\right)\right\}$$

$$\times\frac{\hat{G}(\sigma+\eta+\frac{1-i\nu}{2})}{\hat{G}(\sigma+\eta+\frac{1+i\nu}{2})}\chi\mathrm{per}(\sigma,\eta).$$

Conjecture 3:

$$\chi\mathrm{per}(\sigma,\eta) \equiv \mathrm{const}$$

.

Conjecture 3 allows us to evaluate $\chi\text{per}(\sigma,\eta)$ by choosing $\sigma=\eta=\frac{1}{4}$, $\nu=0$ (i.e. $u\equiv 0$). The result is,

$$\chi\text{per}(\sigma,\eta)=e^{\frac{i\pi}{8}}\frac{2^{-\frac{3}{4}}}{\sqrt{\pi}G^2\left(\frac{1}{2}\right)}.$$

and hence,

$$\chi(\sigma,\nu)=(2\pi)^{i\nu-\frac{1}{2}}\exp\left\{i\pi\left(\eta^2-2\sigma\eta-\sigma^2+\eta-\sigma-\frac{\nu^2}{4}+\frac{1}{8}\right)\right\}$$

$$\times\frac{2^{-\frac{1}{4}}}{\sqrt{\pi}G^2\left(\frac{1}{2}\right)}\frac{\hat{G}(\sigma+\eta+\frac{1-i\nu}{2})}{\hat{G}(\sigma+\eta+\frac{1+i\nu}{2})}.$$

The RH deformation approach

(A. Prokhorov, A. I.)

Put

$$Y(\lambda) := \Psi(\lambda) e^{\left(\frac{ix^2\lambda}{16} + \frac{i}{\lambda}\right)\sigma_3},$$

and let

$$G(\lambda) = e^{-\left(\frac{ix^2\lambda}{16} + \frac{i}{\lambda}\right)\sigma_3} S(\lambda) e^{\left(\frac{ix^2\lambda}{16} + \frac{i}{\lambda}\right)\sigma_3},$$

be the jump matrix for $Y(\lambda)$.

Define the **Malgrange-Bertola differential form** on the space of parameters $\{x, p, q\}$,

$$\omega_{MB}[\partial] = \int_{\Gamma} \mathrm{Tr}\Big(Y_-^{-1} Y_-' (\partial G) G^{-1}\Big) \frac{d\lambda}{2\pi i}$$

$$+\frac{1}{2} \int_{\hat{\Gamma}} \mathrm{Tr}\Big(G' G^{-1} (\partial G) G^{-1}\Big) \frac{d\lambda}{2\pi i}.$$

Proposition. The Malgrange-Bertola differential form ω_{MB} admits the following representation

$$\omega_{MB} = \left(-\frac{xu_x^2}{8} + \frac{x}{4}(\cos u - 1)\right) dx$$

$$- \left(\frac{x^2}{4}u_p \sin u + \frac{x^2}{4}u_x u_{px} + \frac{xu_x u_p}{4}\right) dp$$

$$- \left(\frac{x^2}{4}u_q \sin u + \frac{x^2}{4}u_x u_{qx} + \frac{xu_x u_q}{4}\right) dq.$$

Corollary 1.

$$\omega_{MB}[\partial_x] = \partial_x \ln \tau - \frac{x}{4}.$$

Corollary 2

$$d\omega_{MB} = \frac{v_p u_q - v_q u_p}{4} dq \wedge dp = -\frac{1}{4}\Omega$$

Put

$$\omega = \omega_{MB} + \frac{x}{4}dx + \frac{\alpha d\beta}{4},$$

Then we would have,

- $\omega[\partial_x] = \omega_{JMU}[\partial_x] \equiv \partial_x \ln \tau$
- $d\omega = 0$

Hence the equation

$$\tau = e^{\int \omega},$$

would define the tau-function up to a constant, which **does not depend on p and q.**

Evaluation of the factor $\frac{C_\infty}{C_0}$.

The known asymptotics of $u(x)$ imply that

$$\omega = -d\left(\frac{\alpha^2}{8}\ln x + \frac{\alpha^2}{8}\right) + o(1), \quad x \to 0, \tag{1}$$

and

$$\omega = d\left(2\nu x + \nu^2 \ln x + \nu^2\right) - \frac{i}{4}(b_+ db_- - b_- db_+)$$

$$+\frac{x}{4}dx + \frac{\alpha d\beta}{4} + o(1), \quad x \to \infty.$$

On the other hand, we have that

$$\omega = -d\left(\frac{\alpha^2}{8}\ln x\right) + d\ln C_0 + o(1), \quad x \to 0,$$

and

$$\omega = d\left(2\nu x + \nu^2 \ln x + \frac{x^2}{8}\right) + d\ln C_\infty + o(1), \quad x \to \infty.$$

The comparison of the two sets of the asymptotic relations yields the **key** equation,

$$d \ln \frac{C_\infty}{C_0} = d\left(\nu^2 + \frac{\alpha^2}{8} - i\nu\right) + \frac{\alpha d\beta}{4} - \frac{i}{2} b_+ db_-,$$

or

$$\ln \frac{C_\infty}{C_0} = \nu^2 + \frac{\alpha^2}{8} - i\nu + \frac{1}{4}\int \alpha d\beta - 2ib_+ db_- + c,$$

where c is the numerical constant, **independent on** p **and** q**.**

Remark 1.
The integral

$$\int \alpha d\beta - 2ib_{+}db_{-}$$

is basically the integral

$$\int \eta d\sigma + i\rho d\nu$$

which was calculated within the conformal block approach.

Remark 2.
The formula

$$d \ln \frac{C_\infty}{C_0} = d\left(\nu^2 + \frac{\alpha^2}{8} - i\nu\right) + \frac{\alpha d\beta}{4} - \frac{i}{2} b_+ db_-,$$

tells us that the logarithm of the ratio

$$C_\infty / C_0 \sim \chi(\sigma, \nu)$$

is indeed, up to the elementary function, $\nu^2 + \alpha^2/8 - i\nu$, the **generating function of the canonical transformation** between the Cauchy data (α, β) and the asymptotic at infinity data (b_+, b_-).