Holonomy of braids and its 2-category extension

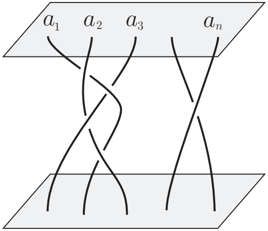
Toshitake Kohno

The University of Tokyo

July 13, 2015
Monte Verità

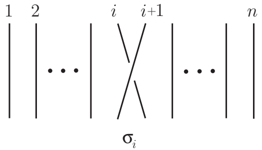
Braid groups

Braid groups were introduced by E. Artin in the 1920's.



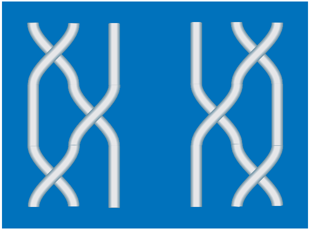
The isotopy classes of geometric braids as above form a group by composition. This is the braid group with n strands denoted by B_n.

Braid relations



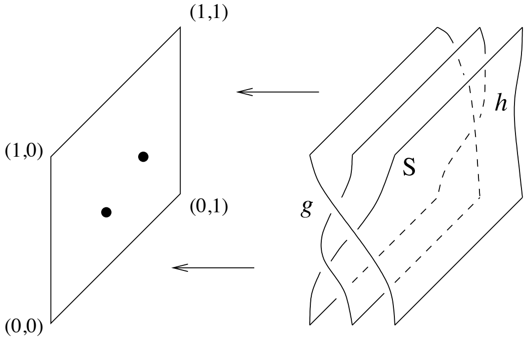
B_n is generated by σ_i, $1 \leq i \leq n-1$ with relations

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$
$$\sigma_i \sigma_j = \sigma_j \sigma_i, \quad |i-j| > 1$$



Braid cobordisms

branched covering with simple branched points



surface braid (Kamda, Carter and Saito)
braid cobordism category $\mathcal{B}C_n$:
- objects : geometric braids with n strands
- morphisms : relative isotopy classes of cobordisms between braids

Plan

- Monodromy representations of logarithmic connections
- Knizhnik-Zamolodchikov (KZ) connection
- Homological representations and hypergeometric integrals
- 2-categories
- Higher holonomy
- Representations of braid cobordism category

Configuration spaces

$\mathcal{F}_n(X)$: configuration space of ordered distinct n points in X.

$$\mathcal{F}_n(X) = \{(x_1, \cdots, x_n) \in X^n \ ; \ x_i \neq x_j \text{ if } i \neq j\},$$
$$\mathcal{C}_n(X) = \mathcal{F}_n(X)/\mathfrak{S}_n$$

Configuration spaces

$\mathcal{F}_n(X)$: configuration space of ordered distinct n points in X.

$$\mathcal{F}_n(X) = \{(x_1, \cdots, x_n) \in X^n \; ; \; x_i \neq x_j \text{ if } i \neq j\},$$
$$\mathcal{C}_n(X) = \mathcal{F}_n(X)/\mathfrak{S}_n$$

Suppose $X = \mathbf{C}$.

$$\pi_1(\mathcal{F}_n(\mathbf{C})) = P_n, \quad \pi_1(\mathcal{C}_n(\mathbf{C})) = B_n$$

We set $X_n = \mathcal{F}_n(\mathbf{C})$

Logarithmic forms on configuration spaces

We set

$$\omega_{ij} = d\log(z_i - z_j), \quad 1 \leq i \neq j \leq n.$$

Consider a total differential equation of the form $d\phi = \omega\phi$ for a logarithmic form

$$\omega = \sum_{i<j} A_{ij}\omega_{ij}$$

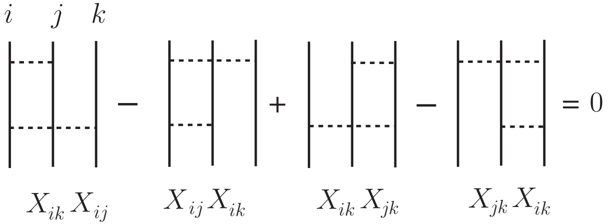
with $A_{ij} \in M_m(\mathbf{C})$.

Infinitesimal pure braid relations

As the flatness condition we infinitesimal pure braid relations

$$[A_{ik}, A_{ij} + A_{jk}] = 0, \quad (i, j, k \ \text{ distinct}),$$
$$[A_{ij}, A_{k\ell}] = 0, \quad (i, j, k, \ell \ \text{ distinct})$$

Algebra of horizontal chord diagrams:



K. T. Chen's iterated integrals of differential forms

$\omega_1, \cdots, \omega_k$: differential forms on M
ΩM : loop space M

$$\Delta_k = \{(t_1, \cdots, t_k) \in \mathbf{R}^k \ ; \ 0 \leq t_1 \leq \cdots \leq t_k \leq 1\}$$

$$\varphi : \Delta_k \times \Omega M \to \underbrace{M \times \cdots \times M}_{k}$$

defined by $\varphi(t_1, \cdots, t_k; \gamma) = (\gamma(t_1), \cdots, \gamma(t_k))$

K. T. Chen's iterated integrals of differential forms

$\omega_1, \cdots, \omega_k$: differential forms on M
ΩM : loop space M

$$\Delta_k = \{(t_1, \cdots, t_k) \in \mathbf{R}^k \ ; \ 0 \leq t_1 \leq \cdots \leq t_k \leq 1\}$$

$$\varphi : \Delta_k \times \Omega M \to \underbrace{M \times \cdots \times M}_{k}$$

defined by $\varphi(t_1, \cdots, t_k; \gamma) = (\gamma(t_1), \cdots, \gamma(t_k))$

The iterated integral of $\omega_1, \cdots, \omega_k$ is defined as

$$\int \omega_1 \cdots \omega_k = \int_{\Delta_k} \varphi^*(\omega_1 \times \cdots \times \omega_k)$$

Iterated integrals as differential forms on loop space

The expression

$$\int_{\Delta_k} \varphi^*(\omega_1 \times \cdots \times \omega_k)$$

is the integration along fiber with respect to the projection $p : \Delta_k \times \Omega M \to \Omega M$.

differential form on the loop space ΩM
with degree $p_1 + \cdots + p_k - k$, where $p_j = \deg \omega_j$

Differentiation on loop spaces

As a differential form on the loop space $d \int \omega_1 \cdots \omega_k$ is

$$\sum_{j=1}^{k} (-1)^{\nu_{j-1}+1} \int \omega_1 \cdots \omega_{j-1} d\omega_j \ \omega_{j+1} \cdots \omega_k$$

$$+ \sum_{j=1}^{k-1} (-1)^{\nu_j+1} \int \omega_1 \cdots \omega_{j-1} (\omega_j \wedge \omega_{j+1}) \omega_{j+2} \cdots \omega_k$$

where $\nu_j = \deg \omega_1 + \cdots + \deg \omega_j - j$.

Universal finite type invariants for braids

We put

$$\omega = \sum_{i<j} \omega_{ij} X_{ij}.$$

Then there is a universal holonomy map

$$\Theta_0 : \pi_1(X_n, \mathbf{x}_0) \longrightarrow \mathbf{C}\langle\langle X_{ij} \rangle\rangle / \mathfrak{a}$$

defined by

$$\Theta_0(\gamma) = 1 + \sum_{k=1}^{\infty} \int_\gamma \underbrace{\omega \cdots \omega}_{k}$$

$\mathfrak{a}$: ideal generated by infinitesimal pure braid relations
$\mathbf{C}\langle\langle X_{ij} \rangle\rangle / \mathfrak{a}$: algebra of horizontal chord diagrams

We put

$$\omega = \sum_{i<j} \omega_{ij} X_{ij}.$$

Then there is a universal holonomy map

$$\Theta_0 : \pi_1(X_n, \mathbf{x}_0) \longrightarrow \mathbf{C}\langle\langle X_{ij} \rangle\rangle / \mathfrak{a}$$

defined by

$$\Theta_0(\gamma) = 1 + \sum_{k=1}^{\infty} \int_\gamma \underbrace{\omega \cdots \omega}_{k}$$

$\mathfrak{a}$: ideal generated by infinitesimal pure braid relations
$\mathbf{C}\langle\langle X_{ij} \rangle\rangle / \mathfrak{a}$: algebra of horizontal chord diagrams
This induces an isomorphism

$$\mathbf{C}\widehat{P_n} \cong \mathbf{C}\langle\langle X_{ij} \rangle\rangle / \mathfrak{a}$$

$\mathbf{C}\widehat{P_n}$: Malcev completion

The space of conformal blocks

Conformal field theory

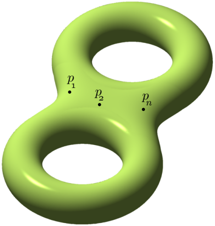
$(\Sigma, p_1, \cdots, p_n)$: Riemann surface with marked points

$\mapsto$

$\mathcal{H}_\Sigma$: complex vector space - the space of conformal blocks

The mapping class group $\Gamma_{g,n}$ acts on $\mathcal{H}_\Sigma$:

Quantum representations



Affine Lie algebra

$\mathfrak{g}$: complex semi-simple Lie algebra
$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbf{C}((\xi)) \oplus \mathbf{C}c$: affine Lie algebra with commutation relation

$$[X \otimes f, Y \otimes g] = [X, Y] \otimes fg + \mathrm{Res}_{\xi=0}\, df\, g\, \langle X, Y\rangle c$$

$\widehat{\mathfrak{g}} = \mathcal{N}_+ \oplus \mathcal{N}_0 \oplus \mathcal{N}_-$ (triangular decomposition)

For the Lie algebra $\mathfrak{g}$ we take
$\alpha_1, \cdots, \alpha_r$: simple roots
$\rho = \frac{1}{2}\sum_{\alpha>0} \alpha$: half sum of positive roots

K : a positive integer (level)
θ : the longest root normalized as $\langle\theta,\theta\rangle = 2$
λ : dominant integral weight s. t. $\langle\lambda,\theta\rangle \le K$ (level K weight)
V_λ : irreducible representation of $\mathfrak{g}$ with highest weight λ

Construction of representations of $\widehat{\mathfrak{g}}$
$\mathcal{M}_\lambda = U(\mathcal{N}_-)V_\lambda$, $\mathcal{N}_+V_\lambda = 0$
  V_λ : irreducible $\mathfrak{g}$-module with highest weight λ
  c acts as $K \cdot$ id.
$\mathcal{H}_\lambda$: irreducible quotient of $\mathcal{M}_\lambda$ called the integrable highest weight modules.

The space of conformal blocks - definition -

Suppose $\lambda_1, \cdots, \lambda_n$ are level K weights.
$p_1, \cdots, p_n \in \Sigma$ (Riemann surface of genus g)
Assign highest weights $\lambda_1, \cdots, \lambda_n$ to $p_1, \cdots, p_n$.
$\mathcal{H}_{\lambda_j}$: irreducible representations of $\widehat{\mathfrak{g}}$ with highest weight λ_j at level K.

The space of conformal blocks - definition -

Suppose $\lambda_1, \cdots, \lambda_n$ are level K weights.
$p_1, \cdots, p_n \in \Sigma$ (Riemann surface of genus g)
Assign highest weights $\lambda_1, \cdots, \lambda_n$ to $p_1, \cdots, p_n$.
$\mathcal{H}_{\lambda_j}$: irreducible representations of $\widehat{\mathfrak{g}}$ with highest weight λ_j at level K.

$\mathcal{M}_p$ denotes the set of meromorphic functions on Σ with poles at most at $p_1, \cdots, p_n$.

The space of conformal blocks - definition -

Suppose $\lambda_1, \cdots, \lambda_n$ are level K weights.
$p_1, \cdots, p_n \in \Sigma$ (Riemann surface of genus g)
Assign highest weights $\lambda_1, \cdots, \lambda_n$ to $p_1, \cdots, p_n$.
$\mathcal{H}_{\lambda_j}$: irreducible representations of $\widehat{\mathfrak{g}}$ with highest weight λ_j at level K.

$\mathcal{M}_p$ denotes the set of meromorphic functions on Σ with poles at most at $p_1, \cdots, p_n$.

The space of conformal blocks is defined as

$$\mathcal{H}_\Sigma(p, \lambda) = \mathcal{H}_{\lambda_1} \otimes \cdots \otimes \mathcal{H}_{\lambda_n} / (\mathfrak{g} \otimes \mathcal{M}_p)$$

where $\mathfrak{g} \otimes \mathcal{M}_p$ acts diagonally via Laurent expansions at $p_1, \cdots, p_n$.

Projectively flat connection

The space of conformal blocks determines a vector bundle over the moduli space $\mathcal{M}_{g,n}$ with a projectively flat connection.

We focus on the case of genus 0.
We assign level K weights $\lambda_1, \cdots \lambda_n$ and λ_∞ at infinity.
The space of conformal block is a quotient space of

$$V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n} \otimes V^*_{\lambda_{n+1}} \ / \ \mathfrak{g}$$

In this case the above connection is the KZ connection.

KZ connections

$\mathfrak{g}$: complex semi-simple Lie algebra.
$\{I_\mu\}$: orthonormal basis of $\mathfrak{g}$ w.r.t. Killing form.
$\Omega = \sum_\mu I_\mu \otimes I_\mu$
$r_i : \mathfrak{g} \to End(V_i)$, $1 \leq i \leq n$ representations.

KZ connections

$\mathfrak{g}$: complex semi-simple Lie algebra.
$\{I_\mu\}$: orthonormal basis of $\mathfrak{g}$ w.r.t. Killing form.
$\Omega = \sum_\mu I_\mu \otimes I_\mu$
$r_i : \mathfrak{g} \to End(V_i)$, $1 \le i \le n$ representations.

Ω_{ij} : the action of Ω on the i-th and j-th components of $V_1 \otimes \cdots \otimes V_n$.

$$\omega = \frac{1}{\kappa} \sum_{i<j} \Omega_{ij} d\log(z_i - z_j), \quad \kappa \in \mathbf{C} \setminus \{0\}$$

ω defines a flat connection for a trivial vector bundle over the configuration space $X_n = \mathcal{F}_n(\mathbf{C})$ with fiber $V_1 \otimes \cdots \otimes V_n$ since we have

$$\omega \wedge \omega = 0$$

Monodromy representations of braid groups

As the holonomy we have representations

$$\theta_\kappa : P_n \to GL(V_1 \otimes \cdots \otimes V_n).$$

In particular, if $V_1 = \cdots = V_n = V$, we have representations of braid groups

$$\theta_\kappa : B_n \to GL(V^{\otimes n}).$$

Monodromy representations of braid groups

As the holonomy we have representations

$$\theta_\kappa : P_n \to GL(V_1 \otimes \cdots \otimes V_n).$$

In particular, if $V_1 = \cdots = V_n = V$, we have representations of braid groups

$$\theta_\kappa : B_n \to GL(V^{\otimes n}).$$

Local system over the configuration space

We write

$$\sum_{i=1}^{n} \lambda_i - \lambda_\infty = \sum_{j=1}^{r} k_j \alpha_j$$

and put $m = \sum_{j=1}^{r} k_j$.

$\pi : X_{n+m} \to X_n$: projection defined by
$(z_1, \cdots, z_n, t_1, \cdots, t_m) \mapsto (z_1, \cdots, z_n)$.
$X_{n,m}$: fiber of π.

$$\Phi = \prod_{1 \le i < j \le n} (z_i - z_j)^{\frac{\langle \lambda_i, \lambda_j \rangle}{\kappa}} \prod_{1 \le i \le m, 1 \le \ell \le n} (t_i - z_\ell)^{-\frac{\langle \alpha_i, \lambda_\ell \rangle}{\kappa}} \times \prod_{1 \le i < j \le m} (t_i - t_j)^{\frac{\langle \alpha_i, \alpha_j \rangle}{\kappa}}$$

Consider the local system $\mathcal{L}$ associated with Φ.

Solutions to KZ equation

Put

$$Y_{n,m} = X_{n,m}/(\mathfrak{S}_{k_1} \times \cdots \times \mathfrak{S}_{k_r})$$

Solutions to KZ equation

Put

$$Y_{n,m} = X_{n,m}/(\mathfrak{S}_{k_1} \times \cdots \times \mathfrak{S}_{k_r})$$

According to Schechtman-Varchenko and others, one can construct horizontal sections of the KZ connections by means of hypergeometric integrals of the form

$$\int_{\Delta} \Phi R(z,t) dt_1 \wedge \cdots \wedge dt_m$$

with some rational function $R(z,t)$.
Δ is a cycle in $H_m(Y_{n,m}, \mathcal{L}^*)$.
We can construct a period map

$$\phi : H_m(Y_{n,m}, \mathcal{L}^*) \to \mathcal{H}^*(p, \lambda)$$

Monodromy and homology of local systems

Theorem

The period map

$$\phi : H_m(Y_{n,m}, \mathcal{L}^*) \longrightarrow \mathcal{H}^*(p, \lambda)$$

is surjective and is equivariant with respect to the action of the pure braid group P_n*. If* K *is sufficiently large relative to* $\lambda_1, \cdots, \lambda_n$ *the period map* ϕ *gives an isomorphism. In particular, the linear representation*

$$\rho_{n,m} : P_n \longrightarrow \mathrm{Aut}\ \mathcal{H}^*(p, \lambda)$$

and the monodromy representation of the KZ equation

$$\overline{\theta}_{\kappa,m} : P_n \to \mathrm{Aut}\ \mathcal{H}^*(p, \lambda)$$

are equivalent.

objects, morphisms, 2-morphisms

$$\begin{tikzcd}[column sep=huge] y & x \arrow[l, bend right, "f"'] \arrow[l, bend left, "g"] \arrow[l, phantom, "\Downarrow\alpha"] \end{tikzcd}$$

vertical composition

$$\begin{tikzcd}[column sep=huge] y & x \arrow[l, bend right=60, "f"'] \arrow[l, "f'" description] \arrow[l, bend left=60, "f''"] \arrow[l, phantom, shift right=3, "\Downarrow\alpha"] \arrow[l, phantom, shift left=3, "\Downarrow\alpha'"] \end{tikzcd} = \begin{tikzcd}[column sep=huge] y & x \arrow[l, bend right, "f"'] \arrow[l, bend left, "f''"] \arrow[l, phantom, "\Downarrow\alpha'\cdot\alpha"] \end{tikzcd}$$

horizontal composition

$$\begin{tikzcd}[column sep=huge] z & y \arrow[l, bend right, "f_1"'] \arrow[l, bend left, "f_1'"] \arrow[l, phantom, "\Downarrow\alpha_1"] & x \arrow[l, bend right, "f_2"'] \arrow[l, bend left, "f_2'"] \arrow[l, phantom, "\Downarrow\alpha_2"] \end{tikzcd} = \begin{tikzcd}[column sep=huge] z & x \arrow[l, bend right, "f_1 f_2"'] \arrow[l, bend left, "f_1' f_2'"] \arrow[l, phantom, "\Downarrow\alpha_1\circ\alpha_2"] \end{tikzcd}$$

Categorification of KZ connections

There is a work in progress to construct 2-holonomy of KZ connection for braid cobordism based on the 2-connection investigated by L. Cirio and J. Martins of the form

$$A = \sum_{i<j} \omega_{ij} \Omega_{ij}$$

$$B = \sum_{i<j<k} (\omega_{ij} \wedge \omega_{ik} \ P_{jik} + \omega_{ij} \wedge \omega_{jk} \ P_{ijk}),$$

where A has values in the algebra of 2-chord diagrams, a categorification of the algebra of horizontal chord diagrams and

$$\delta B = dA + \frac{1}{2} A \wedge A.$$

Consider Chen's formal homology connection

$$\omega \in \Omega^*(M) \otimes \widehat{TH_+(M)}$$

with the following properties.

- $\omega = \sum x^i \otimes x_i + \cdots$, $\{x_i\}$: basis of $TH_+(M)$
  $\deg x_i = p_1 - 1$ for $x_i \in H_{p_i}(M)$
- $\delta\omega + \kappa = 0$
- $\kappa = d\omega + \epsilon(\omega)\omega \wedge \omega, \ \epsilon(\omega) = \pm 1$
- δ is a derivation of degree -1

Theorem

There is a representation of the homotopy 2-groupoid modulo isotopy

$$Hol : \Pi_2(M)/\sim \longrightarrow T\widehat{H_+(M)}_{\leq 2}/\mathcal{J}$$

where $\mathcal{J}$ is the ideal generated by the image of

$$\delta_3 : T\widehat{H_+(M)}_3 \longrightarrow T\widehat{H_+(M)}_2$$

The ideal $\mathcal{J}$ corresponds to the 2-flatness condition.

Consider the case $M = X_n$.
Universal holonomy map from the the homotopy path groupoid

$$\Theta_0 : \Pi_1(X_n) \longrightarrow \mathbf{C}\langle\langle X_{ij} \rangle\rangle$$

given by iterated integrals.

$$T\widehat{H_+(M)}_1 \cong \mathbf{C}\langle\langle X_{ij} \rangle\rangle$$

Theorem

The universal holonomy map Θ_0 can be lifted to a representation of the braid cobordism category

$$Hol : \mathcal{B}C_n \longrightarrow T\widehat{H_+(M)}_{\leq 2}/\mathcal{J}$$

Categorification and related problems

C : cobordism between links L_1 and L_2
$Kh(C) : Kh(L_1) \to Kh(L_2)$
invariants of 2-knots (Khovanov, Jacobson)

Braid group action on categories

- Khovanov-Rouquier-Lauda algebra
- Derived categories of coherent sheaves on Calabi-Yau manifolds
- Fukaya-Seidel category

Problem : Extend the above actions to the braid cobordism category $\mathcal{B}C_n$.